Developments in Multi-Factor and Multi-Cohort Continuous Time Mortality Modelling

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Introduction

- Mortality models have attracted research attention over recent years, particularly discrete time mortality models: (Lee and Carter (1992), Cairns et al. (2006b), Cairns et al. (2009), Renshaw and Haberman (2006)).
- Continuous time affine cohort mortality models have attracted more recent research:
  - single cohort models - single and multi-factor (Milevsky and Promislow (2001), Dahl and Møller (2006), Biffis (2005), Luciano et al. (2008), Schrager (2006), Cairns et al. (2006a), Blackburn and Sherris (2013))
  - multi-cohort models (Jevtic et al. (2013), Xu et al. (2019), Chang and Sherris (2018), Huang et al. (2019)).

Why Continuous Time Multi-factor Multi-Cohort Models?

- Based on mathematical finance models for term structure (interest rates) and credit risk models (default rates) familiar to financial market participants,
- Capturing differing trends and volatility by age,
- Fitting to age-cohort data,
- More realistic correlations across cohorts,
- Arbitrage-free formulation along with real world dynamics to allow calibration of prices of risk,
- Analytical tractability - closed form survival curves for affine class,
- Consistency between mortality dynamics and functional form of the survival curve,
- MLE Estimation, Kalman Filter, Poisson variation, stability of parameter estimates.

Coverage of this Session

- Compare continuous-time multi-factor affine cohort mortality models - with closed-form expressions for survival curves, including the AFNS mortality model with factors for level, slope and curvature for the mortality curve,
    - Fitted with age-cohort data,
    - Brief overview of Kalman filter and estimation of the models, highlighting how Poisson variation can be incorporated into the model estimation,
    - Comparison of fits and prediction using historical US mortality data,
- Present estimation results for an affine multi-factor, multi-cohort mortality model,
    - Quantify the price of mortality risk for the factors using Blackrock CORI indices.

Survival Curve - Continuous Time Affine Mortality Model

- Drawing on term structure of interest rate models - equivalence of average force of mortality rate to yield to maturity for zero coupon bond. Use of similar notation as in yield curve modelling.
- Survival probability $S(x, t, T)$ for single cohort aged x at time t for survival for a duration $(T - t)$ to age $x + (T - t)$, as an affine function of (latent) factors (3 factor case)

$$\begin{aligned} S(x,t,T) &= E[e^{-\int_t^T \mu^i(x,s)ds} | \mathcal{F}_t] \\ &= e^{-\bar{\mu}(t,T)(T-t)} \\ &= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3(t,T)X_3(t)+A(t,T)}, \end{aligned} \quad (1)$$

- $B_j(t, T)$ are factor loadings (functional form derived from mortality dynamics for the latent factors, exponential terms) and $X_j(t)$ are the latent factors (stochastic parameters).

Mortality Rate - Continuous Time Affine Mortality Model

- Average mortality rate - age-period data or age-cohort data

$$\bar{\mu}(t,T) = -\frac{1}{T-t}\log\left[S(t,T)\right] = -\frac{B(t,T)'}{T-t}X_t - \frac{A(t,T)}{T-t}. \tag{2}$$

where vector $B(t,T)$, the factor loadings, and $A(t,T)$ have explicit expressions (derivations similar to term structure models).

- Canonical form for these (Blackburn and Sherris, 2013), where δ_{jj} and σ_{jj} are parameters in the latent factor dynamics (estimated from historical data)

$$B_j(t,T) = -\frac{1-e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1,2,3, \tag{3}$$

$$A(t,T) = \frac{1}{2}\sum_{j=1}^{3}\frac{\sigma_{jj}^2}{\delta_{jj}^3}\left[\frac{1}{2}\left(1-e^{-2\delta_{jj}(T-t)}\right) - 2\left(1-e^{-\delta_{jj}(T-t)}\right) + \delta_{jj}(T-t)\right]. \tag{4}$$

Mortality Rate - AFNS Mortality model

- Mortality model equivalent of the Nelson-Seigel term structure model in arbitrage-free dynamic implementation (Christensen et al., 2011)
- Mortality rate curve has level, slope and curvature factors (independent AFNS mortality model).

