A Study of the Lee-Carter Model with Age-Shifts

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Abstract

The Lee-Carter (LC) model is one of the most popular methods for modeling mortality rates for all ages, because it is easily applied and provides fairly accurate mortality estimations and population projections. However, the parameters of the LC model, including its intercepts and slopes, are assumed to be constant, whereas empirical studies in various countries do not support such an assumption. Therefore, further modifications of the LC model are required to deal with non-constancy in parameters. We propose an age-shift model to modify the LC model and deal with the problem of parameters. We use previously reported data with non-constant parameters from countries such as Japan, Taiwan, Great Britain and the United States to verify if the proposed method can capture their non-constant nature. The proposed method attains smaller estimation errors (with respect to mean absolute percentage error or mean square error). We also apply the proposed age-shift model to the mortality rates of these four countries to evaluate the longevity risk in annuity products by measuring life expectancy. The research findings can benefit the actuary to deal with longevity risk in pricing and valuation.

1. Introduction

The life expectancy of human beings has been increasing significantly since the start of the 20th century. With the United States as an example, life expectancies at birth for both men and women reached the low and high 40s, respectively, in 1900; but by 2000, they had increased to the low and high 70s (Bell et al., 1992; Human Mortality Database, 2006). This trend of increasing increments in life expectancy at birth does not show signs of slowing down (Figure 1). Similar patterns appear in other countries as well; for example, the annual increments of life expectancy at birth in the year 2000 were approximately 0.2 and 0.3 years for Taiwanese men and women, greater than those in the 1980s and 1990s.

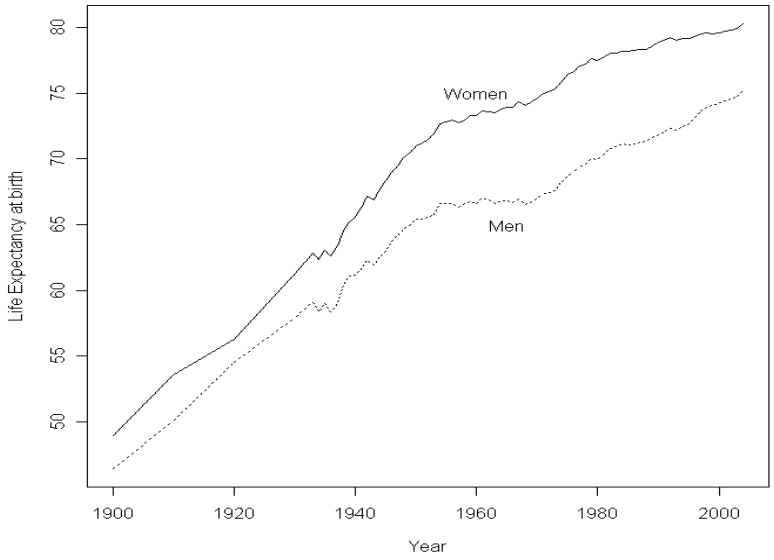


FIGURE 1
Life Expectancy at Birth in United States, 1900–2004

Increased life expectancy indicates the possible risk of underestimating insurance premiums on the basis of period mortality tables for life annuity policies. Traditionally, actuaries have used a fixed and deterministic mortality assumption to price and reserve for life insurance policies. However, because of rapid mortality improvements, the pure premium of annuity products computed from a period mortality table can be as much as 40 percent lower than that

computed from a more accurate cohort life table (Willets, 2004). To construct a cohort mortality table to compute pure premiums for annuity products requires (stochastic) mortality models or mortality projections. Therefore, in recent years, the use of stochastic mortality models to manage mortality risk has become an important tool for actuarial professionals.

Today is not the first time that mortality risk has been studied in the insurance industry. Many previous studies note that mortality risk may cause substantial losses if handled improperly. For example, Equitable Life in the United Kingdom suffered critical interest rate risk and longevity risk because it issued the insurance contracts with guaranteed annuity options (GAOs) in the 1970s and 1980s.Thus, understanding the dynamics of future mortality or interest rate is very important for the actuary to pricing and reserving. Wilkie et al. (2003) and Ballotta and Haberman (2006) both analyze the problem of guaranteed annuity options using a stochastic mortality model. Marceau and Gaillardetz (1999) consider a stochastic mortality and interest rate environment to calculate reserves for a portfolio of term-life insurance and pure endowment policies and Milevsky and Promislow (2001) attempt to value mortality-contingent claims by stochastically modelling the future hazard-plus-interest rate and suggest that both mortality and interest risk can be hedged. These studies all make use of a dynamic mortality model to deal with mortality risk.

