SOCIETY OF ACTUARIES

EXAM MLC Models for Life Contingencies
EXAM MLC SAMPLE WRITTEN-ANSWER QUESTIONS AND SOLUTIONS

Questions

<u>September 17, 2016</u>

- Question 22 was added.

<u>February 12, 2015</u>

- In Questions 12, 13, and 19, the wording was changed slightly to clarify which version of Euler's method (forwards or backwards) was to be used.

<u>April 23, 2014</u>

- In Question 12, an additional sentence was added to clarify when the policy terminates.
- In Question 12, a force of transition was added to the information given.

<u>December 25, 2013</u>

- Questions 4, 7, 9, 10, 20, and 21 were reformatted or had minor wording or formatting changes.
- In questions asking the candidate to show that a calculation resulted in a given numerical value, the requested accuracy was added or modified.
    - This happened in questions 2(a), 2(c), 7(d), 8(a), 9(c)(ii), 10(b), 11(a), 11(b), 12(b), 13(a), 14(a), 18(a), and 20(d)(i).
- Question 21 was restated to give more information.

Solutions

<u>September 17, 2016</u>

- The solution to question 22 was added.

<u>April 14, 2016</u>

- In Question 18(b), one typo was corrected.

<u>February 12, 2015</u>

- In Questions 12, 13, and 19, the wording was changed slightly to clarify which version of Euler's method (forwards or backwards) was used.

<u>April 23, 2014</u>

- In the solution to Question 12, the first line of the solution to part (b) was fixed.

<u>December 25, 2013</u>

- Solutions to questions 4, 7, 9, 10, 20, and 21 were reformatted or had minor wording or formatting changes.
- The rounding of the final answer was changed in the solutions to questions 2(c), 7(d), 8(a), 11(a), 13(a), and 18(a).
- More detail was added to the solutions to questions 4, 16, and 18.

MLC Written Answer Sample Questions

Starting with the Spring 2014 sitting, Exam MLC will include both a written answer portion as well as a multiple choice portion. This document contains a collection of sample written answer questions and solutions. These are intended to assist candidates to prepare for this new style of question, as well as to serve as a learning aid for the exam readings.

This set of questions does not represent a sample exam; the weighting of material by topics and learning objectives does not necessarily conform to the weightings given in the syllabus.

Written answer questions will consist of multiple parts, with points associated with each part. Candidates will be graded primarily on the methods and formulas used, not on their final answers. For all parts of all problems, in order to maximize the credit received, candidates should show as much work as possible, considering the time allotted for the question. Answers lacking justification will receive no credit. Answers should be organized so that the methods, logic, and formulas used are readily apparent. Candidates should not round their answers excessively; enough precision should be provided so that their answers can be accurately graded.

In some cases, candidates are asked to show that a calculation results in a particular number. Typically the answer given will be rounded; candidates should provide a greater level of accuracy than the figure given in the question. This structure of question is intended to assist the candidate by giving an indication when they have done the calculation incorrectly, so they might have an opportunity to correct their work. It also allows a candidate who cannot obtain the correct answer to nonetheless use the answer given in order to proceed with subsequent parts of the problem. (Candidates who are able to solve the problem should use their exact answer for subsequent parts.)

For questions requiring candidates to derive or write down a formula or equation, the resulting expression should be simplified as far as possible, and where numerical values are provided in the problem, they should be used.

For problems requiring a demonstration or proof, the steps taken should be clearly shown and justified as appropriate. Any assumptions made should be indicated.

Some problems may require candidates to sketch a graph. In these problems, candidates should clearly label and mark both axes, show the general form of the function being graphed, and indicate any limiting values, extrema, asymptotes, and discontinuities. The exact shape of the function would not be required for full credit.

The model solutions indicate the correct answer and type of response expected by candidates, though in many cases the level of detail given is beyond that which would be expected for full credit. There may be alternative solutions that would also earn full credit; no attempt has been made to give all possible correct responses.

Solutions also indicate a reference to the Learning Objective(s) tested in the question.

1. (8 points) You are given the following survival function for a newborn:

$$S_0(t) = \frac{(121 - t)^{1/2}}{k}, \quad t \in [0, \omega]$$

(a) (1 point) Show that k must be 11 for $S_0(t)$ to be a valid survival function.

(b) (1 point) Show that the limiting age, ω, for this survival model is 121.

(c) (2 points) Calculate $\mathring{e}_0$ for this survival model.

(d) (2 points) Derive an expression for μ_x for this survival model, simplifying the expression as much as possible.

(e) (2 points) Calculate the probability, using the above survival model, that (57) dies between the ages of 84 and 100.

2. (10 points) You are modeling the status of ABC company using a 3-state Markov model. The states of the model are Healthy (H), Distressed (D), and Liquidated (L).

Transitions between states occur annually according to the following transition probability matrix:

$$\begin{array}{c} \\ \mathbf{H} \\ \mathbf{D} \\ \mathbf{L} \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{H} & \mathbf{D} & \mathbf{L} \end{array} \\ \begin{pmatrix} 0.80 & 0.15 & 0.05 \\ 0.20 & 0.60 & 0.20 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

On January 1, 2014, ABC Company is in the Healthy state and issues a 1,000,000 2-year zero-coupon bond.

(a) (1 point) Show that the probability that ABC will be Liquidated within the next two years is 0.1 to the nearest 0.05. You should calculate the value to the nearest 0.01.

After completing the above calculation, you receive new information that ABC, while currently in the Healthy state, was in the Distressed state for each of the previous 5 years. You are asked to recalculate the probability that ABC will be Liquidated within the next two years, incorporating this new information.

(b) (1 point) Describe the effect of this new information on your calculation.

In the event that ABC is Liquidated within the next two years, it will not be able to pay the face value of the bond and will instead pay X on the maturity date of the bond, where X is uniformly distributed between 0 and 1,000,000.

You are given that $i = 0.08$.

(c) (2 points) Show that the expected present value of the bond payment is 806,000 to the nearest 1,000. You should calculate the value to the nearest 100.

(d) (2 points) Calculate the probability that the present value of the bond payment exceeds its expected present value.

(e) (4 points) Calculate the standard deviation of the present value of the bond payment.

3. (8 points) For Joe, who is age 60, you are given that ${}_tp_{60} = e^{0.5(1-1.05^t)}$.

(a) (3 points)

(i) Calculate q_{80}.

(ii) Calculate μ_{80}.

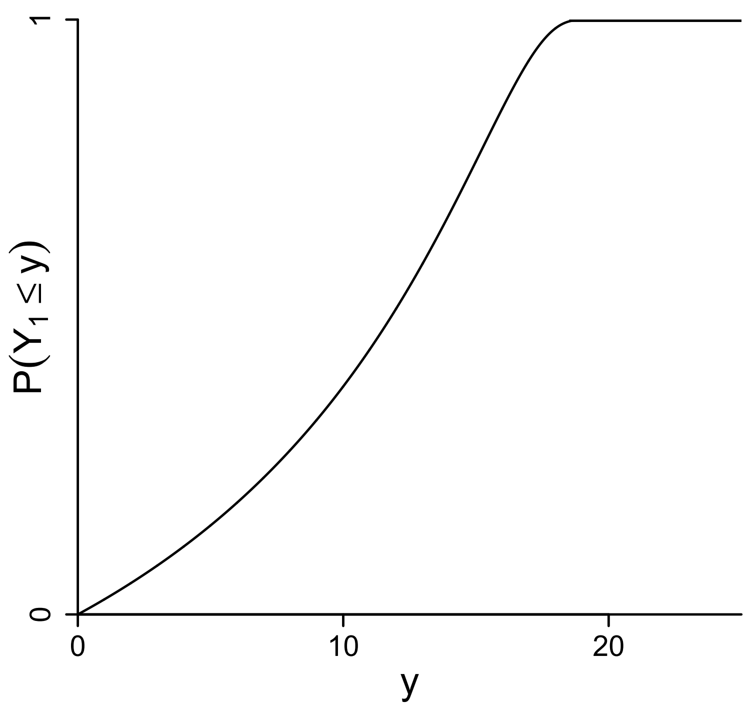
You are also given:

(i) $\delta = 0.05$

(ii) Y_1 is the present value of benefits random variable for a continuous whole life annuity on Joe, paying at a rate of 1 per year.

(iii) Y_2 is the present value of benefits random variable for a continuous 20-year temporary life annuity on Joe, paying at a rate of 1 per year.

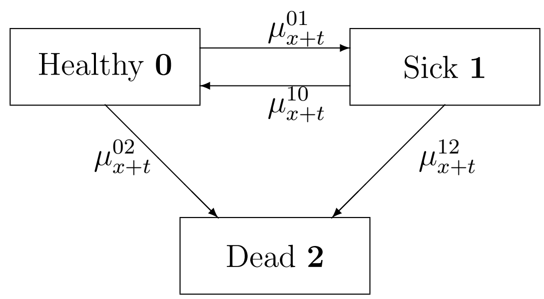
(iv) A graph of the cumulative distribution function for Y_1:



(b) (2 points) Calculate $P\left(\frac{Y_1}{Y_2} > 1\right)$.

(c) (3 points) Sketch the cumulative distribution function for Y_2.

4. (7 points) You are given the following sickness-death model.



You are given the following assumptions for the model (these are the standard assumptions):

- Assumption 1: The state process is a Markov process.
- Assumption 2: The probability of ≥ 2 transitions in any period of length h years is $o(h)$.
- Assumption 3: For all states i and j, and for all ages $x \geq 0$, ${}_tp_x^{ij}$ is a differentiable function of t, and for $i \neq j$,

$$\mu_x^{ij} = \lim_{h \to 0^+} \frac{{}_hp_x^{ij}}{h}$$

(a) (6 points)

(i) Under Assumptions 1 and 2 above, the probability ${}_{t+h}p_x^{00}$ can be expressed as

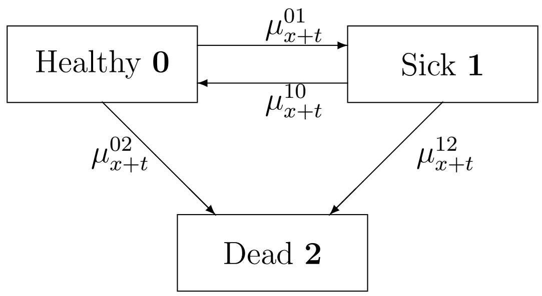
$${}_{t+h}p_x^{00} = {}_tp_x^{00}\ {}_hp_{x+t}^{\overline{00}} + {}_tp_x^{01}\ {}_hp_{x+t}^{10} + o(h)$$

Interpret in words each of the three terms on the right side of this equation.

(ii) Use this equation along with the assumptions above to derive the Kolmogorov forward differential equation for $\frac{d}{dt}{}_tp_x^{00}$.

(b) (1 point) Write down the Kolmogorov forward differential equations for ${}_tp_x^{0j}$, $j = 1, 2$ for this model.

5. (12 points) You are given the following sickness-death model and that $i = 0.05$.



The table below gives some transition probabilities calculated for this model.

x	${}_1p_x^{00}$	${}_1p_x^{01}$	${}_1p_x^{02}$	${}_1p_x^{11}$	${}_1p_x^{10}$	${}_1p_x^{12}$
60	0.96968	0.01399	0.01633	0.93300	0.04196	0.02504
61	0.96628	0.01594	0.01778	0.92477	0.04781	0.02742
62	0.96248	0.01816	0.01936	0.91552	0.05446	0.03002
63	0.95824	0.02067	0.02109	0.90514	0.06199	0.03287

(a) (3 points) Consider the three quantities under this model:

$${}_2p_x^{00} \qquad {}_2p_x^{\overline{00}} \qquad \left({}_1p_x^{00}\right)\left({}_1p_{x+1}^{00}\right)$$

Explain in words, without calculation, why these three quantities may all be different, and rank them in order, from smallest to largest.

(b) (3 points)

(i) Calculate the probability that a life who is healthy at age 60 is healthy at age 62.

(ii) Calculate the probability that a life who is healthy at age 60 is sick at age 62.

(c) (2 points) Calculate the expected present value of a three year annuity-due of 1 per year, with each payment conditional on being healthy at the payment date, for a life who is currently age 60 and is healthy.