$$B^1(t,T) = -(T-t), \quad B^2(t,T) = -\frac{1-e^{-\delta(T-t)}}{\delta}, \qquad (5)$$
$$B^3(t,T) = (T-t)e^{-\delta(T-t)} - \frac{1-e^{-\delta(T-t)}}{\delta},$$

$$\frac{A(t,T)}{T-t} = \sigma_{11}^2\frac{(T-t)}{6} + \sigma_{22}^2\left[\frac{1}{2\delta^2} - \frac{1}{\delta^3}\frac{1-e^{-\delta(T-t)}}{T-t} + \frac{1}{4\delta^3}\frac{1-e^{-2\delta(T-t)}}{T-t}\right] +$$
$$\sigma_{33}^2\left[\frac{1}{2\delta^2} + \frac{1}{\delta^2}e^{-\delta(T-t)} - \frac{1}{4\delta}(T-t)e^{-2\delta(T-t)} - \frac{3}{4\delta^2}e^{-2\delta(T-t)}\right. \qquad (6)$$
$$\left.-\frac{2}{\delta^3}\frac{1-e^{-\delta(T-t)}}{T-t} + \frac{5}{8\delta^3}\frac{1-e^{-2\delta(T-t)}}{T-t}\right].$$

Model Estimation - Kalman Filter

- We model mortality rates but observe deaths - we need a measurement equation capturing the effects of Poisson variation and heterogeneity.
- Mortality rate curve changes stochastically through time, driven by latent factors with trend and uncertainty - we need a state transition equation for the dynamics.
- We then filter the values of latent factors from historical data - deriving means and covariances which are functions of the parameters in the dynamics.
- We can then construct the likelihood (Gaussian) in terms of means and covariance (a function of parameters to be estimated).
- Then numerically select the parameter set that maximises the likelihood using an iterative process.

Mortality Data - Estimating Mortality Models

- Complete age-cohort data for cohorts born in earlier years. US complete cohort data.

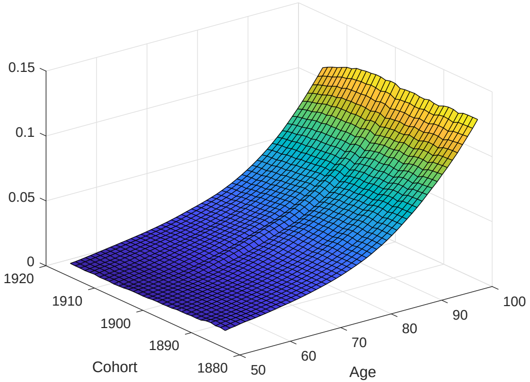


Figure 1: Average Force of Mortality for Males Born from 1883 to 1915

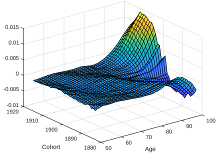
Affine Cohort Mortality Models - Goodness of Fit

Table 1: Comparison of Affine Mortality Models

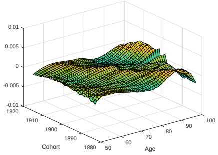
	The Blackburn-Sherris Model		The AFNS Model		The CIR Model
	Independent-Factor	Dependent-Factor	Independent-Factor	Dependent-Factor	
Log Likelihood	9896.419	9938.696	9665.801	9887.878	10045.70
RMSE	0.00250	7.601e-04	6.856e-04	9.160e-04	5.227e-04
No. of Parameters	12	18	10	13	18
AIC	-19570.837	-19643.392	-19113.602	-19551.757	-19857.40
BIC	-18968.292	-19008.277	-18521.914	-18943.783	-19222.29
Probability of Negative Mortality	0.02700	1.011e-32	1.722e-31	4.34e-14	-

- AFNS model fits historical age-cohort data well. Low negative mortality probabilities. CIR the best fit.

