BETA-GAMMA ALGEBRA, DISCOUNTED CASH-FLOWS, AND BARNES' LEMMAS

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Discounted cash-flows

Suppose i.i.d. cash-flows $\{C_n\}$ occur at times $\{T_n\}$, and that one wishes to find the distribution of the discounted value of all future cash-flows. If the discount rate is $r > 0$, then this is

$$X = \sum_{n=1}^{\infty} e^{-rT_n} C_n.$$

Let the waiting times

$$W_1 = T_1, \qquad W_n = T_n - T_{n-1}, \qquad n \geq 2,$$

be i.i.d., making $\{T_n\}$ a renewal process, and assume moreover that $\{T_n\}$ and $\{C_n\}$ are independent. Then the above sum may be rewritten as

$$X = \sum_{n=1}^{\infty} A_1 \cdots A_n C_n \qquad \text{if} \quad A_n = e^{-rW_n}.$$

Such sums of products of random variables occur in a variety of applications and have been studied for several decades.

It is known that in such cases X satisfies the identity in law

$$X \stackrel{\mathrm{d}}{=} A(X + C).$$

A known example is:

$$G_1^{(a)} \stackrel{\mathrm{d}}{=} B^{(a,b)}(G_1^{(a)} + G_2^{(b)}),$$

where all variables on the right are independent and

$B^{(a,b)} \sim \mathbf{Beta}(a, b)$, $G_1^{(a)} \sim \mathbf{Gamma}(a, 1)$, $G_2^{(b)} \sim \mathbf{Gamma}(b, 1)$.

This means that

$$\sum_{n=1}^{\infty} B_1^{(a,b)} \cdots B_n^{(a,b)} G_n^{(a)} \sim \mathbf{Gamma}(a,1)$$

(all variables independent).

The identity

$$G_1^{(a)} \stackrel{\mathrm{d}}{=} B^{(a,b)}(G_1^{(a)} + G_2^{(b)}),$$

is the same as

$$G_1^{(a)} \stackrel{\mathrm{d}}{=} B^{(a,b)} G_2^{(a+b)}$$

This is part of the so-called "beta-gamma algebra". It may be proved with Mellin transforms, *i.e.* by checking that

$$\mathsf{E}[G_1^{(a)}]^p = \mathsf{E}[B^{(a,b)} G_2^{(a+b)}]^p, \qquad p \geq 0.$$

Mellin transform: if $X \geq 0$, then $\mathcal{M}_X(s) = \mathsf{E}X^s$.

Mellin transforms for sums of positive variables

Theorem A. *Suppose* $c > 0$, $\mathrm{Re}(p) > c$ *and*

$$\mathsf{E}(X_1^{-c} X_2^{c-\mathrm{Re}(p)}) < \infty.$$

Then

$$\mathsf{E}(X_1 + X_2)^{-p} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \, \mathsf{E}(X_1^{-z} X_2^{z-p}) \frac{\Gamma(z)\Gamma(p-z)}{\Gamma(p)}.$$

Barnes' Lemmas and properties of beta and gamma variables

Barnes' First Lemma (Barnes, 1908). *For a suitably curved line of integration, so that the decreasing sequences of poles lie to the left and the increasing sequences lie to the right of the contour,*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\,\Gamma(A+z)\Gamma(B+z)\Gamma(C-z)\Gamma(D-z)$$

$$= \frac{\Gamma(A+C)\Gamma(A+D)\Gamma(B+C)\Gamma(B+D)}{\Gamma(A+B+C+D)}.$$

Theorem B. *By Theorem A, Barnes' First Lemma is equivalent to the additivity property of gamma distributions: if* $a, b > 0$ *and* $G_1^{(a)}, G_2^{(b)}$ *are independent, then* $G_1^{(a)} + G_2^{(b)} \stackrel{\mathrm{d}}{=} G_3^{(a+b)}$.

Barnes' Second Lemma (Barnes, 1910). *For a suitably curved line of integration, so that the decreasing sequences of poles lie to the left and the increasing sequences lie to the right of the contour, if* $E = A + B + C + D$,

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dz\, \frac{\Gamma(A+z)\Gamma(B+z)\Gamma(C+z)\Gamma(D-z)\Gamma(-z)}{\Gamma(E+z)}$$

$$= \frac{\Gamma(A)\Gamma(B)\Gamma(C)\Gamma(A+D)\Gamma(B+D)\Gamma(C+D)}{\Gamma(E-A)\Gamma(E-B)\Gamma(E-C)}.$$

Another properties of gamma variables (Dufresne, 1998)

Proposition C. *Suppose all variables are independent.*

For any $a, b, c > 0$,

$$B_1^{(a,b+c)} G_1^{(b)} + G_2^{(c)} \stackrel{\mathrm{d}}{=} G_3^{(b+c)} B_2^{(a+c,b)} \stackrel{\mathrm{d}}{=} G_4^{(a+c)} B_3^{(b+c,a)}.$$

Theorem D. *By Theorem A, Barnes' Second Lemma is equivalent to the property in Proposition C.*

Properties of reciprocal gamma variables

We look at the distribution of

$$H^{(a,b)} = \left(\frac{1}{G_1^{(a)}} + \frac{1}{G_2^{(b)}}\right)^{-1} = \frac{G_1^{(a)} G_2^{(b)}}{G_1^{(a)} + G_2^{(b)}},$$

where $a, b > 0$ and the the gamma variables are independent.