The insurer issues a 3-year term insurance policy to a healthy life age 60. Premiums are payable annually in advance, and are waived if the policyholder is sick at the payment date. The face amount of 10,000 is payable at the end of the year of death.

(d) (4 points) Calculate the annual net premium for this insurance policy.

6. (9 points) Jim and Amy are considering buying a motorcycle to ride together.

   If they do not buy the motorcycle, their future lifetimes are independent, each having constant force of mortality of $\mu = 0.02$.

   If they buy the motorcycle there would be an additional risk that both lives will die simultaneously. This additional risk can be modeled by a common shock with a constant transition intensity of $\gamma = 0.05$. Also, if Jim dies before Amy, Amy will continue to ride the motorcycle, and will consequently continue to experience the additional force of mortality γ. If Amy dies before Jim, Jim will sell the motorcycle, and his mortality will revert to its previous level.

   (a) (2 points)

      (i) Determine the covariance between the future lifetimes of Jim and Amy if they do not buy the motorcycle.

      (ii) State with reasons whether the covariance would increase, decrease or stay the same as in (i) if Jim and Amy do buy the motorcycle.

   Jim and Amy buy the motorcycle.

   (b) (1 point) Sketch a diagram that summarizes the states and transitions in the joint life model including the motorcycle-related mortality. You should clearly label each state and transition in the model and show the amount of each transition intensity.

   (c) (3 points) Calculate the probability that Jim's future lifetime is at least 20 years.

   Jim and Amy purchase a fully continuous joint life insurance policy that pays 100,000 on the first death, unless the death arises from a motorcycle accident, in which case the benefit is only 10,000. Premiums are payable continuously while both lives survive.

   (d) (3 points) Calculate the annual net premium rate for the whole life policy, assuming $\delta = 0.06$.

7. (11 points) For a special deferred term insurance on (40) with death benefits payable at the end of the year of death, you are given:

   (i) The death benefit is 0 in years 1-10; 1000 in years 11-20; 2000 in years 21-30; 0 thereafter.

   (ii) Mortality follows the Illustrative Life Table.

   (iii) $i = 0.06$

   (iv) The random variable Z is the present value, at age 40, of the death benefits.

   (a) (1 point) Write an expression for Z in terms of K_{40}, the curtate-time-until-death random variable.

   (b) (1 point) Calculate $Pr(Z = 0)$.

   (c) (3 points) Calculate $Pr(Z > 400)$.

   You are also given that $E[Z] = 107$.

   (d) (3 points) Show that $Var(Z) = 36{,}000$, to the nearest 1,000. You should calculate the value to the nearest 10.

   (e) (3 points)

      (i) Using the normal approximation without continuity correction, calculate $Pr(Z > 400)$.

      (ii) Explain why your answer to (c) is significantly different from your answer to (e part i).

8. (8 points) You are the actuary for a company that sells a fully discrete 100,000 2-year term life insurance to 60-year-olds. The policy has no cash surrender value. The original pricing assumptions were:

(i) Mortality and lapse rates are given in the table below, where decrement 1 is death and decrement 2 is lapse. Deaths are uniformly distributed in the associated single decrement model. All lapses occur at the end of the year.

x	$q_x'^{(1)}$	$q_x'^{(2)}$
60	0.12	0.10
61	0.18	0.20

(ii) Expenses are 5% of gross premiums.

(iii) $i = 0.07$

(a) (2 points) Show that the expected present value of the death benefit, to the nearest 100, is 23,700. You should calculate the value to the nearest 10.

(b) (2 points) Calculate the annual gross premium for this policy using the equivalence principle.

Your company now offers a policy with the same benefit, but with a single premium. You assume that this new policy will have no lapses. All other assumptions and features remain unchanged.

(c) (1 point) Justify the assumption of no lapses for this new policy.

(d) (2 points) Calculate the expected present value of the death benefit for this new policy.

(e) (1 point) Explain why the expected present value of the death benefit is greater for this new policy than for the original policy.

9. (10 Points) Your company issues fully discrete whole life insurances of 5000 on lives age x with independent future lifetimes.

   (a) (1 Point) Define net premiums, gross premiums and portfolio percentile premiums.

   You are given:

   (i) $i = 0.05$

   (ii) $q_x = 0.030; q_{x+1} = 0.035; q_{x+2} = 0.040$

   (iii) $\ddot{a}_x = 7.963$

   (iv) ${}_{3|}{}^{2}A_x = 0.3230$

   (b) (1 point) Calculate the annual net premium.

   You are also given:

   (v) There are no expenses.

   (vi) The annual gross premium is 410.

   (c) (3 points)

      (i) Calculate the expected loss at issue for one such policy.

      (ii) Show that the standard deviation of the loss at issue for one such policy is 1960 to the nearest 20. You should calculate the value to the nearest 1.

   (d) (3 points) Using the normal approximation, calculate the minimum number of policies needed such that the aggregate loss at issue will be positive less than 1% of the time.

   (e) (2 points) Your company's sales are concentrated in a narrow geographic region. Management believes the future lifetimes may not be independent. Explain, without calculation, if a correlation coefficient of 0.05 between each pair of future lifetimes would increase, decrease, or have no impact on the answer to part (d).

10. (6 points) Your company issues special single premium 3-year endowment insurances.

You are given:

(i) The death benefit is 50,000, payable at the end of the year of death.

(ii) The maturity benefit is 10,000.

(iii) The following mortality table, with deaths uniformly distributed over each year of age:

x	q_x
60	0.11
61	0.12
62	0.20
63	0.28

(iv) $i = 0.06$

(v) The commission is 30% of the premium. There are no other expenses.

(vi) Single gross premiums are determined using the equivalence principle.

(a) (2 points) Calculate the single gross premium using the equivalence principle for (60).

Your company issues one such insurance on (60.25).

(b) (2 points) Show that the single gross premium is 32,500 to the nearest 100, using the equivalence principle. You should calculate the value to the nearest 1.

(c) (1 point) Calculate the probability that the company pays a benefit of more than 20,000.

(d) (1 point) Calculate the gross premium reserve at the end of year 2.

11. (13 points) An insurer plans to issue fully discrete whole life insurances of 100,000 to 100 independent lives age 40. You are given:

(i) Mortality is given by the Illustrative Life Table.

(ii) $i = 0.06$

(iii) Expenses are 50% of the initial gross premium, and 10% of subsequent gross premiums.

(a) (2 points) Show that the annual gross premium per policy calculated under the equivalence principle is 1250 to the nearest 10. You should calculate the value to the nearest 1.

Now suppose that the annual gross premium per policy is determined using a normal approximation so that the probability of a loss on the group of policies is 5%.

(b) (5 points) Show that the annual gross premium per policy to the nearest 10 is 1480. You should calculate the value to the nearest 1.

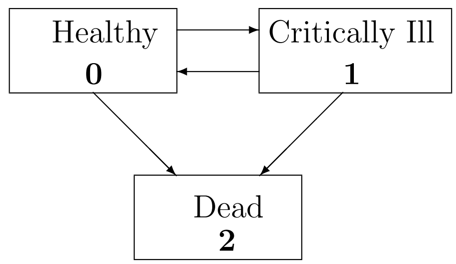
(c) (2 points) Calculate the gross premium reserve per policy at duration 10.

(d) (2 points) Calculate the full preliminary term (FPT) reserve per policy at duration 10.

A marketing officer at the insurance company states that increasing the number of policies sold will allow the insurer to charge a lower premium. In particular, he claims that if the insurer sells a sufficient number of policies, the annual gross premium can be lowered to 1200 per policy while maintaining a 5% probability of loss on the group of policies.

(e) (2 points) Explain in words why the marketing officer's claim is incorrect.

12. (10 points) You are using a Markov model with the following state diagram to model liabilities for a critical illness (CI) insurance product. The product has a 5-year term and pays a benefit of 100,000 immediately on death or on earlier CI diagnosis. The policy terminates if a benefit is paid.



(a) (2 points) Write down the Kolmogorov Forward Equations for ${}_tp_x^{0j}$, $j = 0, 1, 2$ and give the associated boundary conditions.

You are given the following constant transition intensities:

$$\mu_x^{01} = 0.02 \qquad \mu_x^{02} = 0.002 \qquad \mu_x^{10} = 0.001 \qquad \mu_x^{12} = 0.5$$

(b) (3 points) Using a force of interest $\delta = 0.04$, show that the level annual net premium rate, payable continuously while the insured is healthy, is 2000 to the nearest 1000. You should calculate the value to the nearest 100.

(c) (2 points) Write down Thiele's differential equation for the net premium reserve at time t for this policy if the insured is in the healthy state at time t, for $0 < t < 5$.

You apply Euler's backward method to the equation in (c), starting at time $t = 5$ and using a time step of $h = 0.1$.

(d) (3 points)

(i) Using this method, show that the net premium reserve required for an insured in the healthy state at time $t = 4.7$ is 0.

(ii) Explain in words why the net premium reserve required for an insured in the healthy state at time $t = 4.7$ is 0.

13. (9 points) An insurer sells single premium whole life policies to lives age 65 to cover funeral expenses. The death benefit in the first year is a return of the premium, without interest; in subsequent years the death benefit is 25,000. The death benefit is paid immediately on death.

You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) i = 0.06

(iii) Issue expenses of 100 are incurred at issue.

(iv) Claims expenses of 200 are incurred at the time of payment of the death benefit.

(a) (3 points) Show that the single gross premium is 11,230 to the nearest 10 using the equivalence principle and the Uniform Distribution of Deaths fractional age assumption. You should calculate the value to the nearest 1.

You are given the following additional values:

(i) The gross premium reserve at time $\frac{1}{3}$ is ${}_{\frac{1}{3}}V = 11{,}334.98$

(ii) $\mu_{65\frac{1}{3}} = 0.0215$

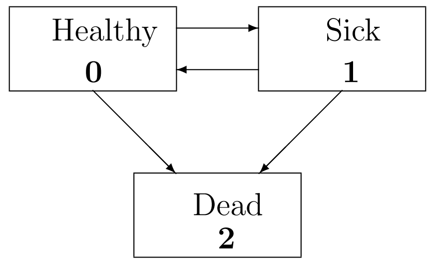
(b) (5 points)

(i) Write an expression for $\frac{d}{dt}\,{}_tV$ for $0 < t \leq 1$ using Thiele's differential equation, where ${}_tV$ is the gross premium reserve for this policy.

(ii) Estimate ${}_{\frac{2}{3}}V$ by applying Euler's forward method to Thiele's differential equation with a time step of $h = 1/3$.

(c) (1 point) Suggest an improvement to the approximation method used in the previous part that would improve its accuracy.

14. (9 points) Consider the following model for income replacement insurance:



An insurer issues a whole-life income replacement policy to a healthy life age 50. The policy provides a disability income of 50,000 per year payable continuously while the life is sick. In addition the policy pays a death benefit of 200,000 at the moment of death. The policyholder pays premiums continuously, at a rate of P per year, while in the healthy state.

Assume no expenses, and an effective interest rate of 5% per year. Annuity and insurance factors for the model are given in the Table below.

(a) (2 points) Show that the annual benefit premium for the policy is 11,410 to the nearest 10. You should calculate the value to the nearest 1.

(b) (2 points) Write down Thiele's differential equations for the net premium reserve at t for both the healthy and sick states.

(c) (2 points) Calculate the net premium reserve for a policy in force at $t = 10$, assuming the policyholder is healthy at that time.

(d) (2 points) Calculate the net premium reserve for a policy in force at $t = 10$, assuming the policyholder is sick at that time.

(e) (1 point) A colleague states that the net premium reserve in the sick state should be less than the net premium reserve in the healthy state, as the sick policyholder has, on average, paid less premium and received more benefit. Explain in words why your colleague is incorrect.

Annuity and insurance factors, 5% effective rate of interest

x	$\bar{a}_x^{00}$	$\bar{a}_x^{01}$	$\bar{a}_x^{11}$	$\bar{a}_x^{10}$	$\bar{A}_x^{02}$	$\bar{A}_x^{12}$
50	11.9520	1.3292	8.9808	3.2382	0.34980	0.39971
⋮	⋮	⋮	⋮	⋮	⋮	⋮
60	8.6097	1.7424	7.1596	1.7922	0.49511	0.56316

15. (7 points) For a Type B Universal Life policy with death benefit equal to 15,000 plus account value issued to (45), you are given:

(i) The premium paid at the beginning of the first year is 3,500.

