SOCIETY OF ACTUARIES

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Case Study: Stochastic Immunization

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Case Study:
Stochastic Immunization

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Objectives

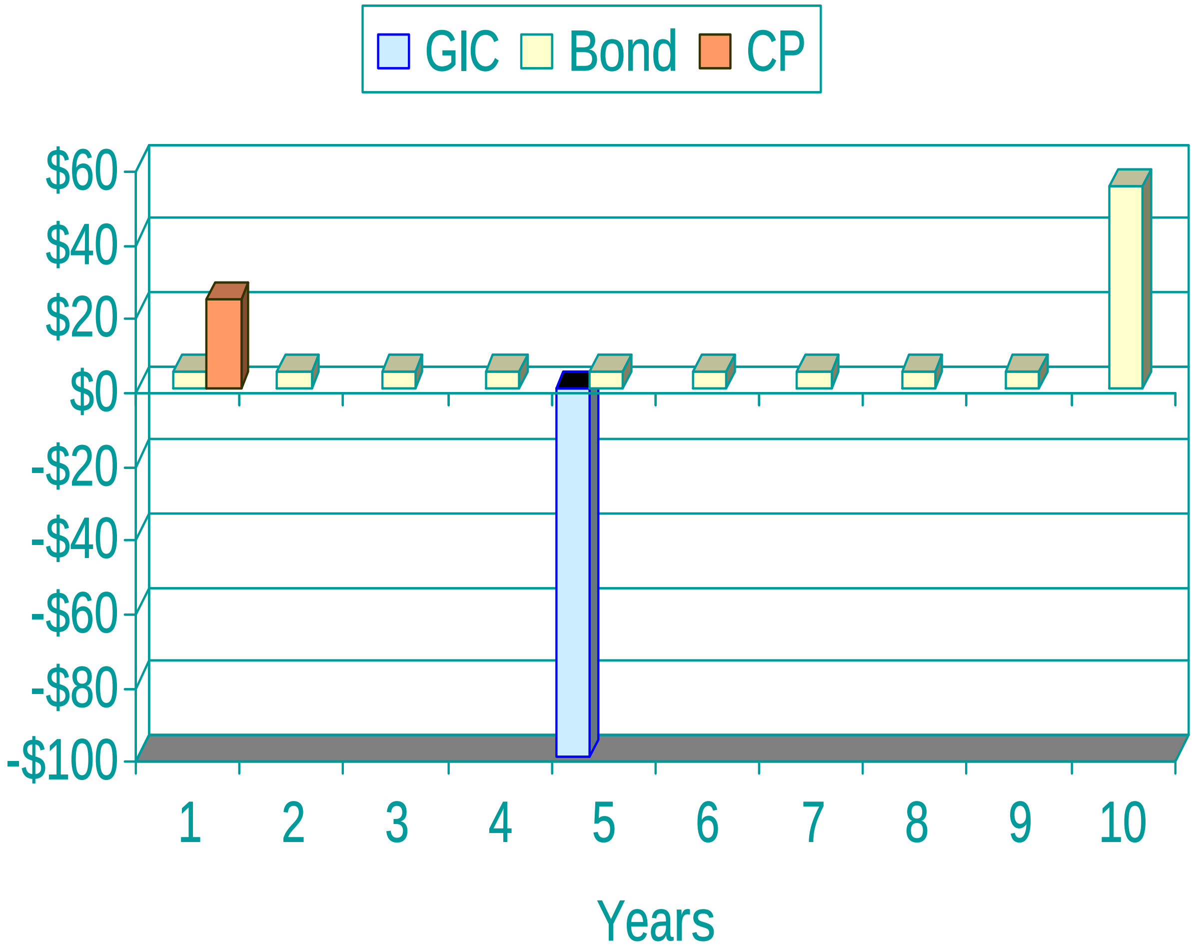
- Quantify potential impact of general yield curve moves
- Review methods to manage interest rate risk
- Discuss practical considerations
- Review Technical Appendix