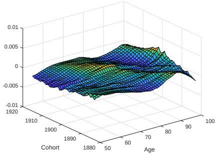
Affine Cohort Mortality Models - Residual Analysis



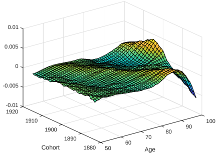
(a) The Independent Blackburn-Sherris Model



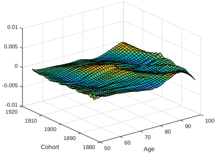
(b) The Dependent Blackburn-Sherris Model



(c) The Independent AFNS Model



(d) The Dependent AFNS Model



(e) The CIR Model

Figure 2: Residuals of Affine Mortality Models

Affine Cohort Mortality Models - MAPE across all cohorts

Mean absolute percentage error (MAPE) - independent AFNS mortality model performs well at older ages

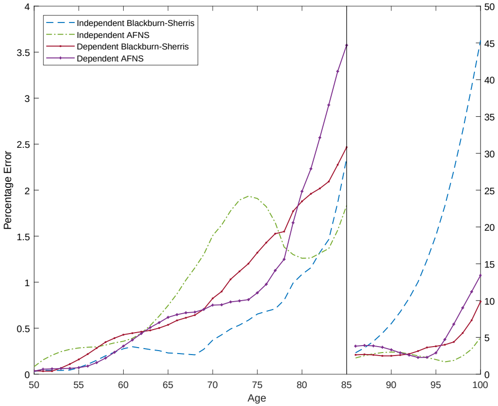


Figure 3: The Models with Gaussian Processes

Affine Cohort Mortality Models - MAPE

Mean absolute percentage error (MAPE) - independent AFNS mortality model performs similar to CIR mortality model at older ages

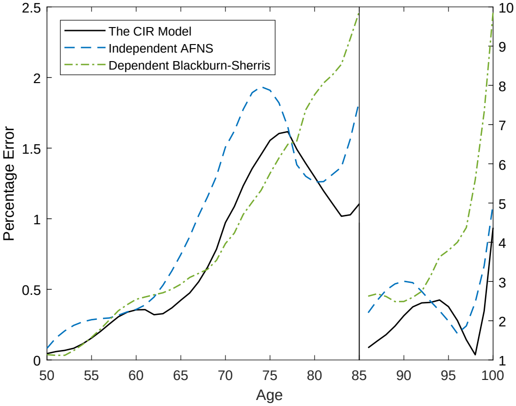


Figure 4: The CIR Model, the Dependent Blackburn-Sherris Model and the Independent AFNS Model

Affine Multi-factor Multi-cohort Continuous Time model

- The best-estimate survival probability, $S^{\bar{Q},i}(x,t,T)$ for cohort i aged x at time t over duration $T-t$, has a closed-form solution:

$$
\begin{aligned}
S^{\bar{Q},i}(x,t,T) &= E^{\bar{Q}}[e^{-\int_t^T \mu^i(x,s)ds}|\mathcal{F}_t] \\
&= e^{B_1(t,T)X_1(t)+B_2(t,T)X_2(t)+B_3^i(t,T)Z^i(t)+A^i(t,T)},
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
B_1(t,T) &= -\frac{1-e^{-\phi_1(T-t)}}{\phi_1}, \\
B_2(t,T) &= -\frac{1-e^{-\phi_2(T-t)}}{\phi_2}, \\
B_3^i(t,T) &= -\frac{1-e^{-\phi_3^i(T-t)}}{\phi_3^i}, \\
A^i(t,T) &= \frac{1}{2}\sum_{j=1}^{2}\frac{\sigma_j^2}{\phi_j^3}\left[\frac{1}{2}(1-e^{-2\phi_j(T-t)})-2(1-e^{-\phi_j(T-t)})+\phi_j(T-t)\right] \\
&\quad+\frac{1}{2}\frac{(\sigma_3^i)^2}{(\phi_3^i)^3}\left[\frac{1}{2}(1-e^{-2\phi_3^i(T-t)})-2(1-e^{-\phi_3^i(T-t)})+\phi_3^i(T-t)\right].
\end{aligned} \tag{8}
$$