(ii) Expense charges in each year are 4% of premium plus 60, charged at the start of the year.

(iii) The cost of insurance rate is equal to 130% of the mortality rate at the attained age based on the *Illustrative Life Table*.

(iv) $i^c = 6\%$ and $i^q = 5\%$ for all years

(v) The account value at the end of the second year is equal to 6,528.75.

(a) (3 points) Calculate the premium paid at the beginning of the second year.

(b) (3 points) The corridor factor requirement is a minimum of 2.5 at each age. Calculate the largest amount of premium this policyholder can pay at the beginning of the second year in order to maintain the same amount of additional death benefit (ADB) of 15,000 at the end of the year.

(c) (1 point) Explain the purpose of a corridor factor requirement.

16. (14 points) An insurer issues a fully discrete four year term insurance contract to a life aged 50. The face amount is 100,000 and the gross annual premium for the contract is 660 per year.

The company uses the following assumptions to analyze the emerging surplus of the contract.

Interest:	7% per year
Initial Expenses:	80 immediately before first premium
Renewal expenses:	14 incurred on each premium date, including the first
Mortality:	Illustrative Life Table
Lapses:	None

(a) (8 points) Assume first that the insurer sets level reserves of ${}_tV^L = 50$, $t = 0, 1, 2, 3$, for each policy in force.

(i) Calculate the profit vector for the contract.

(ii) Calculate the profit signature for the contract.

(iii) Calculate the Net Present Value (NPV) assuming a risk discount rate of 10% per year.

(b) (4 points) The insurer is considering a different reserve method. The reserves would be set by zeroizing the emerging profits under the profit test assumptions, subject to a minimum reserve of 0. Calculate ${}_tV^Z$ for $t = 3, 2, 1, 0$.

(c) (2 points) State with reasons whether the NPV in part (a)(iii) would increase or decrease using the zeroized reserves from part (b), but keeping all other assumptions as in part (a).

17. (5 points) An insurer issues a fully continuous whole life policy with face amount S on a life aged x. Gross premiums are calculated according to the equivalence principle.

    Premiums and reserves are calculated using the same assumptions.

    (a) (3 points) Assume that expenses of $100e\%$ of the premium are payable continuously throughout the term of the contract. There are no other expenses.

        Show that the expense reserve is zero throughout the term of the contract.

    (b) (2 points) Now assume that there are additional acquisition expenses of $100f\%$ of the premium incurred at issue. Explain in words why, in this case, the expense reserve is negative for $t > 0$.

18. (13 points) A life insurance company issues a Type B universal life policy to a life age 60. The main features of the contract are as follows.

   Premiums: 3,000 per year, payable yearly in advance.

   Expense charges: 4% of the first premium and 1% of subsequent premiums.

   Death benefit: 10,000 plus the Account Value, payable at the end of the year of death.

   Cost of insurance rate: 0.022 for $60 \leq x \leq 64$; discounted at 3% to start of policy year.

   Cash surrender values: 90% of the account value for surrenders after 2 or 3 years, 100% of the account value for surrenders after four years or more.

   The company uses the following assumptions in carrying out a profit test of this contract.

   Interest rate: 6% per year.

   Credited interest: 5% per year.

   Survival model: $q_{60+t} = 0.02$ for $t = 0, 1, 2, 3$.

   Withdrawals: None in the first three years; all contracts assumed to surrender at the end of the fourth year.

   Initial expenses: 200 pre-contract expenses.

   Maintenance expenses: 50 incurred annually at each premium date including the first.

   Risk discount rate: 8% per year.

   There are no reserves held other than the account value.

   (a) (4 points) Show that the projected final account value for an insured surrendering at the end of the fourth year is 12,400 to the nearest 100. You should calculate the value to the nearest 1.

   (b) (7 points) Calculate the profit margin for a new policy.

   (c) (2 points) Calculate the NPV for a policy that is surrendered at the end of the second year.

19. (6 points) An insurer issues fully continuous 10-year term insurance policies with face amount 100,000 to lives age 50. Level gross premiums of 300 per year are payable continuously throughout the term of the contract. There is no cash value on lapse.

Gross premium reserves are calculated on the following basis:

Mortality:	$\mu_x = Bc^x$, where $B = 10^{-5}$, $c = 1.1$
Lapse:	Lapse transition intensity is 0.05 per year.
Interest:	4% per year compounded continuously
Expenses:	50 per year, incurred continuously.

(a) (2 points)

(i) Write down Thiele's differential equation for ${}_tV$ for this contract.

(ii) Write down a boundary condition for Thiele's differential equation.

(b) (4 points) Using Euler's backward method with a time step of $h = 0.2$ years, estimate ${}_{9.6}V$.

20. (10 points) Company XYZ offers a pension plan for their Chief Executive Officer (CEO), currently age 63. The pension plan pays a lump sum benefit of 250,000 at the end of the year of retirement or 500,000 at the end of the year of death while employed.

You are given:

(i) The CEO's birthday is January 1.

(ii) The following multiple decrement table for CEOs, where decrement 1 is retirement and decrement 2 is death:

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$
63	100,000	10,000	800
64	89,200	20,000	950
65	68,250	67,050	1,200

(iii) $i = 0.05$

(a) (3 points) Describe in words and calculate:

(i) $q_{64}^{(1)}$

(ii) ${}_2p_{63}^{(\tau)}$

(b) (2 points) Calculate the expected present value of the retirement benefit.

Company XYZ renegotiates the CEO's pension benefit to be tied to the Company's condition each year. The condition of the company follows a two state Markov model with states Healthy (H) and Distressed (D) and annual transition probabilities:

$$\begin{array}{c} \\ \mathbf{H} \\ \mathbf{D} \end{array} \begin{array}{c} \begin{array}{cc} \mathbf{H} & \mathbf{D} \end{array} \\ \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

The Company's condition is independent of the CEO's status.

If at the end of a year, the Company is Healthy, the pension benefits are increased by 25% from the previous year. If at the end of a year, the Company is Distressed, the pension benefits are decreased by 15% from the previous year. These adjustments are also made for the year during which the CEO leaves.

The company is currently Distressed.

(c) (1 point) Calculate the probability that the company is Distressed two years from now.

(d) (4 points)

(i) Show that if the company is Distressed at the end of the first year and Healthy at the end of the second year, the benefit for death between ages 64 and 65 is 531,000, to the nearest 1,000. You should calculate the value to the nearest 10.

(ii) Calculate the probability that the payment exceeds 370,000 if the CEO dies or retires between ages 64 and 65.

21. (9 points) Lauren enters a defined benefit pension plan on January 1, 2000 at age 45, with a starting annual salary of 50,000. You are given:

- The annual retirement benefit is 1.7% of the final three-year average salary for each year of service.
- Her normal retirement date is December 31, 2019, at age 65.
- All retirements occur on December 31. The reduction in the benefit for early retirement is 5% for each year prior to normal retirement date.
- Lauren expects to receive a 4% salary increase each January 1.
- The benefit is payable as a single life annuity.

(a) (3 points) Show that 63 is the first age at which Lauren expects she can retire with at least a 25% replacement ratio.

Lauren retires at age 65. Her pay increases have been different than assumed. She can choose from these three actuarially equivalent retirement benefits, each payable annually:

- A single life annuity-due of 24,000.
- A life annuity-due with ten years guaranteed of X.
- A 50% joint and survivor annuity-due with initial benefit Y. (Y is paid for Laurens lifetime. Upon Lauren's death a reversionary benefit of $Y/2$ is paid to Lauren's spouse.)

You are also given:

- Lauren's spouse Chris is ten years younger.
- Lauren and Chris have independent future lifetimes.
- Mortality follows the Illustrative Life Table.
- $i = 0.06$

(b) (4 points)

(i) Calculate X.

(ii) Calculate Y.

You are given that at $i = 0.06$, the duration of a whole life annuity of 1 on (65) is 7.039 and the duration of a life annuity of 1 on (65) with 10 years certain is 6.977.

(c) (1 point) State with reasons whether you would expect X to increase or decrease if i were 0.03. The benefit would still be actuarially equivalent to the life annuity-due of 24,000, using $i = 0.03$ for both.

The company offers Lauren another actuarially-equivalent retirement option: a life annuity-due of Z, with the provision that if she died within the first 10 years, the balance of the annuity payments of Z for the first 10 years would be paid immediately at her death.

(d) (1 point) Explain, without calculating the benefit, whether $Z > X, Z = X,$ or $Z < X$.

22. On December 31, 2015, Nancy, who is age 55, has 25 years of service. Her salary in 2015 was 50,000.

The annual accrued benefit as of any date is 1.6% of the three-year final average salary as of that date, multiplied by years of service as of that date. The pension is payable only to retired participants as a monthly single life annuity, with the first payment due at retirement.

The valuation assumptions are as follows:

- There are no benefits except for retirement benefits.
- Exits from employment follow the Illustrative Service Table, except that all lives surviving in employment to age 61 retire at that time.
- Retirements at age 60 occur at exact age 60 while other decrements at age 60 occur throughout the year after the retirement at exact age 60 and prior to exact age 61.
- After retirement, mortality follows the Illustrative Life Table.
- Annuities are valued using the 2-term Woolhouse formula.
- Nancy has received a 3% salary increase on January 1 of each year for the last five years.
- Future salaries are expected to increase by 3% each year on January 1.
- All contributions are paid on January 1 each year.
- $i = 0.06$

(a) (1 point) A pension plan may be classified as a Defined Contribution plan or a Defined Benefit plan. State which type of plan Nancy has and briefly describe the other type of plan.

(b) (3 points)

(i) The actuarial accrued liability for Nancy at December 31, 2015 using the Projected Unit Credit (PUC) funding method is 150,000 to the nearest 1000. Calculate the actuarial accrued liability to the nearest 1.

(ii) Calculate the normal cost for 2016 using PUC.

(c) (3 points)

(i) The actuarial accrued liability for Nancy at December 31, 2015 using the Traditional Unit Credit (TUC) funding method is 126,000 to the nearest 1000. Calculate the actuarial accrued liability to the nearest 1.

(ii) Calculate the normal cost for 2016 using TUC.

(d) (1 point) For Nancy, the normal cost during 2016 using PUC is less than the normal cost during 2016 using TUC. Explain why this will be true for all employees near retirement.

1. (a) One of the requirements for a survival function is that $S_0(0) = 1$:

$$\begin{aligned} S_0(0) &= 1 \\ \frac{(121-0)^{1/2}}{k} &= 1 \\ (121)^{1/2} &= k \\ k &= 11 \end{aligned}$$

(b) The limiting age for this survival function is the smallest value of ω such that $S_0(\omega) = 0$:

$$\begin{aligned} S_0(\omega) &= 0 \\ \frac{(121-\omega)^{1/2}}{k} &= 0 \\ \omega &= 121 \end{aligned}$$

(c)

$$\begin{aligned} \mathring{e}_0 &= \int_0^{\infty} S_0(t)\, dt \\ &= \int_0^{121} \frac{(121-t)^{1/2}}{11}\, dt \\ &= \frac{1}{11}\left[\left(-\frac{2}{3}\right)(121-t)^{3/2}\right]_0^{121} \\ &= \frac{2}{33}(121)^{3/2} \\ &= 80.6667 \end{aligned}$$

(d)

$$\begin{aligned} \mu_x &= \frac{-\frac{d}{dx} S_0(x)}{S_0(x)} \\ &= \frac{-\frac{d}{dx}\frac{(121-x)^{1/2}}{k}}{\frac{(121-x)^{1/2}}{k}} \\ &= \frac{-\frac{d}{dx}(121-x)^{1/2}}{(121-x)^{1/2}} \\ &= \frac{\frac{1}{2}(121-x)^{-1/2}}{(121-x)^{1/2}} \\ &= \frac{1}{2(121-x)} \end{aligned}$$

(e)

$$
\begin{aligned}
S_{57}(t) &= \frac{S_0(57+t)}{S_0(57)} \\
&= \frac{\frac{(121-(57+t))^{1/2}}{11}}{\frac{(121-57)^{1/2}}{11}} \\
&= \frac{(121-(57+t))^{1/2}}{(121-57)^{1/2}} \\
&= \sqrt{\frac{64-t}{64}}
\end{aligned}
$$

Then we can calculate

$$
\begin{aligned}
{}_{27|16}q_{57} &= S_{57}(27) - S_{57}(43) \\
&= \sqrt{\frac{64-27}{64}} - \sqrt{\frac{64-43}{64}} \\
&= 0.1875
\end{aligned}
$$

This problem tests Learning Objective 1.