A mortality model that can provide accurate predictions becomes essential to sound premium calculations. Among all mortality models, the Lee-Carter (LC) model, proposed by Lee and Carter in 1992, is one of the most popular choices, because it is easy to implement and outperforms other models with respect to its prediction errors (e.g., Koissi et al., 2006; Melnikov and Romaniuk, 2006). In addition, various researchers extend the LC model to attain a broader interpretation (Brouhns et al., 2002; Renshaw and Haberman, 2003), and many countries use the LC model as the base mortality model for their population projections. The Continuous

Mortality Investigation Bureau (CMIB, 2006) in Britain even suggests the LC model as a means to compute stochastic mortality rather than the reduction factor (RF) model previously proposed by the CMIB.

However, the LC model still has room for improvement. To simplify the discussion, we consider that the LC model assumes the logarithms of the mortality rates are approximately a linear function of time. The slopes and intercepts are functions of ages and constant over time. However, many studies show that these time-invariant parameters are not necessarily fixed in time, which causes inaccurate mortality predictions, especially for older age groups.

In the remainder of this study, we first provide the empirical results of applying the LC model to data from the Human Mortality Database (HMD) and discuss the problems associated with the parameters. We also propose a method to deal with the problems in the LC model and use the HMD data to evaluate our approach. In the fourth section, we compute life expectancy using the proposed method and compare it with that derived from the LC model, then discuss the limitations of our research and some suggestions for further work in the final section.

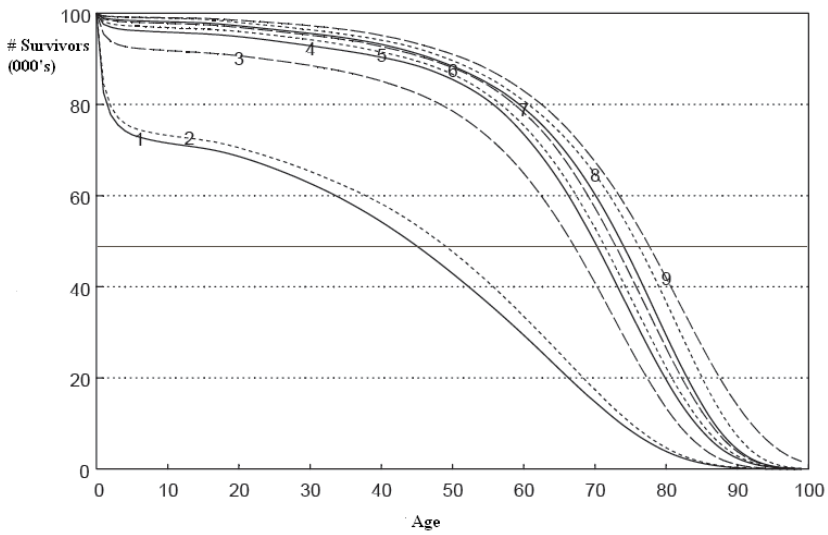
2. Empirical Analysis of the Lee-Carter Model

Lee and Carter (1992) propose the following mortality model for the central death rate $m_{x,t}$:

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{x,t}\,, \qquad (2.1)$$

where parameter α_x describes the average age-specific mortality, κ_t represents the general mortality level, and the decline in mortality at age x is captured by β_x. The term $\varepsilon_{x,t}$ denotes the deviation of the model from the observed log-central death rates and should be white

noise with 0 mean and relatively small variance (Lee, 2000). The parameter estimates can be derived from matrix operations, such as the singular value decomposition. Equivalently, applying the constraints $\Sigma_t \kappa_t = 0$ and $\Sigma_x \beta_x = 1$, the estimate of parameter α_x is the average log-central death rate over time t, such that $\hat{\alpha}_x = \sum_{t=t_1}^{t_1+T-1} \ln(m_{x,t}) / T$, where t_1 is the starting year and T is the number of years in the data. The parameters α_x and β_x are functions of age x and do not change with time, and the parameter κ_t is a linear function of time. Also, if missing values exist, an approximation method and some modifications (Wilmoth, 1996) can be used for parameter estimation.