Sample Portfolio

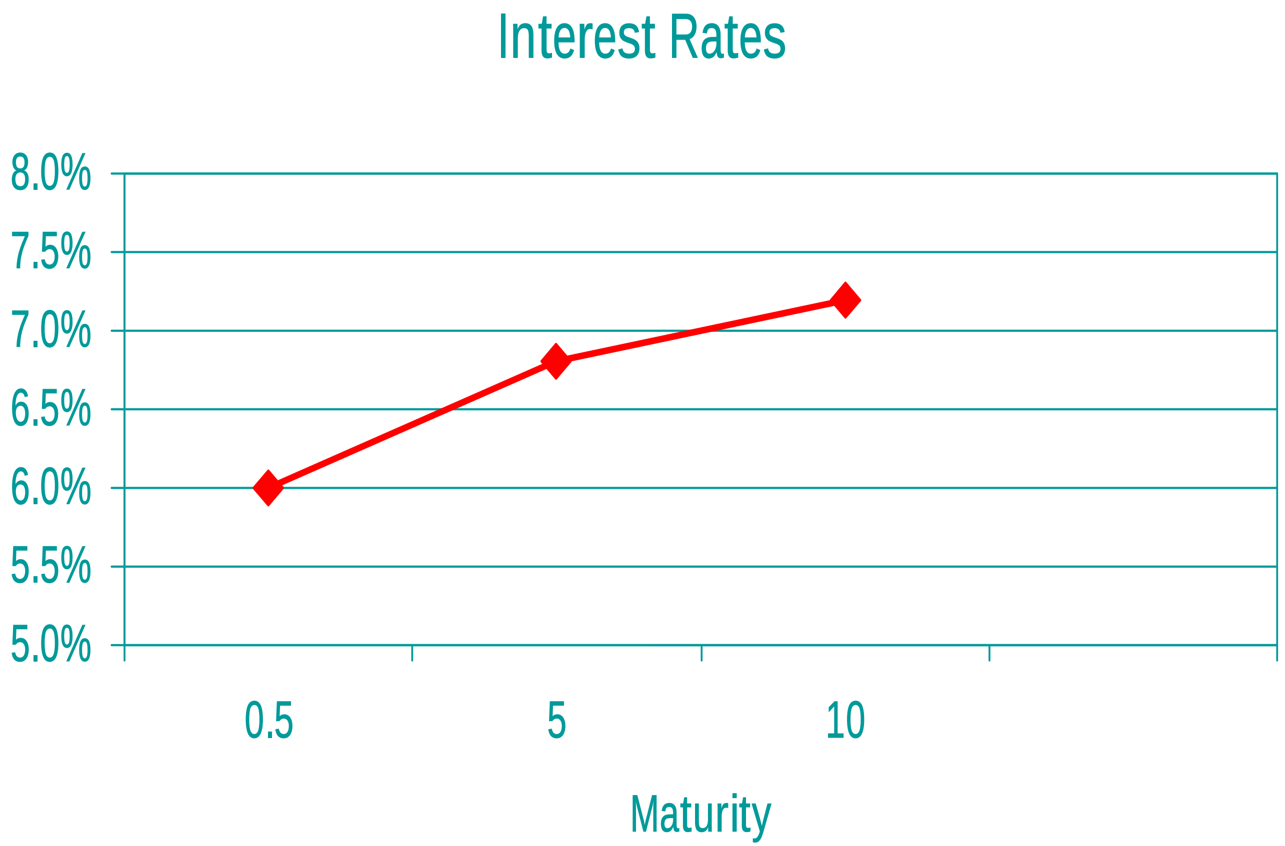
- Assets:
  - $50.0 million, 9%, 10 Year Bond
  - $25.0 million, 6 Month CP

- Liabilities:
  - $100.0 million, 5 Year GIC

Portfolio Cash Flows



Valuation Yield Curve
(.060, .068, .072)