2. (a) The paths leading to Liquidation within the next 2 years are:

$H \to L$ with probability 0.05,

$H \to D \to L$ with probability $(0.15)(0.2) = 0.03$, and

$H \to H \to L$ with probability $(0.8)(0.05) = 0.04$.

The probabilities of these paths sum to 0.12.

(b) The calculation in the previous part is based on a Markov model, under which the previous states are irrelevant. Hence, this information does **not** impact the calculation.

(c) Let L be a random variable whose value is 1 in the event of liquidation and 0 otherwise; then

$$\begin{aligned} E[PV] &= E[PV|L=1] \cdot P(L=1) + E[PV|L=0] \cdot P(L=0) \\ &= \left(500,000\, v^2\right)(0.12) + \left(1,000,000\, v^2\right)(0.88) \\ &= 805,899 \approx 805,900 \end{aligned}$$

(d) The PV of the bond payment will exceed the expected present value of the bond if the bond payment exceeds $(805,899)(1.08)^2 = 940,000$. The probability of this happening is

$$\begin{aligned} P(\text{Payment} > 940,000) &= P(\text{Payment} > 940,000 \,|\, L=1)P(L=1) \\ &\quad + P(\text{Payment} > 940,000 \,|\, L=0)P(L=0) \\ &= (0.06)(0.12) + (1)(0.88) \\ &= 0.8872 \end{aligned}$$

(e) The variance of the PV is

$$Var(PV) = E[Var(PV|L)] + Var(E[PV|L]).$$

In the case of no liquidation, the PV of the bond payment is a constant, so the variance of its PV is 0. In the event of liquidation, the bond payment is uniformly distributed on 0 to 1,000,000, so that the PV of the bond payment is uniformly distributed on 0 to 1,000,000 v^2. In this case, the variance of the PV of the bond payment is $\frac{\left(1{,}000{,}000 v^2\right)^2}{12}$. Then the variance of the PV, given the liquidation status, is:

$$Var(PV|L) = \begin{cases} 0 & \text{with prob. } 0.88 \\ \frac{\left(1{,}000{,}000 v^2\right)^2}{12} & \text{with prob. } 0.12 \end{cases}$$

so that its expected value is

$$E[Var(PV|L)] = (0.88)(0) + (0.12) \cdot \frac{(1,000,000 v^2)^2}{12} = 7,350,298,528$$

In the case of no liquidation, the expected value of the PV of the bond payment is $1,000,000\,v^2$. In the case of liquidation, the PV of the bond payment is uniformly distributed on 0 to $1{,}000{,}000\,v^2$, so that the expected value of its PV is $500,000\,v^2$. Then the expectation of the PV, given the liquidation status, is:

$$E[PV \mid L] = \begin{cases} 1,000,000\,v^2 & \text{with prob. } 0.88 \\ 500,000\,v^2 & \text{with prob. } 0.12 \end{cases}$$

so that its variance is

$$Var(E[PV|L]) = (1,000,000\,v^2 - 500,000\,v^2)^2 \cdot (0.88) \cdot (0.12) = 19,404,788,114$$

Thus the standard deviation of the PV is

$$\sqrt{Var(PV)} = \sqrt{E[Var(PV|L)] + Var(E[PV|L])} = 163,570.$$

This problem tests Learning Objectives 1 and 2.

3. (a) (i)

$$q_{80} = 1 - p_{80} = 1 - \frac{{}_{21}p_{60}}{{}_{20}p_{60}} = 1 - \frac{e^{0.5(1-1.05^{21})}}{e^{0.5(1-1.05^{20})}} = 0.06418$$

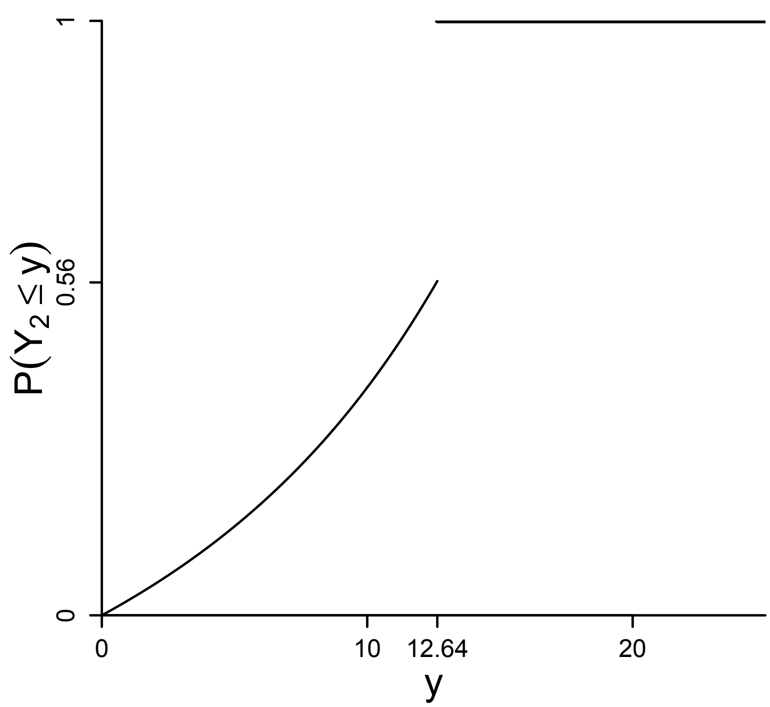
(ii)

$$\begin{aligned}\mu_{60+t} &= \frac{-\frac{d}{dt}\,{}_tp_{60}}{{}_tp_{60}} \\ &= \frac{-e^{0.5(1-1.05^t)}(0.5)(-1.05^t)(\ln(1.05))}{e^{0.5(1-1.05^t)}} \\ &= (0.5)(1.05^t)(\ln(1.05))\end{aligned}$$

so that $\mu_{80} = 0.0647$.

(b) $P\left(\frac{Y_1}{Y_2} > 1\right) = P(Y_1 > Y_2)$, which is the probability that the policyholder lives at least 20 years: ${}_{20}p_{60} = 0.4375$.

(c) The plot is shown here, where $\bar{a}_{\overline{20|}} = 12.64$.



This problem tests Learning Objectives 1 and 2.

4. (a) (i) The terms on the right side of the equation are probabilities representing the following: (1) the process is in state 0 at t, and stays there to $t+h$; (2) the process is in state 1 at t and moves back to state 0 by time $t+h$, and (3) the process is in state 0 at t and moves at least twice before $t+h$ to state 1 and back to state 0. In each of these probabilities, the process is in state 0 at time 0.

(ii)

$$
\begin{aligned}
{}_{t+h}p_x^{00} &= {}_tp_x^{00}\ {}_hp_{x+t}^{\overline{00}} + {}_tp_x^{01}\ {}_hp_{x+t}^{10} + o(h) \text{ (using assumptions 1 and 2)}\\
\Rightarrow {}_{t+h}p_x^{00} &= {}_tp_x^{00}\,(1 - \ {}_hp_{x+t}^{01} - \ {}_hp_{x+t}^{02} - o(h)) + {}_tp_x^{01}\ {}_hp_{x+t}^{10} + o(h)\\
\Rightarrow {}_{t+h}p_x^{00} &= {}_tp_x^{00}\,(1 - \ {}_hp_{x+t}^{01} - \ {}_hp_{x+t}^{02}) + {}_tp_x^{01}\ {}_hp_{x+t}^{10} + o(h)\\
\Rightarrow \frac{{}_{t+h}p_x^{00} - {}_tp_x^{00}}{h} &= {}_tp_x^{01}\,\frac{{}_hp_{x+t}^{10}}{h} - {}_tp_x^{00}\,\frac{{}_hp_{x+t}^{01} + \ {}_hp_{x+t}^{02}}{h} + \frac{o(h)}{h}\\
\Rightarrow \lim_{h\to 0}\frac{{}_{t+h}p_x^{00} - {}_tp_x^{00}}{h} &= {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right) \quad \text{(assumption 3)}\\
\Rightarrow \frac{d}{dt}\,{}_tp_x^{00} &= {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{d}{dt}\,{}_tp_x^{01} &= {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right)\\
\frac{d}{dt}\,{}_tp_x^{02} &= {}_tp_x^{00}\,\mu_{x+t}^{02} + {}_tp_x^{01}\,\mu_{x+t}^{12}
\end{aligned}
$$

This problem tests Learning Objective 1.

5. (a) The ordering is

$$ {}_2p_x^{\overline{00}} \leq {}_1p_x^{00}\,{}_1p_{x+1}^{00} \leq {}_2p_x^{00} $$

The reason is that ${}_2p_x^{\overline{00}}$ is the probability that (x) is healthy throughout the period from age x to age $x+2$, given that she is healthy at age x, while ${}_1p_x^{00}\,{}_1p_{x+1}^{00}$ is the probability that (x) is healthy at age $x+1$ and age $x+2$, but includes the possibility of periods of sickness between the integer ages. Similarly, ${}_2p_x^{00}$ is the probability that (x) is healthy at age $x+2$ (given healthy at x), which must be no less than ${}_1p_x^{00}\,{}_1p_{x+1}^{00}$ as it allows for the additional event that (x) is sick at age $x+1$, but recovers by age $x+2$.

(b) (i)

$$ {}_2p_{60}^{00} = p_{60}^{00}\,p_{61}^{00} + p_{60}^{01}\,p_{61}^{10} = 0.93765 $$

(ii)

$$ {}_2p_{60}^{01} = p_{60}^{00}\,p_{61}^{01} + p_{60}^{01}\,p_{61}^{11} = 0.02839 $$

(c)

$$ \text{EPV} = 1 + p_{60}^{00}\,v + {}_2p_{60}^{00}\,v^2 = 2.7740 $$

(d) EPV of the death benefit, where $S = 10000$,

$$ S\left(p_{60}^{02}\,v + \left(p_{60}^{00}\,p_{61}^{02} + p_{60}^{01}\,p_{61}^{12}\right)v^2 + \left({}_2p_{60}^{00}\,p_{62}^{02} + {}_2p_{60}^{01}\,p_{62}^{12}\right)v^3\right) $$
$$ = 0.047955S = 479.55 $$

So

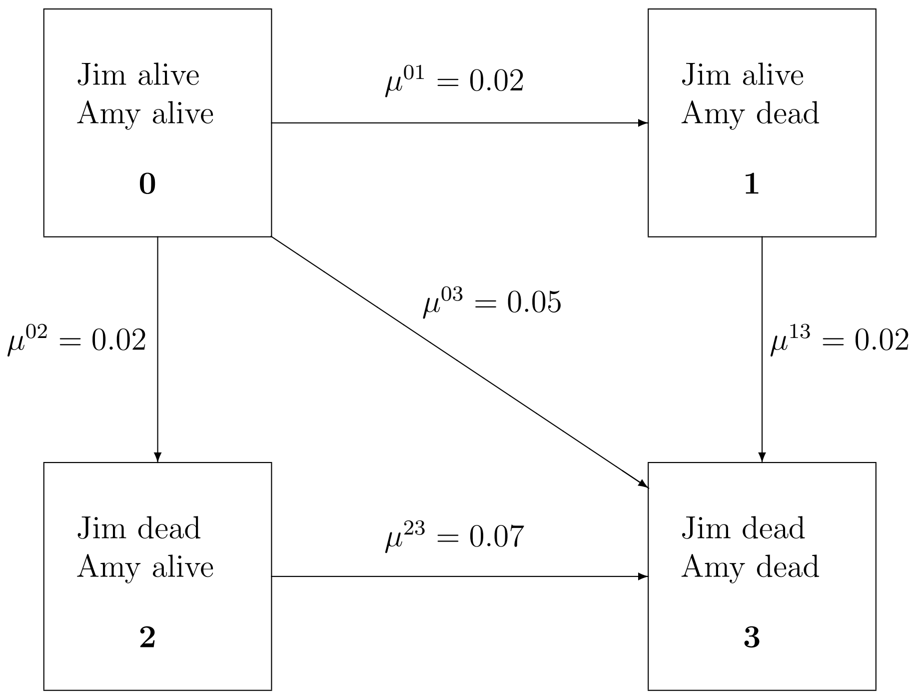
$$ P = \frac{479.55}{2.774} = 172.87 $$

This problem tests Learning Objectives 1 and 3.