Notes: The number "9" indicates approximately the year 2000 (i.e., 1910 + 90) and "8" is the year 1990 (1910 + 80). Other numbers follow similarly.

FIGURE 2
Survival Curves of Taiwanese Men (Complete Life Tables)

The LC model contains relatively few parameters, and it provides fairly good estimates and predictions of the observed mortality rates in many countries, such as the United States and Japan. In turn, the LC model has gained significant attention since it was introduced. However, future mortality rates under the original LC model extrapolate past

trends. In particular, if the mortality rates follow the assumption of equation (2.1), the indication is that mortality improvements at all ages will follow a fixed pattern. But this assumption is unlikely to be true, because usually younger people experience greater improvements in their early years, as do the elderly recently. For example, in Taiwan[4], more than 20 percent of newborns died before age 5 years and less than 50 percent of them survived beyond age 50 among Taiwanese men in 1920 . In 2000, fewer than 20 percent of the newborns died before age 60 and about 50 percent of them survived beyond age 80 (Figure 2). The elderly have been experiencing larger mortality reductions in recent years, and the younger age groups enjoy the largest reductions in the early 20th century.

Many countries (e.g., Great Britain and Japan) have experienced a similar mortality reduction shift. Therefore, the slope β_x of each age in equation (2.1) is not necessarily a constant of time; otherwise, there would not be a shift in age for the largest mortality reduction（Booth et al., 2002). Another limitation of applying the LC model is the limiting mortality rates of each age. Because the logarithm mortality rates in equation (2.1) could be linear functions of time in the case that k_t is projected linearly, the mortality rates of all ages eventually go to 0.

Several modifications have been proposed to cope with the limitations of the LC model. The reduction shift of ages for different time periods can be treated as a "cohort" effect, so introducing a cohort effect into the LC model represents a popular approach. The original LC model is close to a combination of the age effect and the interaction of age and time, so a possible modification brings in additional terms related to the cohort effect. For example, Booth et al. (2002) propose adding more than one interaction terms of age and

[4] The data of mortality experience in Taiwan is obtained from the Ministry of Interior.

time, such that

$$\ln(m_{x,t}) = \alpha_x + \sum_{j=1}^{J} \beta_x(j)\ \kappa_t(j) + \varepsilon_{x,t}\ , \qquad (2.2)$$

where $\beta_t(j)\ \kappa_t(j)$ is the jth interaction term between age and time, $j = 1, 2, \ldots, J$.

Renshaw and Haberman (2003) investigate the LC model with age-specific enhancement for mortality forecasts. Hyndman and Ullah (2005) further suggest using principal component (PC) decomposition to solve for the paired parameters ($\beta_t(j),\ \kappa_t(j)$). The idea behind this approach is similar to that proposed by Bell (1997), according to which the LC model displays similar behavior for both one PC and two PCs.

In 2006, the U.K.'s Continuous Mortality Investigation Bureau (CMIB) used the proposal of Renshaw and Haberman (2006) considering the cohort based on the LC model to project mortality rates. The proposed modification is similar to that offered by Hyndman and Ullah (2005), which is

$$\ln(\mu_{x,t,c}) = \alpha_x + \beta_x(t)\ \kappa_t + \beta_x(c)\ \kappa_c^* + \varepsilon_{x,t,c}\ , \qquad (2.3)$$

where μ is the force of mortality, and κ_c^* is the cohort effect. Equation (2.3), called the LC age-period-cohort (APC) model, can be used to predict future mortality rates. In this case, the CMIB suggests using the likelihood method for parameter estimations and the classical multivariate time series method for predictions. Note that the model in equation (2.3) can be treated as a special case of the APC model that includes only one main effect (age) and two second-order interaction terms (age-period and age-cohort).