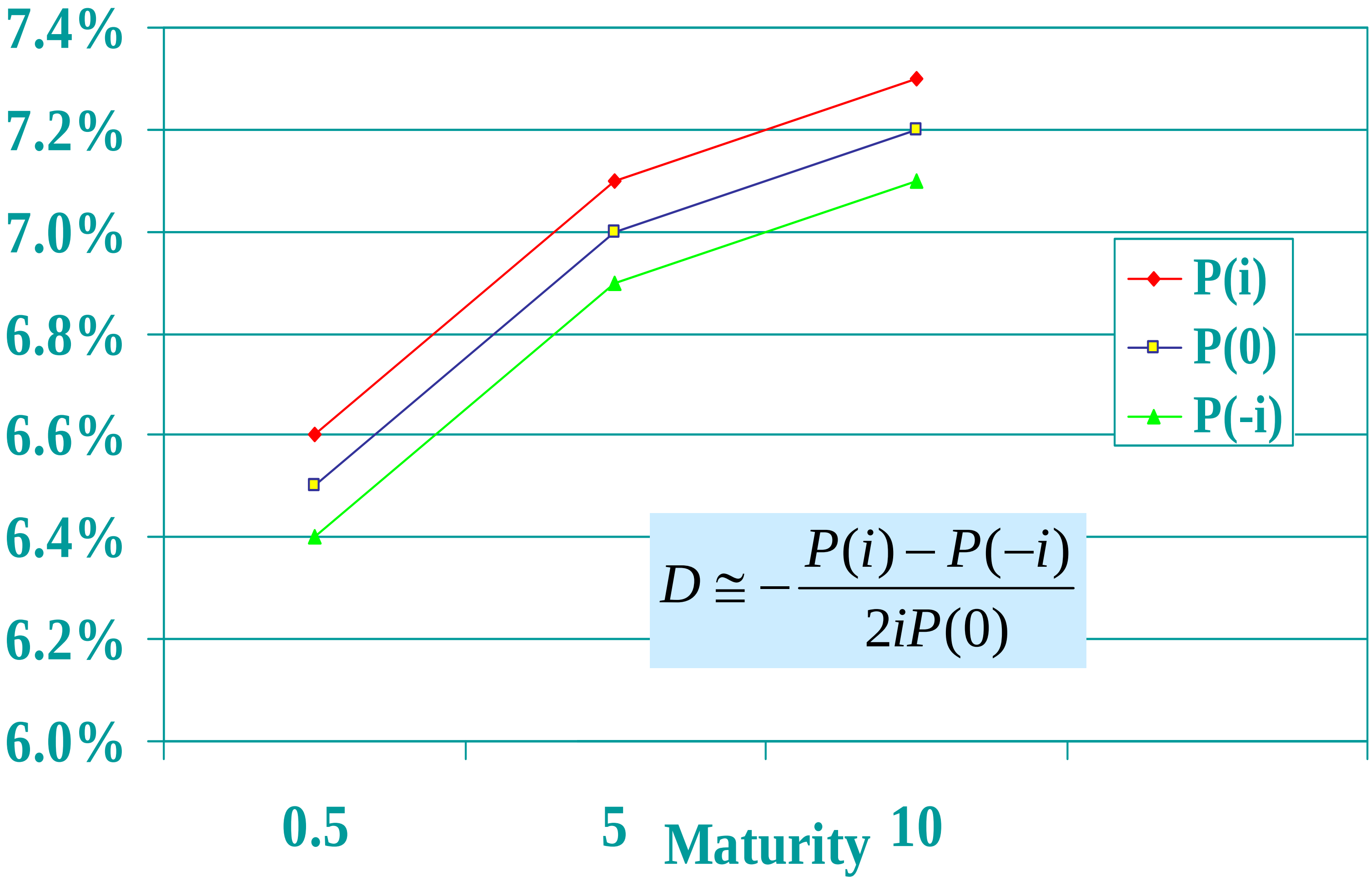
Traditional Price Function Model
(Parallel shift approach)

- Yield curve shift:
  - (.060, .068, .072) ⇨ (.060 + i, .068 + i, .072 + i)

- Price function:
  - P(i)

- Modified duration:
  - D = - P'(0) / P(0)