6. (a) (i) Because Jim and Amy have independent future lifetimes prior to the motorcycle purchase, the covariance of their future lifetimes is 0.

(ii) The common shock due to the motorcycle purchase will cause the covariance of their future lifetimes to increase. Suppose Jim and Amy are both alive at t, and we know that Jim is dead at $t+k$, then there is a probability that Jim died from the motorcycle accident, which means that Amy would also have died. Hence, a shorter lifetime for Jim is associated with a shorter lifetime for Amy, which implies positive covariance.

(b)



(c) In terms of the diagram above, the probability that Jim lives at least 20 more years is ${}_{20}p^{00} + {}_{20}p^{01}$, where

$$ {}_{20}p^{00} = \exp\left[-\int_0^{20} 0.09\,dt\right] = \exp\left[-0.09t\,\big|_0^{20}\right] = e^{-1.8} \text{ and} $$

$$ \begin{aligned} {}_{20}p^{01} &= \int_0^{20} {}_tp^{00}\,\mu^{01}\,{}_{20-t}p^{11}\,dt \\ &= \int_0^{20} e^{-0.09t}(0.02)e^{-0.02(20-t)}\,dt = e^{-0.4}\int_0^{20} 0.02\,e^{-0.07t}\,dt \\ &= e^{-0.4}\,0.02\,\frac{1-e^{-1.4}}{0.07} = \frac{0.02}{0.07}(e^{-0.4}-e^{-1.8}) \end{aligned} $$

so that

$$ {}_{20}p^{00} + {}_{20}p^{01} = \frac{2}{7}e^{-0.4} + \frac{5}{7}e^{-1.8} = 0.3096 $$

(d) The annual net premium rate for this policy is

$$ \begin{aligned} P &= \frac{100,000\left[\int_0^\infty e^{-\delta t}\,{}_tp^{00}\left(\mu^{01}+\mu^{02}\right)\,dt\right] + 10,000\left[\int_0^\infty e^{-\delta t}\,{}_tp^{00}\,\mu^{03}\,dt\right]}{\int_0^\infty e^{-\delta t}\,{}_tp^{00}\,dt} \\ &= \frac{100,000\left[\int_0^\infty e^{-0.06t}\,e^{-0.09t}\,(0.04)\,dt\right] + 10,000\left[\int_0^\infty e^{-0.06t}\,e^{-0.09t}\,(0.05)\,dt\right]}{\int_0^\infty e^{-0.06t}\,e^{-0.09t}\,dt} \\ &= \frac{100,000\left[\frac{0.04}{0.15}\right] + 10,000\left[\frac{0.05}{0.15}\right]}{\frac{1}{0.15}} \\ &= 4,500 \end{aligned} $$

This problem tests Learning Objectives 1 and 3.

7. (a) We can write Z as

$$Z = \begin{cases} 0 & K_{40} < 10 \\ \dfrac{1000}{1.06^{K_{40}+1}} & 10 \leq K_{40} < 20 \\ \dfrac{2000}{1.06^{K_{40}+1}} & 20 \leq K_{40} < 30 \\ 0 & K_{40} \geq 30 \end{cases}$$

(b)

$$\begin{aligned} Pr(Z = 0) &= Pr(K_{40} < 10) + Pr(K_{40} \geq 30) \\ &= [(l_{40} - l_{50}) + l_{70}]/l_{40} \\ &= (9,313,166 - 8,950,901 + 6,616,155)/9,313,166 \\ &= 0.75 \end{aligned}$$

(c) If $\dfrac{1000}{1.06^{K_{40}+1}} > 400$, then

$$\begin{aligned} \left(\frac{1}{1.06}\right)^{K_{40}+1} &> \frac{400}{1000} \\ -(K_{40}+1)ln(1.06) &> ln(0.4) \\ K_{40}+1 &< 15.73 \\ K_{40} &< 14.73 \\ K_{40} &\leq 14 \text{ since } K_{40} \text{ is an integer.} \end{aligned}$$

If $\dfrac{2000}{1.06^{K_{40}+1}} > 400$, then

$$\begin{aligned} \left(\frac{1}{1.06}\right)^{K_{40}+1} &> \frac{400}{2000} \\ -(K_{40}+1)ln(1.06) &> ln(0.2) \\ K_{40}+1 &< 27.62 \\ K_{40} &< 26.62 \\ K_{40} &\leq 26 \text{ since } K_{40} \text{ is an integer.} \end{aligned}$$

$$\begin{aligned} Pr(Z > 400) &= Pr(\text{Benefit} = 1000 \text{ and } Z > 400) + Pr(\text{Benefit} = 2000 \text{ and } Z > 400) \\ &= Pr(10 \leq K_{40} \leq 14) + Pr(20 \leq K_{40} \leq 26) \\ &= (l_{50} - l_{55})/l_{40} + (l_{60} - l_{67})/l_{40} \\ &= (8,950,901 - 8,640,861)/9,313,166 + (8,188,074 - 7,201,635)/9,313,166 \\ &= 0.0333 + 0.1059 = 0.1392 \end{aligned}$$

(d)

$$
\begin{aligned}
E[Z^2] &= 1000^2({}_{10}E_{40}\, v^{10})({}^2A_{50} - {}_{10}E_{50}\, v^{10}\, {}^2A_{60}) + 2000^2({}_{20}E_{40}\, v^{20})({}^2A_{60} - {}_{10}E_{60}\, v^{10}\, {}^2A_{70}) \\
&= 1000^2 \left[({}^2A_{50})({}_{10}E_{40}\, v^{10}) + 3({}^2A_{60})({}_{20}E_{40}\, v^{20}) - 4({}^2A_{70})({}_{30}E_{40}\, v^{30})\right] \\
&= 1000^2 \left[0.09476(0.53667)v^{10} + 3(0.17741)(0.27414)v^{20} \right. \\
&\qquad \left. -4(0.30642)(0.53667)(0.23047)v^{30}\right] \\
&= 1000^2[0.02840 + 0.04549 + .02640] \\
&= 47,495
\end{aligned}
$$

Then $Var(Z) = E[Z^2] - [E(Z)]^2 = 47,495 - (107)^2 = 36,046 \approx 36,050.$

(e) (i) Under the normal approximation without continuity correction, Z', our approximation, is normal with mean 107 and standard deviation $\sqrt{36046} = 190$. Then $(Z' - 107)/190$ is a standard normal random variable.

$$
\begin{aligned}
Pr(Z' > 400) &= Pr((Z' - 107)/190) > 1.54 \\
&= 1 - \Phi(1.54) \\
&= 1 - 0.9382 \\
&= 0.0618
\end{aligned}
$$

(ii) The normal approximation is almost never a good approximation for a single observation, unless the underlying distribution is approximately normal. Here the underlying distribution, with 75% probability of 0, is far from normal.

This problem tests Learning Objective 2.

8. (a) The EPV of the death benefit is

$$100{,}000\left(v\,q_{60}^{(1)} + v^2\,p_{60}^{(\tau)}\,q_{61}^{(1)}\right) = 100{,}000\left(\frac{0.12}{1.07} + \frac{(1-0.12)(1-0.10)(0.18)}{(1.07)^2}\right)$$
$$= 23{,}667 \approx 23{,}670$$

(b) The annual gross premium is G, where

$$\begin{aligned}
\text{EPV(premiums)} &= \text{EPV(benefits)} + \text{EPV(expenses)}\\
G\left(1 + v\,p_{60}^{(\tau)}\right) &= 23{,}667 + (0.05)P\left(1 + v\,p_{60}^{(\tau)}\right)\\
0.95G\left(1 + v\,p_{60}^{(\tau)}\right) &= 23{,}667\\
G &= \frac{23{,}667}{0.95\left(1 + \frac{(0.9)(0.88)}{1.07}\right)}\\
G &= 14{,}316
\end{aligned}$$

(c) Since this is a single premium product, after the premium is paid, the only potential cash flow will be the benefit payment from the insurer to the insured. Thus, the insured would have no reason to lapse this policy; lapsing could only be detrimental to the policyholder, it could never be beneficial in terms of the possible future cash flows. This is opposed to the case where the policyholder pays an annual premium; in this case, the insured might decide that the potential future benefit from the insurer is not sufficient to continue to pay the annual premium.

(d) $EPV = 100{,}000\left(v\,q_{60}^{(1)} + v^2\,p_{60}^{(\tau)}\,q_{61}^{(1)}\right) = 100{,}000\left(\frac{0.12}{1.07} + \frac{(0.88)(0.18)}{(1.07)^2}\right) = 25{,}050$

(e) Because lapses have been eliminated, mortality will act on a larger portion of the population, resulting in a larger number of deaths and hence a greater EPV of death benefits.

This problem tests Learning Objectives 2 and 3.

9. (a) Net premiums are premiums determined by the equivalence principle, ignoring expenses.

Gross premiums are the premiums payable by the policyholder.

Portfolio percentile premiums are gross premiums determined such that the probability of an aggregate loss at issue on the portfolio is less than some desired probability.

(b) $A_x = 1 - d\ddot{a}_x = 1 - (0.05/1.05)(7.963) = 0.6208$
$P = (\text{Benefit})(A_x)/(\ddot{a}_x) = (5000)(0.6208)/7.963 = 390$

(c) For one policy,

$$E(L) = DB * A_x - P * \ddot{a}_x$$
$$= (5000)(0.6208) - (410)(7.963) = -160.83$$

$${}^2A_x = {}_3|{}^2A_x + {}^2A^{\,1}_{x:\overline{3}|}$$

$${}^2A^{\,1}_{x:\overline{3}|} = q_x v^2 + p_x q_{x+1} v^4 + p_{x+1} p_{x+2} q_{x+2} v^6$$
$$= 0.03/1.05^2 + (0.97)(0.035)/1.05^4 + (0.97)(0.965)(0.04)/1.05^6 = 0.0831$$

$${}^2A_x = 0.3230 + 0.0831 = 0.4061$$

$$Var(L) = (DB + P/d)^2 \times ({}^2A_x - A_x^2)$$
$$= [5000 + 410/(0.05/1.05)]^2[0.4061 - 0.6208^2] = 3,835,668$$

Standard deviation = 1958.

(d) For a portfolio of n policies,

$$Pr(L^{agg} > 0) = Pr\left(Z > \frac{0 - (-E(L) * n)}{\sqrt{Var(L) * n}}\right)$$

$$Pr\left(Z > \frac{0 - (-160.83n)}{1958\sqrt{n}}\right) < 0.01$$
$$(160.83n)/(1958\sqrt{n}) < 2.326$$
$$n > 801.88$$

You need at least 802 policies.

(e) For a fixed number of policies, a positive correlation coefficient would increase the variance of the aggregate loss at issue with no change in the expected aggregate loss at issue. Therefore it would increase the number of policies needed such that the aggregate loss at issue would be positive less than 1% of the time.

This problem tests Learning Objectives 2 and 3.