Not many studies focus on the limiting mortality rates of 0, with the exception of the RF model suggested by the CMIB (1999. 2006). The RF model takes the following form:

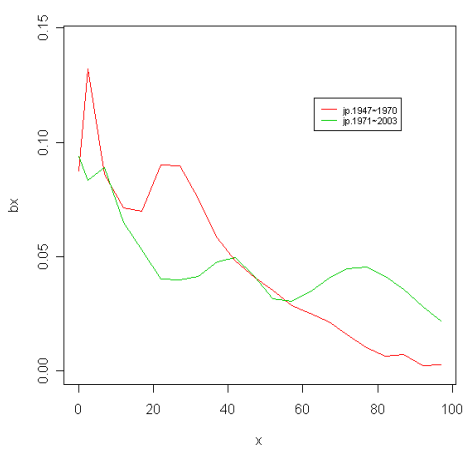
$$\frac{q_{x,t}}{q_{x,0}} = RF(x,t) = \alpha(x) + [1-\alpha(x)][1-f(x)]^{t/20}, \qquad (2.4)$$

where $q_{x,t}$ is the mortality rate of age at time t. The limiting value $q_{x,t}$ in equation (2.4) is $\alpha(x)$ times the original mortality $q_{x,0}$, not necessarily 0. Although the RF model relaxes the restriction for the limiting values, the LC model achieves a better fit. The CMIB therefore decided to study and use the LC-related model, not the RF model, in 2006. For the rest of this study, we focus on the modification to the age shift in mortality reduction rather than the limiting mortality rates.

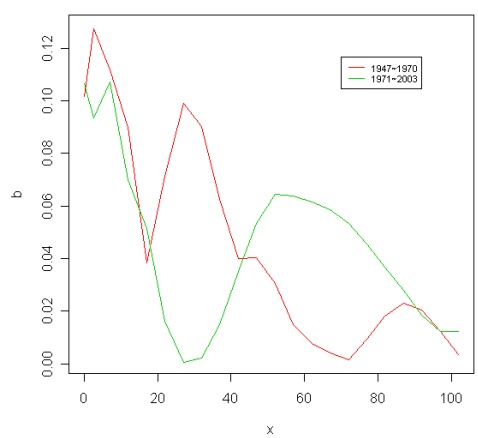
3. Approach for Modifying the Lee-Carter Model

We propose an approach to handle age shifts in mortality reduction. Similar to Bell (1997) and Hyndman and Ullah (2005), we apply the principal component (PC) approach to the logarithm of central mortality rates for data from the HMD and specifically select data from Europe, America and Asia. In particular, we choose data from four countries—Great Britain, the United States, Japan and Taiwan—to explore possible patterns in $\kappa_t(j)$ and κ_c^*. To be consistent, the data we use are from the years 1947–2003 for all four countries.

Japanese Man



British Man



American Man

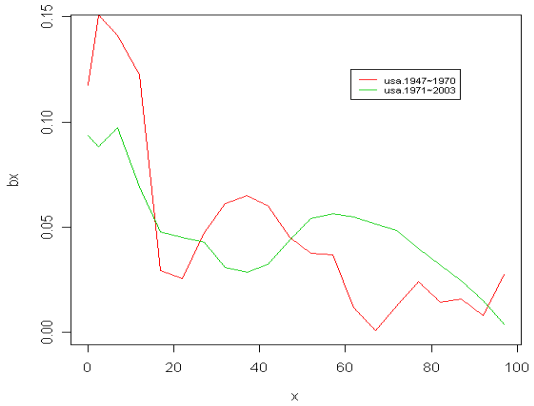


FIGURE 3
Estimates of β_x for Japanese, British and American Men

Age shifts exist in the mortality reductions—that is, β_xs are not constants of time in the original LC model—as we show by dividing the data into two periods: 1947–1970 and 1971–2003. The age groups considered are basically five-year groups, including 5–9, 10–14, …, 95–99, as well as 0, 1–4, and 100+ (ages 100 and older). The data pertaining to British and Japanese men serve as demonstrations, as we show in Figure 3; the estimates of β_xs are obviously different in the two time periods. In particular, the elderly reach greater mortality reductions in 1971–2003, and younger adults (ages 20–40) generally experience larger reductions in 1947–1970. These results are exactly what we expected; therefore, the age shift of mortality reductions exists among these four countries.