Approximating Duration



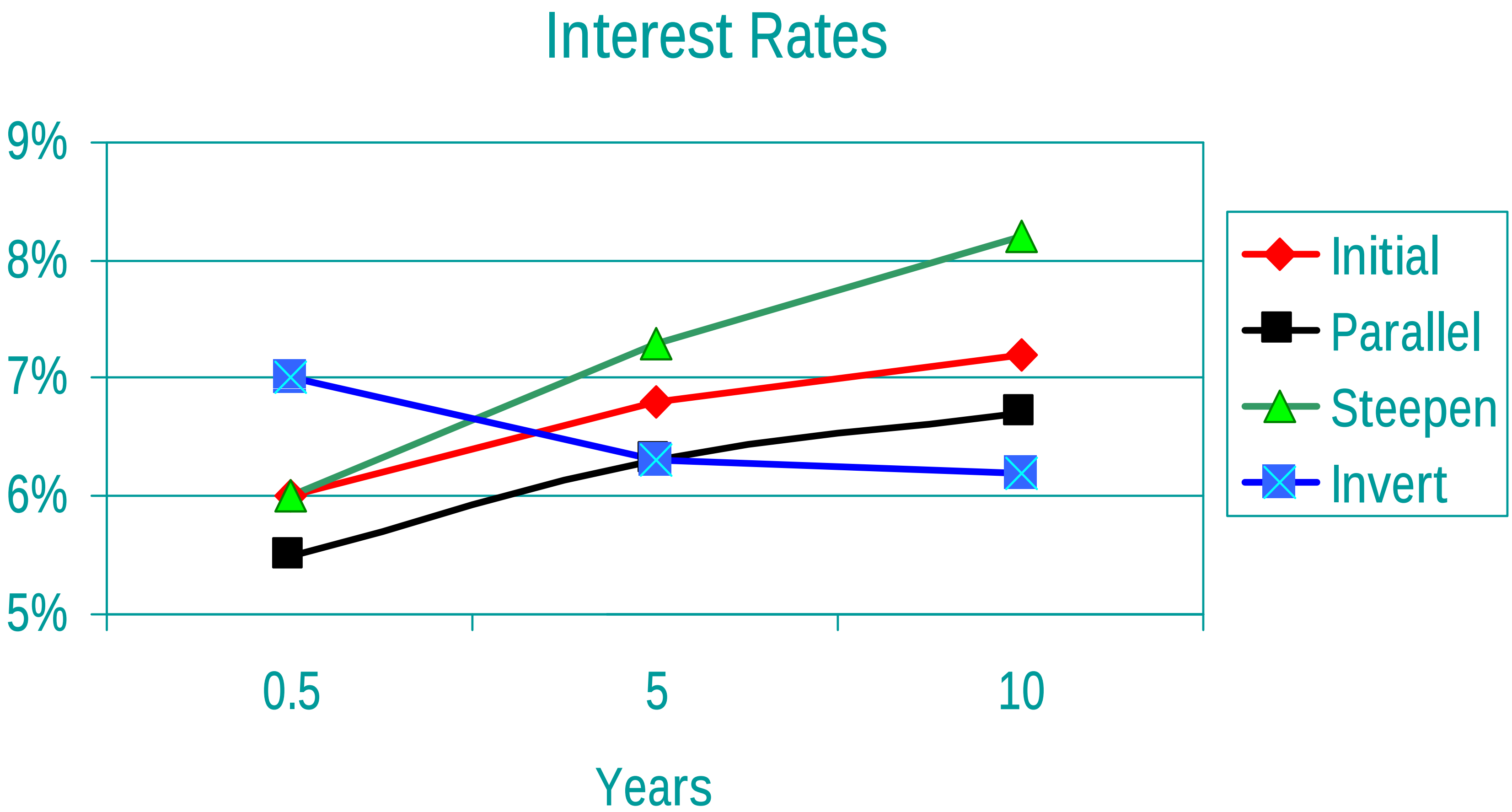
Portfolio Valuation
(On Initial curve using 10 bp shocks)

	MV	Dur
Assets	$80.69	4.93
Liabilities	71.39	4.88
Surplus	$9.29	5.31
Surplus Ratio		11.5%
Surplus Ratio Duration		.05

Approximating Market Values

- General:
  - $P(i) \sim P(0)\ (1 - Di)$
- Example:
  - $S(i) \sim 9.29\ (1 - 5.31i)$
  - $R(i) \sim .115\ (1 - .05i)$

Illustrative Yield Curves



Surplus Ratio Volatility

	Initial	Parallel	Steepen	Invert
Assets	$80.69	82.72	77.03	84.42
Liabs	-71.39	-73.16	-69.55	-73.49
Surplus	$9.29	9.56	7.48	10.94
Surplus Ratio	11.5%	11.6%	9.7%	13.0%

Non-Parallel Shift Problems

- Surplus Ratio is not immunized
- Dollar surplus is more volatile than duration implies
- Conclusions:
  - Predictive ability of duration is limited
  - Need a more general duration model
- Questions:
  - How to model general risk?
  - How to manage general risk?

General Price Function Model
(Non-parallel shift approach)

- Yield Curve Shift:

  (.060, .068, .072) ⇨ (.060+i, .068+j, .072+k)

- Price Function:

  P(i,j,k)