10. (a)

$$\begin{aligned} EPV &= (50,000)(q_{60}v + p_{60}q_{61}v^2 + p_{60}p_{61}q_{62}v^3) + (10,000)p_{60}p_{61}p_{62}v^3 \\ &= (50,000)\left[0.11/1.06 + (0.89)(0.12)/1.06^2 + (0.89)(0.88)(0.20)/1.06^3\right] \\ &\qquad + (10,000)(0.89)(0.88)(0.8)/1.06^3 \\ &= 21,778 \end{aligned}$$

$$\begin{aligned} G &= EPV + 0.3G \\ &= EPV/(1-0.3) = 31,111 \end{aligned}$$

(b) The l_x column below is the product of (1-q_x)'s.

k	l_{60+k}	$l_{60.25+k}$	$d_{60.25+k}$	$(50,000)d_{60.25+k}v^{k+1}$
0	1.0000	0.9725	0.1092	5150.94
1	0.8900	0.8633	0.1193	5308.83
2	0.7832	0.7440	0.1613	6771.53
3	0.6266	0.5827		
4	0.4511			

EPV of death benefits = (5150.94+5308.83+6771.53)/0.9725=17,718.56
EPV of maturity benefit = $(10,000)(0.5827/0.9725)/1.06^3 = 5,030.81$
EPV = 17,718.56+5,030.81 = 22,749.37

G=22,749.37/(1-0.3)=32,499

(c) The probability the company pays more than 20,000 is the probability that it pays the death benefit of 50,000.

probability=1-0.5827/0.9725 = 0.401

(d) Since there are no future expenses, the gross premium reserve is just the expected present value of the future death and maturity benefits:

V=(50,000)(0.1613/0.7440)/1.06+(10,000)(0.5827/0.7440)/1.06 = 17,617

This problem tests Learning Objectives 2, 3, and 4.

11. (a) The annual gross premium is G, where

$$\begin{aligned} 0.9G\ddot{a}_{40} &= 100,000A_{40} + 0.4G \\ G &= \frac{100,000A_{40}}{0.9\ddot{a}_{40} - 0.4} \\ G &= \frac{16,132}{(0.9)(14.8166) - 0.4} \\ G &= 1247 \end{aligned}$$

(b) The future loss at issue RV for a single policy is given by

$$\begin{aligned} L_0 &= 100,000v^{K_{40}+1} + 0.4G - 0.9G\ddot{a}_{\overline{K_{40}+1|}} \\ &= 100,000v^{K_{40}+1} + 0.4G - 0.9G\left(\frac{1 - v^{K_{40}+1}}{0.0566}\right) \\ &= 0.4G - \frac{0.9G}{0.0566} + 100,000v^{K_{40}+1} + \frac{0.9Gv^{K_{40}+1}}{0.0566} \\ &= -15.501G + v^{K_{40}+1}\left(100,000 + 15.901G\right), \end{aligned}$$

where G is the annual gross premium. Then

$$\begin{aligned} E[L_0] &= -15.501G + A_{40}\left(100,000 + 15.901G\right) \\ &= -15.501G + 0.16132\left(100,000 + 15.901G\right) \\ &= 16,132 - 12.9359G \end{aligned}$$

and

$$\begin{aligned} Var[L_0] &= \left({}^2A_{40} - A_{40}^2\right)\left(100,000 + 15.901G\right)^2 \\ &= \left(0.04863 - (0.16132)^2\right)\left(100,000 + 15.901G\right)^2 \\ &= (0.0226)\left(100,000 + 15.901G\right)^2 \end{aligned}$$

The mean and variance for the aggregate loss variable are

$$E[L_0^{agg}] = 100 \cdot E[L_0] \quad \text{and} \quad Var(L_0^{agg}) = 100 \cdot Var(L_0)$$

Then the probability of an aggregate loss for the block of business is

$$P\left(L_0^{agg} > 0\right) = P\left(\frac{L_0^{agg} - E[L_0^{agg}]}{\sqrt{Var(L_0^{agg})}} > \frac{0 - E[L_0^{agg}]}{\sqrt{Var(L_0^{agg})}}\right) \approx P\left(Z > \frac{-E[L_0^{agg}]}{\sqrt{Var(L_0^{agg})}}\right),$$

so that $\dfrac{-E[L_0^{agg}]}{\sqrt{Var(L_0^{agg})}} = 1.645.$

Then $1,293.59G - 1,613,200 = 16.45((0.15035)(100,000 + 15.901G))$ so that $G = 1,483$.

(c) The gross premium reserve at time 10 is

$$ {}_{10}V^{g} = 100,000A_{50} - 0.9(1,483)\ddot{a}_{50} = 7,198 $$

(d) The FPT reserve at time 10 is

$$ {}_{10}V^{FPT} = 100,000A_{50} - \frac{100,000A_{41}}{\ddot{a}_{41}}\ddot{a}_{50} = 9,667 $$

(e) The marketing officer is generally correct that increasing the number of policies sold will allow the insurer to lower the gross annual premium while maintaining a 5% probability of loss. This is due to the fact that increasing the number of policies sold diversifies the mortality risk (i.e., the deviation from the expected value) so that it will tend to zero. As a result, the limiting value of the gross annual premium that will result in a 5% probability of loss to the insurer is that given by the equivalence principle, 1,247 in this case. Diversification cannot lower the premium any further than this; in particular, a premium of 1,200 is unattainable in this manner.

This problem tests Learning Objectives 3 and 4.

12. (a) The Kolmogorov Forward Equations for this model are

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\left(\mu_{x+t}^{01} + \mu_{x+t}^{02}\right), \qquad {}_0p_x^{00} = 1$$
$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\left(\mu_{x+t}^{10} + \mu_{x+t}^{12}\right), \qquad {}_0p_x^{01} = 0$$
$$\frac{d}{dt}{}_tp_x^{02} = {}_tp_x^{00}\,\mu_{x+t}^{02} + {}_tp_x^{01}\,\mu_{x+t}^{12}, \qquad {}_0p_x^{02} = 0$$

(b) The level annual premium rate, payable continuously, is P, where

$$P\int_0^5 e^{-\delta t}\,{}_tp_x^{\overline{00}}\,dt = 100,000\int_0^5 e^{-\delta t}\,{}_tp_x^{\overline{00}}(\mu_{x+t}^{01} + \mu_{x+t}^{02})\,dt$$
$$P = 100,000\,\frac{\int_0^5 0.022e^{-0.062t}\,dt}{\int_0^5 e^{-0.062t}\,dt}$$
$$P = 100,000(0.022)\,\frac{\int_0^5 e^{-0.062t}\,dt}{\int_0^5 e^{-0.062t}\,dt}$$
$$P = 100,000(0.022) = 2,200$$

(c) Thiele's differential equation is given by

$$\begin{aligned}\frac{d}{dt}{}_tV^{(0)} &= {}_tV^{(0)}\,\delta + P - \mu_{x+t}^{01}(100,000 + {}_tV^{(1)} - {}_tV^{(0)}) - \mu_{x+t}^{02}(100,000 + {}_tV^{(2)} - {}_tV^{(0)})\\ &= {}_tV^{(0)}\,(0.04) + 2,200 - (0.02)(100,000 + 0 - {}_tV^{(0)}) - (0.002)(100,000 + 0 - {}_tV^{(0)})\\ &= {}_tV^{(0)}\,(0.04) + 2,200 - (0.022)(100,000 - {}_tV^{(0)})\\ &= {}_tV^{(0)}(0.062)\end{aligned}$$

(d) (i) We know that, as we approach time $t = 5$, the reserve approaches 0. Then Euler's backward method with $h = 0.1$ gives

$$\begin{aligned}{}_5V^{(0)} - {}_{4.9}V^{(0)} &= (h)({}_5V^{(0)})(0.062)\\ 0 - {}_{4.9}V^{(0)} &= (0.1)(0)(0.062)\\ {}_{4.9}V^{(0)} &= 0\end{aligned}$$

Similarly,

$$\begin{aligned}{}_{4.9}V^{(0)} - {}_{4.8}V^{(0)} &= (h)({}_{4.9}V^{(0)})(0.062)\\ 0 - {}_{4.8}V^{(0)} &= (0.1)(0)(0.062)\\ {}_{4.8}V^{(0)} &= 0\end{aligned}$$

and

$$
\begin{aligned}
{}_{4.8}V^{(0)} - {}_{4.7}V^{(0)} &= (h)({}_{4.8}V^{(0)})(0.062) \\
0 - {}_{4.7}V^{(0)} &= (0.1)(0)(0.062) \\
{}_{4.7}V^{(0)} &= 0
\end{aligned}
$$

(ii) The reserve is 0 at time $t = 4.7$ because the premium rate being paid matches the expected rate of payment of claims. This is in turn due to the constant forces of transition among the states.

The problem tests Learning Objectives 1, 3, and 4.

13. (a) Letting G denote the single gross premium, we have the equation of value:

$$
\begin{aligned}
G &= 100 + (G+200)\,\bar{A}^{\,1}_{65:\overline{1|}} + (25,000+200)_{1|}\bar{A}_{65} \\
&= 100 + (25,000+200)\bar{A}_{65} - (25000+200)\,\bar{A}^{\,1}_{65:\overline{1|}} + (G+200)\bar{A}^{\,1}_{65:\overline{1|}} \\
&= 100 + (25,200)\bar{A}_{65} + (G-25,000)\,\bar{A}^{\,1}_{65:\overline{1|}}
\end{aligned}
$$

Then applying the UDD assumption gives:

$$G = 100 + (25,200)\left(\frac{i}{\delta}\right) A_{65} + (G-25,000)\left(\frac{i}{\delta}\right) A^{\,1}_{65:\overline{1|}}$$

$$G = 100 + (25,200)\left(\frac{0.06}{\ln(1.06)}\right)(0.4398) + (G-25,000)\left(\frac{0.06}{\ln(1.06)}\right)\left(\frac{0.02132}{1.06}\right)$$

$$G = 11,227$$

(b) (i) For $0 < t \leq 1$:

$$\frac{d}{dt}\,{}_tV = \delta\,{}_tV - \mu_{65+t}\,(G+200-{}_tV)$$

(ii) Applying Euler's forward method, for $t+h \leq 1$, yields

$${}_{t+h}V \simeq {}_tV + h\,\delta\,{}_tV - h\mu_{65+t}\,(G+200-{}_tV)$$

So

$$
\begin{aligned}
{}_{\frac{2}{3}}V &\simeq {}_{\frac{1}{3}}V + \left(\frac{1}{3}\right)(\delta)\,{}_{\frac{1}{3}}V - \left(\frac{1}{3}\right)\mu_{65\frac{1}{3}}\left(G+200-{}_{\frac{1}{3}}V\right) \\
&\simeq 11,554
\end{aligned}
$$

(c) Euler's method could be made more accurate by using a smaller value for h, such as 0.1 or 0.01.

This problem tests Learning Objectives 3 and 4.

14. Let $S = 200,000$ and $B = 50,000$, $\delta = \log(1.05)$.

(a)

$$P = \frac{S\bar{A}_{50}^{02} + B\bar{a}_{50}^{01}}{\bar{a}_{50}^{00}} = 11,414$$

(b)

$$\frac{d}{dt}{}_tV^{(0)} = \delta\, {}_tV^{(0)} + P - \mu_{50+t}^{02}\left(S - {}_tV^{(0)}\right) - \mu_{50+t}^{01}\left({}_tV^{(1)} - {}_tV^{(0)}\right)$$
$$\frac{d}{dt}{}_tV^{(1)} = \delta\, {}_tV^{(1)} - B - \mu_{50+t}^{12}\left(S - {}_tV^{(1)}\right) - \mu_{50+t}^{10}\left({}_tV^{(0)} - {}_tV^{(1)}\right)$$

(c)

$${}_{10}V^{(0)} = S\bar{A}_{60}^{02} + B\bar{a}_{60}^{01} - P\bar{a}_{60}^{00} = 87,871$$

(d)

$${}_{10}V^{(1)} = S\bar{A}_{60}^{12} + B\bar{a}_{60}^{11} - P\bar{a}_{60}^{10} = 450,156$$

(e) The colleague is considering the policy retrospectively. The net premium reserves are prospective, which means that the net premium reserve is the EPV at $t = 10$ of future benefits minus future premiums. As the benefits are more expensive in the sick state ($\bar{a}_x^{11} > \bar{a}_x^{01}$, and $\bar{A}_x^{12} > \bar{A}_x^{02}$), and there is less value for future premiums (as they are only paid in the healthy state) then it is apparent that the net premium reserve in the sick state must be greater than the net premium reserve in the healthy state.

This problem tests Learning Objectives 3 and 4.