Next, we apply the PC approach to the logarithm of central mortality rates. The LC model can be treated as the one-PC model, and the first PC is a linear function of time. According to Bell (1997), the logarithms of mortality rates contain one, two or three PCs, depending on the data. We use the Japanese data to describe our findings. In Figures 4 and 5, we provide the graphs of the first two PCs for Japanese male and female logarithm mortality rates. The first two PCs account for 98.72 percent (men) and 99.52 percent (women) of the

variations, and the two-PC models explain approximately 5 percent more variation than the one-PC models. Therefore, in this research we modify LC model based on two PCs.

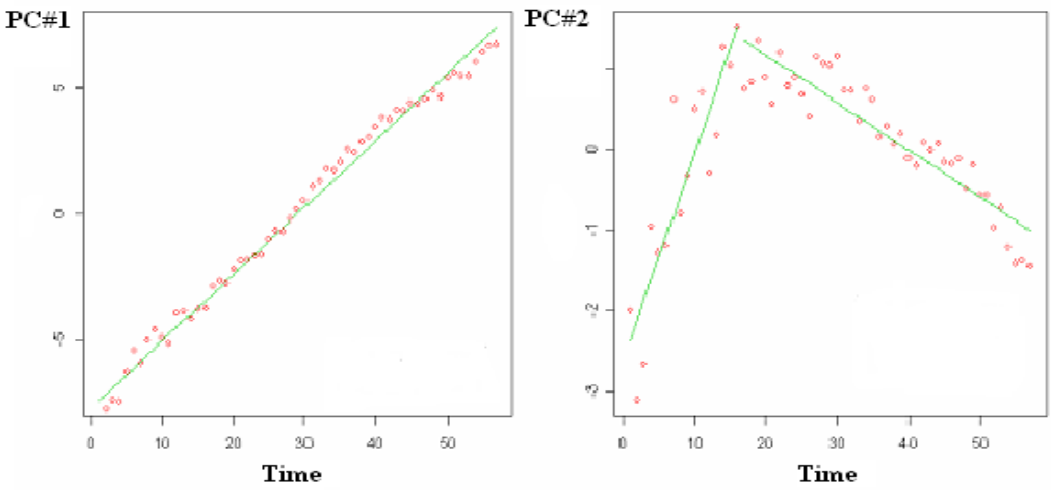


FIGURE 4

The First Two PCs of Japanese Men (Logarithm of Central Mortality Rates)

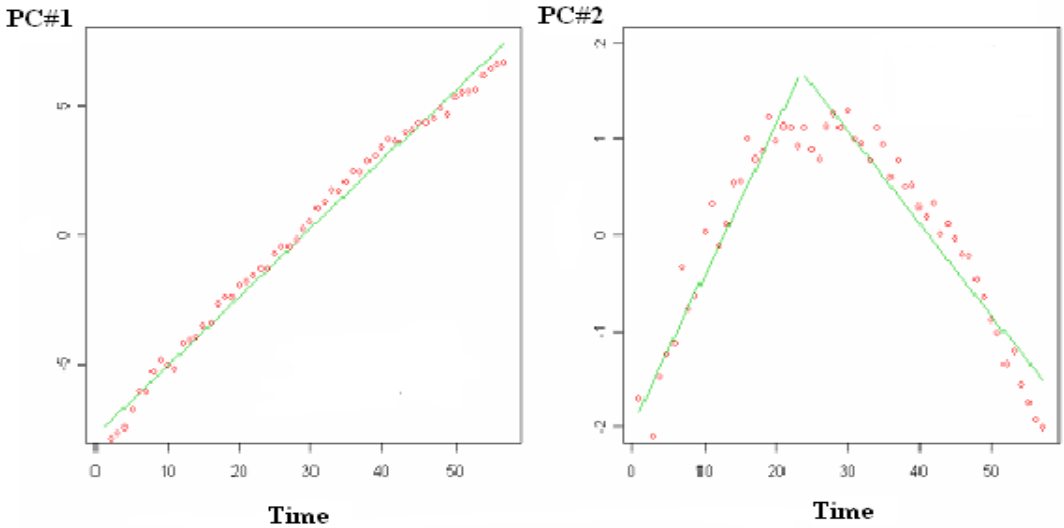


FIGURE 5

The First Two PCs of Japan Women (Logarithm of Central Mortality Rates)