The distribution of $H^{(a,b)}$ turns out to be directly related to the "beta product distribution".

Proposition E. *The distribution of the product of independent $B^{(a,b)}$ and $B^{(c,d)}$ extends to a four-parameter family called the "beta product" distribution. It is a proper probability distribution on $(0,1)$ if, and only if, the parameters (a,b,c,d) satisfy:*

$a, c, b+d, \mathrm{Re}(a+b), \mathrm{Re}(c+d) > 0,$ *and either*

(i) (real case) b, d *are real and* $\min(a, c) < \min(a+b, c+d),$ *or*

(ii) (complex case) $\mathrm{Im}(b) = -\mathrm{Im}(d) \neq 0$ *and* $a+b = \overline{c+d}$*.*

"$B^{(a,b,c,d)}$" will represent a variable with that distribution.

The density of $B^{(a,b,c,d)}$ is

$$\frac{\Gamma(a+b)\Gamma(c+d)}{\Gamma(a)\Gamma(c)\Gamma(b+d)} u^{a-1}(1-u)^{b+d-1} \,{}_2F_1(a+b-c, d; b+d; 1-u)\mathbf{1}_{\{0<u<1\}}.$$

Theorem F. *(a) If* $\mathrm{Re}(p) > -\min(a,b)$*, then*

$$\mathsf{E}\left(H^{(a,b)}\right)^p \;=\; \frac{(a)_p(b)_p(a+b)_p}{(a+b)_{2p}}.$$

(b) For any $0 < a, b < \infty$*,*

$$H^{(a,b)} \;=\; \frac{G_1^{(a)}G_2^{(b)}}{G_1^{(a)}+G_2^{(b)}} \;\overset{\mathrm{d}}{=}\; \tfrac{1}{4}B^{(a,\frac{b-a}{2},b,\frac{a-b+1}{2})}G^{(a+b)},$$

where the variables on the right are independent. This is the same as:

$$\frac{1}{G^{(a)}}+\frac{1}{G^{(b)}} \;\overset{\mathrm{d}}{=}\; \frac{4}{B^{(a,\frac{b-a}{2},b,\frac{a-b+1}{2})}}\cdot\frac{1}{G^{(a)}+G^{(b)}}$$

(c) For $a, b > 0$ *and* $\mathrm{Re}(s) > -4$,

$$\mathsf{E}e^{-sH^{(a,b)}} = {}_3F_2(a, b, a+b; \tfrac{a+b}{2}, \tfrac{a+b+1}{2}; -\tfrac{s}{4}).$$

(d) For any $a, b > 0$,

$$(G_1^{(a+b)})^2 H^{(a,b)} \stackrel{\mathrm{d}}{=} G_2^{(a)} G_3^{(b)} G_4^{(a+b)},$$

where the variables on either side are independent.

Corollary G. *(a) The identity in law*

$$\frac{1}{G_1^{(a)}} \stackrel{\mathrm{d}}{=} A\left(\frac{1}{G_2^{(a)}} + \frac{1}{G_3^{(b)}}\right),$$

with independent variables on the right, has a solution A if, and only if, one of the three cases below occurs:

(i) $0 < a < b < \infty$, $b > \frac{1}{2}$. *Then*

$$A \stackrel{\mathrm{d}}{=} \frac{1}{4B^{(\frac{a+b}{2}, \frac{b-a}{2}, \frac{a+b+1}{2}, \frac{a+b-1}{2})}}.$$

(ii) $a = b > \frac{1}{2}$. *Then*

$$A \stackrel{\mathrm{d}}{=} \frac{1}{4B^{(a+\frac{1}{2}, a-\frac{1}{2})}}.$$

(iii) $a = b = \frac{1}{2}$. *Then* $A = \frac{1}{4}$ *and*

$$\frac{4}{G_1^{(\frac{1}{2})}} \stackrel{\mathrm{d}}{=} \frac{1}{G_2^{(\frac{1}{2})}} + \frac{1}{G_3^{(\frac{1}{2})}}.$$

(b) In any one of the three cases above, let

$$A_n = \frac{1}{4B_n^{(\frac{a+b}{2}, \frac{b-a}{2}, \frac{a+b+1}{2}, \frac{a+b-1}{2})}}, \qquad n = 1, 2, \ldots$$

Then, if all variables are independent,

$$\sum_{n=0}^{\infty} A_1 \cdots A_n \frac{1}{G_n^{(b)}} \stackrel{\mathrm{d}}{=} \frac{1}{G_0^{(a)}}.$$

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