15. (a) Let π be the premium paid at the beginning of the second year. The account value at the end of year 1 is

$$\text{AV}_1 = [3,500(1-0.04) - 60 - 1.30q_{45}(1/1.05)(15,000)]\,(1.06) = 3,419.26$$

where we plug $q_{45} = 4.00/1,000$. The account value, given to be 6,528.75, at the end of year 2 is

$$\text{AV}_2 = [3,419.26 + \pi(1-0.04) - 60 - 1.30q_{46}(1/1.05)(15,000)]\,(1.06)$$

where $q_{46} = 4.31/1,000$. Solving then for π, we get

$$\begin{aligned}\pi &= \frac{(6,528.75/1.06) - 3,419.26 + 60 + 1.30(4.31/1,000)(1/1.05)(15,000)}{0.96} \\ &= \frac{2,879.984}{0.96} = 2,999.983 \approx 3,000\end{aligned}$$

(b) We solve for the largest π so that $(\text{AV}_2 + 15,000)/\text{AV}_2 \geq 2.5$, or equivalently, $\text{AV}_2 \leq 10,000$. We thus have

$$\begin{aligned}\pi &\leq \frac{(10,000/1.06) - 3419.26 + 60 + 1.30(4.31/1,000)(1/1.05)(15,000)}{0.96} \\ &= \frac{6,154.748}{0.96} = 6,411.20\end{aligned}$$

(c) The policy must have a large enough death benefit component to qualify as a life insurance contract. For UL contracts, this is tested using the corridor factor, defined by the ratio of total death benefit to account value at death.

This problem tests Learning Objectives 3 and 4.

16. (a) (i) The profit test table is as follows. Row headers have same meaning as in AMLCR. The profit vector is the final column, Pr.

t	${}_{t-1}V$	P	E	I	EDB	E_tV	Pr_t
0			80			50.00	-130.00
1	50	660	14	48.72	592	49.70	103.02
2	50	660	14	48.72	642	49.68	53.04
3	50	660	14	48.72	697	49.65	-1.93
4	50	660	14	48.72	758	0.00	-13.28

The values in the I column are calculated as $I_t = ({}_{t-1}V + P_t - E_t)(i) = (50 + 660 - 14)(0.07) = 48.72$.

$EDB_t = 100,000q_{50+t-1}$ so that $EDB_1 = 100,000(0.00592) = 592, EDB_2 = 100,000(0.00642) = 642, EDB_3 = 100,000(0.00697) = 697,$ and $EDB_4 = 100,000(0.00758) = 758$.

$E_tV = {}_tVp_{50+t-1}$ so that $E_1V = 50(1 - 0.00592) = 49.70, E_2V = 50(1 - 0.00642) = 49.68, E_3V = 50(1 - 0.00697) = 49.65,$ and $E_4V = 0(1 - 0.00758) = 0$.

Then $Pr_t = {}_{t-1}V + P - E + I - EDB - E_tV$ so that

$$Pr_0 = -80 - 50 = -130.00$$
$$Pr_1 = 50 + 660 - 14 + 48.72 - 592 - 49.70 = 103.02$$
$$Pr_2 = 50 + 660 - 14 + 48.72 - 642 - 49.68 = 53.04$$
$$Pr_3 = 50 + 660 - 14 + 48.72 - 697 - 49.65 = -1.93$$
$$Pr_4 = 50 + 660 - 14 + 48.72 - 758 = -13.28$$

(ii) The profit signature is

$$(-130, 103.02, 53.04p_{50}, -1.93\,{}_2p_{50}, -13.28\,{}_3p_{50})'$$

$$= (-130, 103.02, 52.73, -1.91, -13.03)'$$

(iii) The NPV is

$$-130 + 103.02v_{10\%} + 52.73v^2_{10\%} - 1.91v^3_{10\%} - 13.03v^4_{10\%} = -3.10$$

(b) Start from final year; ${}_tV^Z$ is the greater of 0 and the amount needed to give

$Pr_{t+1} = 0$.

$$\left({}_3V^Z + 660 - 14\right)(1.07) - 758 = 0 \Rightarrow {}_3V^Z = 62.41$$
$$\left({}_2V^Z + 660 - 14\right)(1.07) - 697 - p_{52}\,{}_3V^Z = 0 \Rightarrow {}_2V^Z = 63.32$$
$$\left({}_1V^Z + 660 - 14\right)(1.07) - 642 - p_{51}\,{}_2V^Z = 0 \Rightarrow {}_1V^Z = 12.80$$
$$\left({}_0V^Z + 660 - 14\right)(1.07) - 592 - p_{50}\,{}_1V^Z = 0 \Rightarrow {}_0V^Z = -80.84,$$

but since the calculated value of ${}_0V^Z$ is negative, we instead set it to 0.

(c) Initial reserves are lower; we expect the NPV would increase as surplus emerges faster. Note that interest rate assumed (7%) is less than risk discount rate (10%) so releasing surplus sooner increases NPV.

This problem tests Learning Objective 4.

17. (a) The expense reserve at t, for a policy in force, is

$$_tV^E = eP\bar{a}_{x+t} - (P - P^N)\bar{a}_{x+t}$$

where P is the gross premium, P^N is the benefit premium, so $P - P^N$ is the expense loading in the gross premium.

Now

$$P = \frac{S\bar{A}_x}{(1-e)\bar{a}_x} \quad \text{and} \quad P^N = \frac{S\bar{A}_x}{\bar{a}_x}$$

$$\Rightarrow P = P^N \frac{1}{1-e}$$

$$\Rightarrow P - P^N = eP$$

$$\Rightarrow {}_tV = 0 \quad \text{for all } t.$$

(b) The expense reserve is the EPV of future expenses minus the EPV of future expense loadings, that is, $P - P^N$. As there are initial expenses as well as renewal expenses allowed for in the gross premium, it will be greater than eP, but the future expenses are still $eP\,\bar{a}_{x+t}$ at t. Hence, the expense reserve will be negative.

This problem tests Learning Objective 4.

18. (a) $AV_t = (AV_{t-1} + P(1-e) - COI)(1+i)$

$$AV_1 = (3,000\,(0.96) - 10,000(0.022)v_{3\%})\,(1.05) = (2,880.0 - 213.59)\,(1.05) = 2,799.73$$
$$AV_2 = (2,799.73 + 3,000(0.99) - 213.59)\,(1.05) = 5,833.95$$
$$AV_3 = (5,833.95 + 3,000(0.99) - 213.59)\,(1.05) = 9,019.87$$
$$AV_4 = (9,019.87 + 3,000(0.99) - 213.59)\,(1.05) = 12,365.10 \approx 12,365$$

(b) Profit test table: (note, numbers rounded for presentation)

t	AV_{t-1}	P	Exp	Int	EDB	ESB	EAV_t	Pr_t	Π_t	NPV(t)
0	–	–	200	–	–	–	–	-200	-200	-200
1	0	3,000	50	177	256	–	2,744	127.27	127.27	-82.16
2	2,800	3,000	50	345	317	–	5,717	60.76	59.54	-31.11
3	5,834	3,000	50	527	380	–	8,839	91.12	87.51	38.35
4	9,020	3,000	50	718	447	12,118	–	122.97	115.74	123.43

Int_t is calculated as $(AV_{t-1} + P_t - Exp_t)(0.06)$ so that

$$Int_1 = (3000 - 50)(0.06) = 177.00$$
$$Int_2 = (2799.73 + 3000 - 50)(0.06) = 344.98$$
$$Int_3 = (5833.95 + 3000 - 50)(0.06) = 527.04$$
$$Int_4 = (9020 + 3000 - 50)(0.06) = 718.20$$

$$EDB_t = (10,000 + AV_t)(0.02)$$
$$EDB_1 = (10,000 + 2,799.73)(0.02) = 255.99$$
$$EDB_2 = (10,000 + 5,833.95)(0.02) = 316.68$$
$$EDB_3 = (10,000 + 9,019.87)(0.02) = 380.40$$
$$EDB_4 = (10,000 + 12,365.10)(0.02) = 447.30$$

The only surrenders are in year 4; in this year, everyone who lives to the end of the year surrenders, so $ESB_4 = AV_4(1 - 0.02) = 12,117.80$.

For years 1 - 3, $EAV_t = AV_t(1-q)$:

$$EAV_1 = 2,799.73(0.98) = 2,743.74$$
$$EAV_2 = 5,833.95(0.98) = 5,717.27$$
$$EAV_3 = 9,019.87(0.98) = 8,839.47$$

$Pr_t = AV_{t-1} + P_t - Exp_t + Int_t - EDB_t - ESB_t - EAV_t$
$Pr_0 = -200$
$Pr_1 = 3,000 - 50 + 177.00 - 255.99 - 2,743.74 = 127.27$
$Pr_2 = 2,799.73 + 3,000 - 50 + 344.98 - 316.68 - 5,717.27 = 60.76$
$Pr_3 = 5,833.95 + 3,000 - 50 + 527.04 - 380.40 - 8,839.47 = 91.12$
$Pr_4 = 9,019.87 + 3,000 - 50 + 718.20 - 447.30 - 12,117.80 = 122.97$

The profit signature Π is

$(-200, 127.27, 60.76(0.98), (91.12)(0.98)^2, (122.97)(0.98)^3)'$

$= (-200, 127.27, 59.54, 87.51, 115.74)'$

$NPV(t) = \sum_{k=0}^{t} \Pi_k \, v_{8\%}^k$ for $t = 0, 1, 2, 3, 4$.

EPV of premiums is

$3,000(1 + 0.98v_{8\%} + (0.98v_{8\%})^2 + (0.98v_{8\%})^3) = 10,433.84$

Hence, the profit margin is

$$\frac{NPV}{\text{EPV Prems}} = \frac{123.43}{10,433.84} = 1.18\%$$

(c) Now there is no uncertainty with respect to survival.
The NPV is

$$\begin{aligned}
\text{PV initial expenses} &= -200 \\
\text{PV profit in first year } ((3,000 - 50)(1.06) - 2,799.73) &= 327.27 \\
\text{PV profit in second year } ((2,799.73 + 3,000 - 50)(1.06) - 0.9(5,833.95)) &= 844.16 \\
\Rightarrow \quad NPV &= -200 + 327.27v_{8\%} + 844.16v_{8\%}^2 = 826.76
\end{aligned}$$

This problem tests Learning Objective 4.

19. (a) (i) $\frac{d}{dt}{}_tV = \delta\, {}_tV + P - 50 - \mu^m_{50+t}(S - {}_tV) + \mu^l\, {}_tV$ where $\delta = 0.04,\ P = 300,$ $\mu^m_y = 10^{-5} \times 1.1^y,\ S = 100,000,\ \mu^l = 0.05.$

(ii) At the termination of the contract, there is no reserve required, so a boundary condition is ${}_{10}V = 0.0$. *(Note that the initial reserve may not be zero as we are not told that the premium is an equivalence principle premium.)*

(b) Apply Euler's backward method from ${}_{10}V$ with $h = 0.2$, given ${}_{10}V = 0$:

$$\frac{{}_{t+h}V - {}_tV}{h} \approx \delta_{t+h}V + P - 50 + \mu^m_{50+t+h}(S - {}_{t+h}V) + 0.05_{t+h}V$$
$$\Rightarrow {}_tV \approx {}_{t+h}V - h\left(\delta\, {}_{t+h}V + 0.05_{t+h}V + \mu^m_{50+t+h}\ {}_{t+h}V + P - 50 - \mu^m_{50+t+h}\, S\right)$$

So

$$\begin{aligned}{}_{9.8}V &\approx {}_{10}V - 0.2\,(\delta\, {}_{10}V + 0.05_{10}V + \mu^m_{60}\ {}_{10}V + P - 50 - \mu^m_{60}\, S) \\ &\approx 10.90 \\ {}_{9.6}V &\approx {}_{9.8}V - 0.2\,(\delta\, {}_{9.8}V + 0.05_{9.8}V + \mu^m_{59.8}\ {}_{9.8}V + P - 50 - \mu^m_{59.8}\, S) \\ {}_{9.6}V &\approx 20.44\end{aligned}$$

This problem tests Learning Objective 4.

20. (a) (i) $q_{64}^{(1)}$ is the probability that a CEO age 64 terminates employment by retiring (or decrements due to decrement 1) within one year.

$q_{64}^{(1)} = d_{64}^{(1)}/l_{64}^{(\tau)} = 20,000/89,200 = 0.22422$

(ii) ${}_2p_{63}^{(\tau)}$ is the probability that a CEO age 63 does not terminate employment by retirement or death (or does not decrement due to decrement 1 or decrement 2) in the next two years.