As we expected, the first PC from both the male and female data is very close to a straight line of time. In contrast, the second PC looks like a straight line of time but behaves

quite differently before and after a certain cut-off point. On the basis of this pattern, we propose the following modified model:

$$\ln(\frac{m_{x,t}}{m_{x,0}}) = \beta_x \kappa_t + \beta_x^* \kappa_t^*$$
$$\equiv \beta_x (a + bt) + \beta_x^* \{(a_1 + b_1 I[t < t_0]) + (a_2 + b_2 I[t \geq t_0])\}, \quad (3.1)$$

where t_0 is the cut-off point (or jump).

The idea of adding $\beta_x^* \kappa_x^*$ in equation (3.1) is similar to adding $\kappa_t(j)$ in equation (2.2) and κ_c^* in equation (2.3). The parameters for β_x and β_x^*, namely, a, b, a_1, b_1, a_2, b_2, and t_0, can be estimated using an ordinary regression, after finding the first two PCs from the principal component analysis. Although we introduce a two PCs approach to deal with the issue, the extra number of parameters needed increases at least 50 percent compared with the original LC model. If all the components of $\kappa_t(j)$ in equation (2.2) and κ_c^* in equation (2.3) are linear functions of time or cohort, both equations can be simplified to (2.1). In other words, the parameters $\kappa_t(j)$ and κ_c^* cannot be simply linear functions of time, or they could not be used to describe the age shift in the mortality reduction. We use empirical data to determine possible forms for $\kappa_t(j)$ and κ_c^*.

Note that the effect of the cut-off point in κ_x^* is equivalent to introducing an "age-shift" in the mortality reductions; that is, the mortality reductions before and after the cut-off point differ. The mortality improvement is more significant after the cut-off point. In all four countries, the elderly have the largest mortality reductions after the cut-off point, whereas the 20–60 age groups experience the smallest reductions. Only Taiwanese men still achieve large reductions in the younger age groups, in addition to the elderly groups. In the

next section, we use an empirical study to evaluate our modification to the LC model.

Similar to Hyndman and Ullah (2005), the computation of our approach is fairly straightforward. The number of age-shifts is not limited to one, and we can use an idea similar to cubic spline interpolation to find the optimal polynomial between two age-shifts (though we prefer using a linear function). However, just as in equations (2.2) and (2.3), we cannot decide if there are future age-shifts solely on the basis of the current PCs or past trends. Therefore, equation (3.1) requires further modification to include the possibility of future age-shifts. This is not part of the paper ands is left for future work.

4. Empirical Comparisons and Applications

In this section, we use the empirical data to evaluate our modification in equation (3.1) in comparison with the LC model. We first compare the estimation errors of the original LC model and the proposed modification in equation (3.1) for the data from Great Britain, Japan, Taiwan and the United States. We use the mean absolute percentage error (MAPE) to examine the goodness fit of the original and modified models, defined as

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\frac{\left|Y_i - \hat{Y}_i\right|}{Y_i} \times 100\%, \qquad (4.1)$$

where Y_i and $\hat{Y}_i$ are the observed and estimated values, and n is the number of observations.

In Table 1, we list the MAPE of these four countries. The modified method in equation (3.1) produces smaller MAPE than the original LC model in all four countries, for both men and women. Except in the United States, the MAPE of the modified method are less than 20 percent of those of the LC model; in the United States, the ratio between the modified method and the LC method is approximately 30 percent. These reduction ratios are

obvious, though the number of additional parameters is 50 percent in the modified method. Also, we can compare these two methods using the Akaike information criterion or Bayesian information criterion.

TABLE 1

MAPE of Original LC Model and Proposed Modification

	Method	Britain	Japan	Taiwan	United
Male	Lee-Carter	5.73%	7.04%	5.63%	4.47%
	Proposed	1.31%	1.71%	0.85%	1.45%
Female	Lee-Carter	5.54%	9.35%	5.11%	3.50%
	Proposed	1.12%	1.70%	1.02%	1.07%

More parameters likely will yield smaller MAPE or estimation errors, though this trend is not always the case in modeling mortality rates. In a previous study, Yue et al. (2007) compare the APC model and the RF model to the LC model and find that the LC model usually achieves the smallest MAPE, even though the number of parameters used in the APC and RF models is at least 50 percent more than the LC model, similar to the comparison of our modified method to the LC model.