Multi-cohort Mortality Model Estimation

- We use U.S. male mortality data from Human Mortality Database for 1934 to 2013, aged 50 to 100, cohorts born 1884 to 1913.
- Restructured on a cohort basis. Common cohort factors estimated with Kalman filter using all cohorts. Cohort specific factors estimated numerically by grouping cohorts.
- Sample survival probability for cohort i aged x at time t over duration $T-t$ is

$$\tilde{S}^i(x,t,T) = \prod_{s=1}^{T-t}(1-\tilde{q}_x^i(t+s-1)), \tag{9}$$

  where $\tilde{q}_x^i(t)$ is the observed death rate at time t.
- Corresponding sample average force of mortality is

$$\tilde{\mu}^i(x,t,T) = -\frac{1}{T-t}\log\tilde{S}^i(x,t,T). \tag{10}$$

Multi-cohort Mortality Model Estimation

- We use the Kalman filter estimation for the two common factors - parameter estimates shown in Table 2.

ϕ_1	-0.14313
ϕ_2	-0.07904
σ_1	0.00006
σ_2	0.00018
$\varepsilon_1(\times 10^7)$	2.74881
$\varepsilon_2(\times 10^7)$	1.99699
Log likelihood	24440
RMSE	0.00051

Table 2: Kalman filter parameter estimates, log likelihood and RMSE.

Multi-cohort Mortality Model Estimation

- Parameters for the cohort specific factors are estimated by minimising calibration error in the model after including the cohort factor.
- Grouping by 10 cohorts.
- Cohort parameters are shown in Table 3.

i cohort	ϕ_3^i	σ_3^i	$Z^i(0)$
1884-1893	0.06791	0.00558	0.00163
1894-1903	0.05228	0.00719	0.00106
1904-1913	0.05463	0.00122	-0.00079

Table 3: Estimation results for cohort specific factors with 10-year grouping of cohorts.

Calibrating Price of Risk - Blackrock CoRI

- BlackRock introduced the CoRI Indexes in June 2013 to help investors estimate and track the cost of \$1 of annual lifetime income at retirement.
- The CoRI consists of twenty indexes corresponding to twenty cohorts born from 1941 to 1960 in U.S.
- For cohorts with an age below 65 the index is the discounted cost of purchasing inflation-adjusted lifetime retirement income at age 65, and for other cohorts it is the cost of purchasing inflation-adjusted retirement income for remaining life.
- The CoRI indexes are constructed based on real-time market data, do not include any fees or premium taxes that would be associated with the price of an annuity. Available on NYSE.
- Investors can use the CoRI index as a risk metric directly or invest in the BlackRock CoRI Funds that track the index.

Implied Price of Longevity Risk based on CoRI indices

- The calibrated risk premiums are shown in Table 4.

$\hat{\lambda}_{\mu,1}$	$\hat{\lambda}_{\mu,2}$	$\hat{\lambda}^1_{\mu,3}$	$\hat{\lambda}^2_{\mu,3}$
0.3601	0.0892	0.1099	0.0973

Table 4: Calibrated market price of longevity risk.

- Note that all prices of risk are positive
- Prices of risk are consistent with and of similar magnitude to other studies.
- Cohort prices of risk calibrated for 1941 to 1950 cohorts ($\hat{\lambda}^1_{\mu,3}$) and 1951 to 1960 cohorts ($\hat{\lambda}^2_{\mu,3}$).
- There are similar prices of risk for each cohort risk factor across all cohorts.

Summing Up

- Empirical results support the independent-factor AFNS cohort mortality model:
  - Parsimonious, captures the variation in cohort mortality rates in US data, producing a better fit at older ages than the independent-factor Blackburn-Sherris model, and has good predictive performance.
  - As a Gaussian model it is easy to implement with closed-form expressions for survival probabilities, easy to estimate using the Kalman filter, and can be readily implemented using simulation.
  - Negative mortality rates have very low probability.
  - Factors (Level, Slope, Curvature) better fit historical data dynamics and have intuitive factor interpretation.
  - Multi-factor age-cohort models, and particularly the AFNS model, are well suited for financial and insurance applications.
- Ongoing research: incorporating incomplete cohorts into estimation, better capturing Poisson variation, age-dependence in trend and covariance, machine learning for model estimation.

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