${}_2p_{63}^{(\tau)} = l_{65}^{(\tau)}/l_{63}^{(\tau)} = 68,250/100,000 = 0.6825$

(b) $EPV = 250,000(q_{63}^{(1)} * (1/1.05) + p_{63}^{(\tau)} q_{64}^{(1)} * (1/1.05)^2 + {}_2p_{63}^{(\tau)} q_{65}^{(1)} * (1/1.05)^3)$
= 213,962

(c) The probability the Company is Distressed in two years is (0.2)(0.3) + (0.8)(0.8) = 0.06 + 0.64 = 0.70

(d) (i) The death benefit would be 500,000*0.85*1.25=531,250.

(ii) Assuming the CEO leaves between ages 64 and 65, the probability of decrementing due to retirement is $d_{64}^{(1)}/d_{64}^{(\tau)} = 20,000/20,950 = 0.954654$. The probability of decrementing due to death is $d_{64}^{(2)}/d_{64}^{(\tau)} = 950/20,950 = 0.045346$.

There are 4 possible paths if retirement occurs between ages 64 and 65. The possible states and associated probabilities are
$D \to H \to H : (0.2 * 0.7) = 0.14$
$D \to H \to D : (0.2 * 0.3) = 0.06$
$D \to D \to H : (0.8 * 0.2) = 0.16$
$D \to D \to D : (0.8 * 0.8) = 0.64$

The retirement benefits associated with these are
$D \to H \to H : 250,000 * 1.25 * 1.25 = 390,625$
$D \to H \to D : 250,000 * 1.25 * 0.85 = 265,625$
$D \to D \to H : 250,000 * 0.85 * 1.25 = 265,625$
$D \to D \to D : 250,000 * 0.85 * 0.85 = 180,625$

The death benefits associated with these are
$D \to H \to H : 500,000 * 1.25 * 1.25 = 781,250$
$D \to H \to D : 500,000 * 1.25 * 0.85 = 531,250$
$D \to D \to H : 500,000 * 0.85 * 1.25 = 531,250$
$D \to D \to D : 500,000 * 0.85 * 0.85 = 361,250$

The probability that the benefit exceeds 370,000 is 0.954654 * 0.14 + 0.045346 * (0.14 + 0.06 + 0.16) = 0.15

This problem tests Learning Objectives 1 and 5.

21. (a) FS = final salary; YOS = years of service; ERF = early retirement factor

$$\begin{aligned}\text{Benefit} &= 0.017 * YOS * (FS + (FS/1.04) + (FS/1.04^2))/3 * ERF \\ &= 0.017 * YOS * FS * (1 + 1/1.04 + 1/1.04^2)/3 * ERF \\ &= 0.01636 * YOS * ERF * FS\end{aligned}$$

At age 63, YOS=18, ERF=0.90, Benefit = (0.01636)(18)(0.90)FS = 0.265FS. Since the Benefit exceeds 0.25FS, the replacement ratio exceeds 0.25.
At age 62, YOS=17, ERF=0.85, Benefit = (0.01636)(17)(0.85)FS = 0.236FS, so 63 is the first age with a replacement ratio of at least 0.25.

(b) (i) $\ddot{a}_{65} = 9.8969$
The lump sum value of the benefit = $\ddot{a}_{65} * 24,000 = 237,526$
The life with ten years guaranteed annuity factor = $\ddot{a}_{\overline{10}|} + ({}_{10}E_{65})\ddot{a}_{75}$ = 7.8017 + 0.39994 * 7.2170 = 10.6881
$X = 237,526/10.6881 = 22,223.$

(ii) $\ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{xy} = 12.2758 - 8.8966 = 3.3792$
The 50% joint and survivor annuity factor = 0.50 * 3.3792 + 9.8969 = 11.5865
$Y = 237,526/11.5865 = 20,500$

(c) Both $\ddot{a}_{65}$ and $\ddot{a}_{\overline{65:\overline{10}|}}$ would increase as i decreased. Because the duration of $\ddot{a}_{65}$ is longer, it would increase proportionately more. X would increase to keep the options actuarially equivalent.

(d) The payments, undiscounted, under this new option would be the same as the payments for the life annuity with 10 years guaranteed. The payments would never be later than under the life annuity with 10 years guaranteed, and some would be earlier if Lauren died within 10 years. The actuarial present value per dollar of benefit would be higher under the new option, so $Z < X$.

This problem tests Learning Objective 5.

22. (a) Nancy has a Defined Benefit plan. A Defined Contribution pension plan specifies how much an employer will contribute, as a percentage of salary, into a plan. The employee may also contribute. The contributions are accumulated in a notional account which is available to the employee when he or she leaves the company. The contributions may be set to meet a target benefit level, but the actual retirement income may be well below or above the target, depending on investment experience.

To complete Part (b) and (c), we note that Nancy will retire at age 60 or 61. Therefore, we will need to know the accrued benefit at both 60 and 61. We will also need to know the monthly annuity values at age 60 and 61. Using the 2-term Woolhouse approximation we have

$$\ddot{a}_{60}^{(12)} = \ddot{a}_{60} - \frac{11}{24} = 10.6871 \text{ and } \ddot{a}_{61}^{(12)} = \ddot{a}_{61} - \frac{11}{24} = 10.4458$$

(b) (i) Under the Projected Unit Credit cost method, the actuarial accrued liability is the actuarial present value of the accrued benefit. The accrued benefit is equal to the projected benefit at the decrement date multiplied by service as of the valuation date and by the accrual rate.

We have the following information:

Projected Final Average Salary at 60	$50,000\left[(1.03)^3 + (1.03)^4 + (1.03)^5\right]/3$ $= 56,292$
Projected Final Average Salary at 61	$50,000\left[(1.03)^4 + (1.03)^5 + (1.03)^6\right]/3$ $= 56,292(1.03) = 57,981$
Service at valuation date	25
Accrual Rate	0.016
Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(25) = 22,517
Projected Benefit for retirement at 61 (PB_{61})	(57,981)(0.016)(25) = 23,192

The actuarial accrued liability is the actuarial present value (as of the valuation date) of the projected benefit and is given by

$$
\begin{aligned}
AAL_{55} &= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot q_{60}^{(r)} \cdot {}_{60-55}p_{55}^{(\tau)} \cdot v^5 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot q_{61}^{(r)} \cdot {}_{61-55}p_{55}^{(\tau)} \cdot v^6 \\
&= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{60}^{(\tau)}} \cdot \frac{l_{60}^{(\tau)}}{l_{55}^{(\tau)}} \cdot v^5 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot 1 \cdot \frac{l_{61}^{(\tau)}}{l_{55}^{(\tau)}} \cdot v^6 \\
&= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{55}^{(\tau)}} \cdot v^5 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{l_{61}^{(\tau)}}{l_{55}^{(\tau)}} \cdot v^6 \\
&= (22,517)(10.6871)\left(\frac{3552}{27,006}\right)(1.06)^{-5} + (23,192)(10.4458)\left(\frac{19,991}{27,006}\right)(1.06)^{-6} \\
&= 150,072
\end{aligned}
$$

(ii) We now need the accrued benefit at age 56:

Projected Benefit for retirement at 60 (PB_{60})	(56,292)(0.016)(26) = 23,417
Projected Benefit for retirement at 61 (PB_{61})	(57,981)(0.016)(26) = 24,120

The actuarial accrued liability at December 31, 2016 is given by

$$
\begin{aligned}
AAL_{56} &= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot q_{60}^{(r)} \cdot {}_{60-56}p_{56}^{(\tau)} \cdot v^4 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot q_{61}^{(r)} \cdot {}_{61-56}p_{56}^{(\tau)} \cdot v^5 \\
&= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{60}^{(\tau)}} \cdot \frac{l_{60}^{(\tau)}}{l_{56}^{(\tau)}} \cdot v^4 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot 1 \cdot \frac{l_{61}^{(\tau)}}{l_{56}^{(\tau)}} \cdot v^5 \\
&= PB_{60} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{56}^{(\tau)}} \cdot v^4 + PB_{61} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{l_{61}^{(\tau)}}{l_{56}^{(\tau)}} \cdot v^5 \\
&= (23,417)(10.6871)\left(\frac{3552}{26,396}\right)(1.06)^{-4} + (24,120)(10.4458)\left(\frac{19,991}{26,396}\right)(1.06)^{-5} \\
&= 169,282
\end{aligned}
$$

Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$ where:

C_t = Normal Cost for year t to $t+1$ and
${}_tV$ is the Actuarial Accrued Liability at time t

Note that the EPV of benefits for mid-year exit is zero. Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$

$$150,072 + C_t = 0 + (1.06)^{-1}\left(\frac{26,396}{27,006}\right)(169,282)$$

$C_t = 6003$

(c) (i) Under the Traditional Unit Credit cost method the actuarial accrued liability (AAL) is the actuarial present value of the accrued benefit on the valuation date.

The formula for the accrued benefit, B, is

$$B_{55} = 0.016 \cdot 25 \cdot 50,000 \cdot \left(\frac{1 + (1.03)^{-1} + (1.03)^{-2}}{3} \right) = 19,423$$

and thus

$$\begin{aligned} AAL_{55} &= B_{55} \cdot \ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{55}^{(\tau)}} \cdot v^5 + B_{55} \cdot \ddot{a}_{61}^{(12)} \cdot \frac{l_{61}^{(\tau)}}{l_{55}^{(\tau)}} \cdot v^6 \\ &= B_{55} \left(\ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{55}^{(\tau)}} \cdot v^5 + \ddot{a}_{61}^{(12)} \cdot \frac{l_{61}^{(\tau)}}{l_{55}^{(\tau)}} \cdot v^6 \right) \\ &= 19,423 \left[(10.6871) \left(\frac{3552}{27,006} \right) (1.06)^{-5} + (10.4458) \left(\frac{19,991}{27,006} \right) (1.06)^{-6} \right] \\ &= 126,277 \end{aligned}$$

(ii) For the NC, we must calculate the expected accrued benefit, B_{56}, one year after the valuation date.

$$B_{56} = 0.016 \cdot 26 \cdot 50,000 \cdot \left(\frac{1.03 + 1 + (1.03)^{-1}}{3} \right) = 20,806$$

$$\begin{aligned} AAL_{56} &= B_{56} \left(\ddot{a}_{60}^{(12)} \cdot \frac{d_{60}^{(r)}}{l_{56}^{(\tau)}} \cdot v^4 + \ddot{a}_{61}^{(12)} \cdot \frac{l_{61}^{(\tau)}}{l_{56}^{(\tau)}} \cdot v^5 \right) \\ &= 20,806 \left[(10.6871) \left(\frac{3552}{26,396} \right) (1.06)^{-4} + (10.4458) \left(\frac{19,991}{26,396} \right) (1.06)^{-5} \right] \\ &= 146,698 \end{aligned}$$

Then:

${}_tV + C_t = EPV$ of benefits for mid-year exits $+ v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$ where:

C_t = Normal Cost for year t to $t+1$ and
${}_tV$ is the Actuarial Accrued Liability at time t

Note that the EPV of benefits for mid-year exit is zero. Then:

$${}_tV + C_t = EPV \text{ of benefits for mid-year exits} + v \cdot {}_1p_x^{(\tau)} \cdot {}_{t+1}V$$

$$126,277 + C_t = 0 + (1.06)^{-1} \left(\frac{26,396}{27,006} \right) (146,698)$$

$$C_t = 8891$$

(d) Under both funding approaches, the contribution rate tends to increase as the member acquires more service and gets closer to retirement. The TUC contributions start rather smaller than the PUC contributions and, rise more steeply, ending at considerably more than the PUC contribution. This is true because the TUC contributions do not project future salary increases or future service credit while PUC contributions are based on salary with projected future salary increases and assuming future service credits will be earned. Therefore, the TUC contributions in any given year must reflect the full increase in the salary and the additional year of service now reflected in the accrued benefit at the end of the year of service which were not reflected in the accrued benefit at the end of the year of service. The PUC contribution will not need to reflect the additional year of service as it was already reflected in the accrued liability at the beginning of the year.

This problem tests Learning Objective 5.