In addition to estimation errors, we compute the life expectancies at various ages using the LC and our proposed model to evaluate the differences in pricing annuity products. The second PC for Japan and Great Britain reveals an obvious pattern of age shifting, such that the coefficients for ages older than 60 versus younger ages (e.g., 20–45) have different signs. Therefore, younger and older ages groups have different mortality reductions when we add the second PC in the model. In particular, in Figures 4 and 5, the slope of the second PC differs before and after the jump point. (This difference is one of the reasons we call the cut-off point a “jump,” because it is like jumping from one side to the other.) Therefore, the younger population would experience greater mortality reductions in the past, and the older

population has a larger reduction now. The coefficients of the second PC for the Taiwanese and U.S. data behave similarly, but the coefficients of younger or older ages do not always have the same signs.

We compute life expectancies at ages 50, 55, 60, 65, 70 and 75 years, with a limiting age of 100, using the LC model and the proposed method. As expected, because the proposed method has the second PC to emphasize the mortality reduction for the older ages, life expectancies are always greater than those using the LC model. Also, the coefficients of the second PC for the older ages do not always have the same sign in the Taiwanese and U.S. data. Therefore, the differences in the life expectancy for all ages are always smaller than 5 percent in Taiwan and the United States. Only data from Japan and Great Britain show larger differences in life expectancy (Figures 6 and 7). The Japanese data reveal the largest differences, such that life expectancy at age 65 years using the proposed method is 13 percent, which is 7 percent more than that achieved using the LC model for women and men. The data pertaining to British men also display larger differences, and other cases look similar to those in the British female data, with few differences.

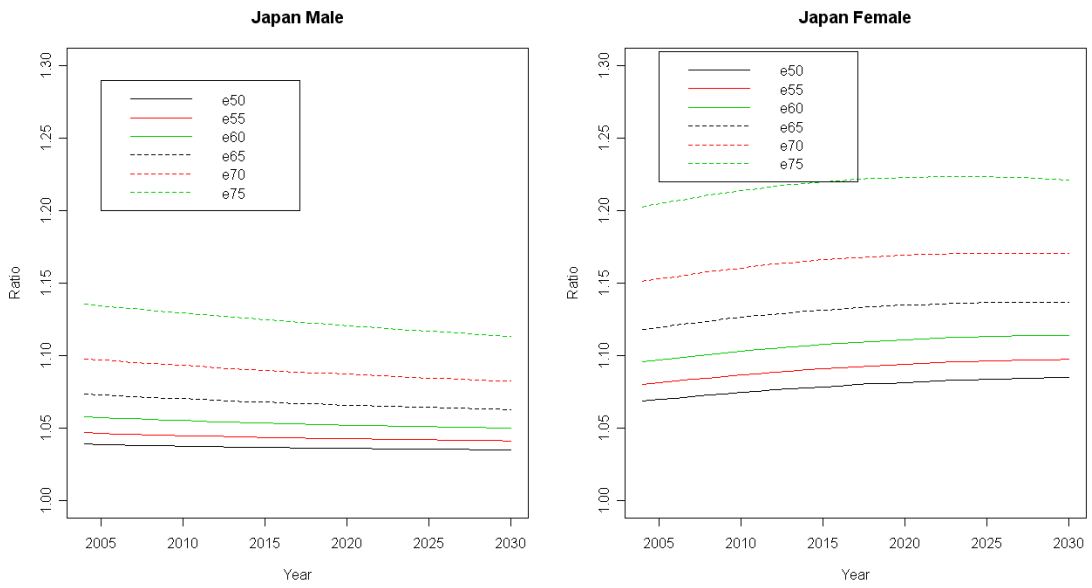


FIGURE 6

Ratio of Life Expectancy between Proposed and LC Models (Japan)

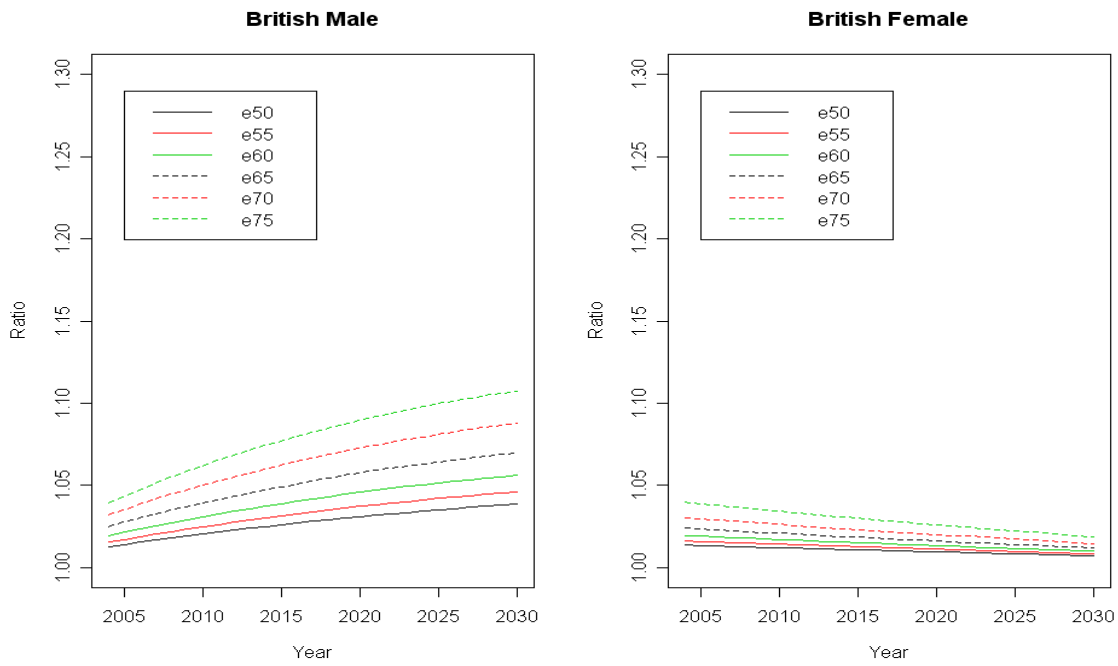


FIGURE 7

Ratio of Life Expectancy between Proposed and LC Models (Great Britain)

5. Conclusions and Discussions

The Lee-Carter model has received significant attention in the effort to model mortality rates since 1992. Because its computation is fairly straightforward and it reaches good accuracy in its predictions, the LC model probably represents the most popular approach for population projections. However, two main restrictions affect the use of the LC model: the constant assumption for the parameters and the limiting mortality of 0. These limitations have prompted lots of discussions and many proposed modifications.

In this study, we propose another modification designed to deal with the age shifts in the mortality reductions. The proposed age-shift model using principal component analysis has significantly improved the model fitting based on the empirical study. Using mortality data from Great Britain, Japan, Taiwan and the United States, we find that the modified method achieves much lower MAPE compared with the LC model. This model is easy to apply for actuarial works. In this research we use this model to calculate the life expectancy. Because the coefficients of the second PC in the Japanese data have the same signs, the life expectancy (and pure premium for annuity products) calculated would undergo a significant increase if the proposed method, instead of the LC model, were applied. For the Taiwanese and U.S. data, though the estimation errors can be reduced, the proposed method and the LC model result in similar life expectancies.

The modified method can improve model fit, but there remains room for further improvement. Our proposed approach introduces age-shifts of mortality improvements at a cut-off time on the second PC. Throughout this study, we assume there are two PCs, and only the second PC can have an age-shift at some time point. We do not discuss the possibility of three or more PCs, nor do we provide a methodology or criterion for determining the number of PCs in the model.

We also allow only the second PC to have a jump. However, we fail to address the issue of the number of age-shifts and their optimal locations. The discussion would be even more complicated if more than one PC could jump—similar to considering the variable selection problem (e.g., number of PCs vs. number of age-shifts) and the change-point problem (i.e., the optimal locations for the age-shifts) at the same time. Many possible combinations may need to be considered, which may require multiple computations.

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