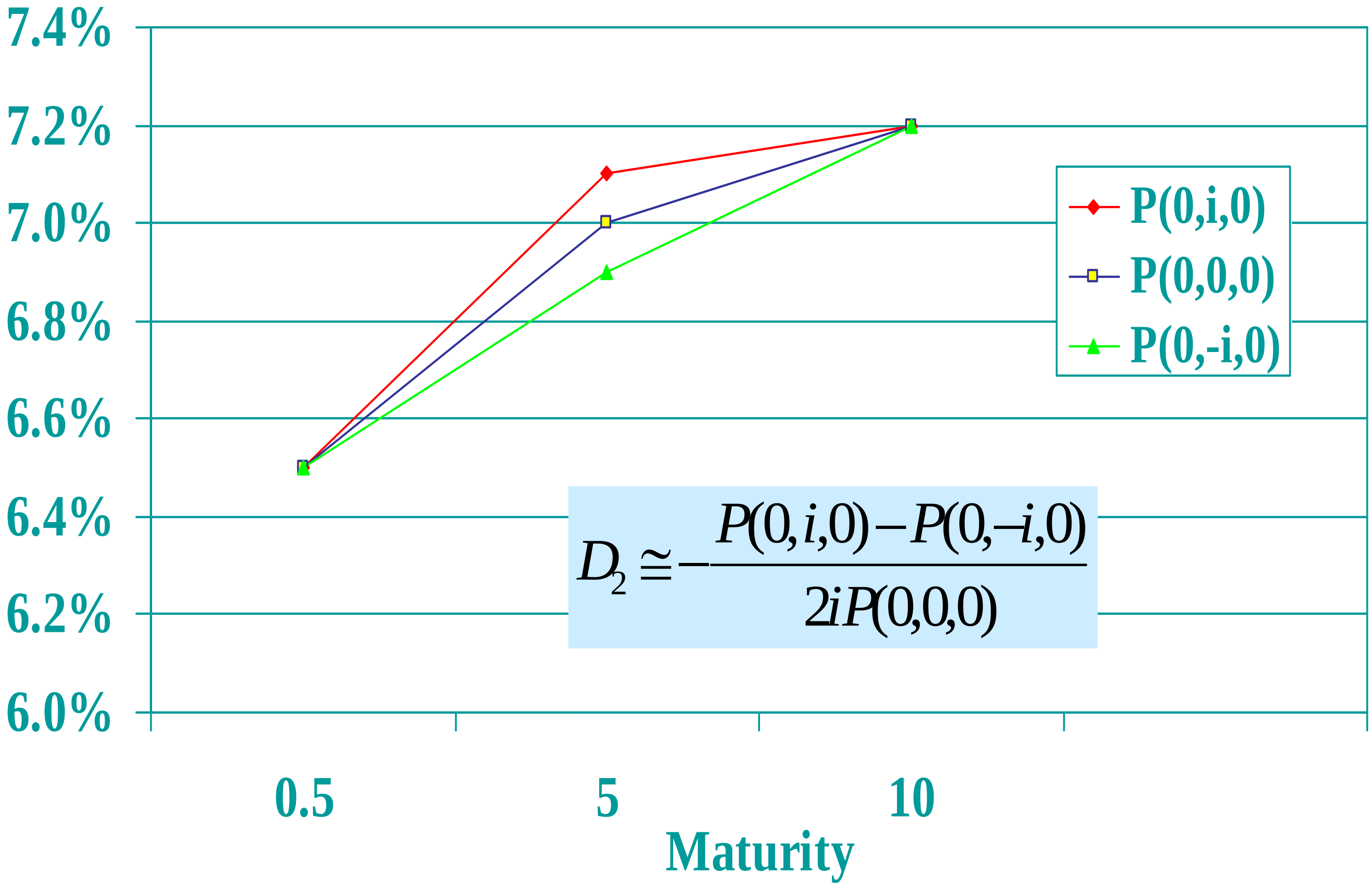
- Partial Durations (PDs):

  $D_1 = D_{0.5} = -P_1(0,0,0) / P(0,0,0)$

  $D_2 = D_{5.0} = -P_2(0,0,0) / P(0,0,0)$

  $D_3 = D_{10.0} = -P_3(0,0,0) / P(0,0,0)$

Approximating Partial Durations - Example



Portfolio Partial Durations
(Initial curve using 5 bp shocks)

	Assets	Liabs	Surplus
D_1	0.53	-0.33	4.03
D_2	0.23	5.20	-38.56
D_3	6.56	0.00	39.84
Total	4.93	4.88	5.31

Approximating Market Values
(Non-parallel shift approach)

- General:
  - $P(i,j,k) \sim P(0,0,0)\,(1 - D_1 i - D_2 j - D_3 k)$
- Example:
  - $S(i,j,k) \sim 9.29\,(1 - 4.03i + 38.56j - 39.84k)$
  - Let (i,j,k) = (-2 bp, 19 bp, -20 bp)

    ⇒ (.0598, .0699, .0700)

  - MV Approx. = $10.72 million, +15.39%
  - MV Actual = $10.72 million, **+15.39%**

Equivalent Parallel Shift

- Which Parallel Shift, **i**^{E}, is durationally equivalent to (i, j, k)?

- General:

$$i^{E} = \frac{D_1 i + D_2 j + D_3 k}{D}$$

- Example: **i**^{E} = .76i - 7.26j + 7.50k

  = -290 bp

- How large can **i**^{E} become?

When is i^E relatively extreme?

- When $(i,j,k) \sim (D_1,D_2,D_3)$

- For example:

$(-.0002, .0019, -.0020) =$ **$-.00005$** $\times (4.03, -38.56, 39.84)$

- How likely is this to occur?

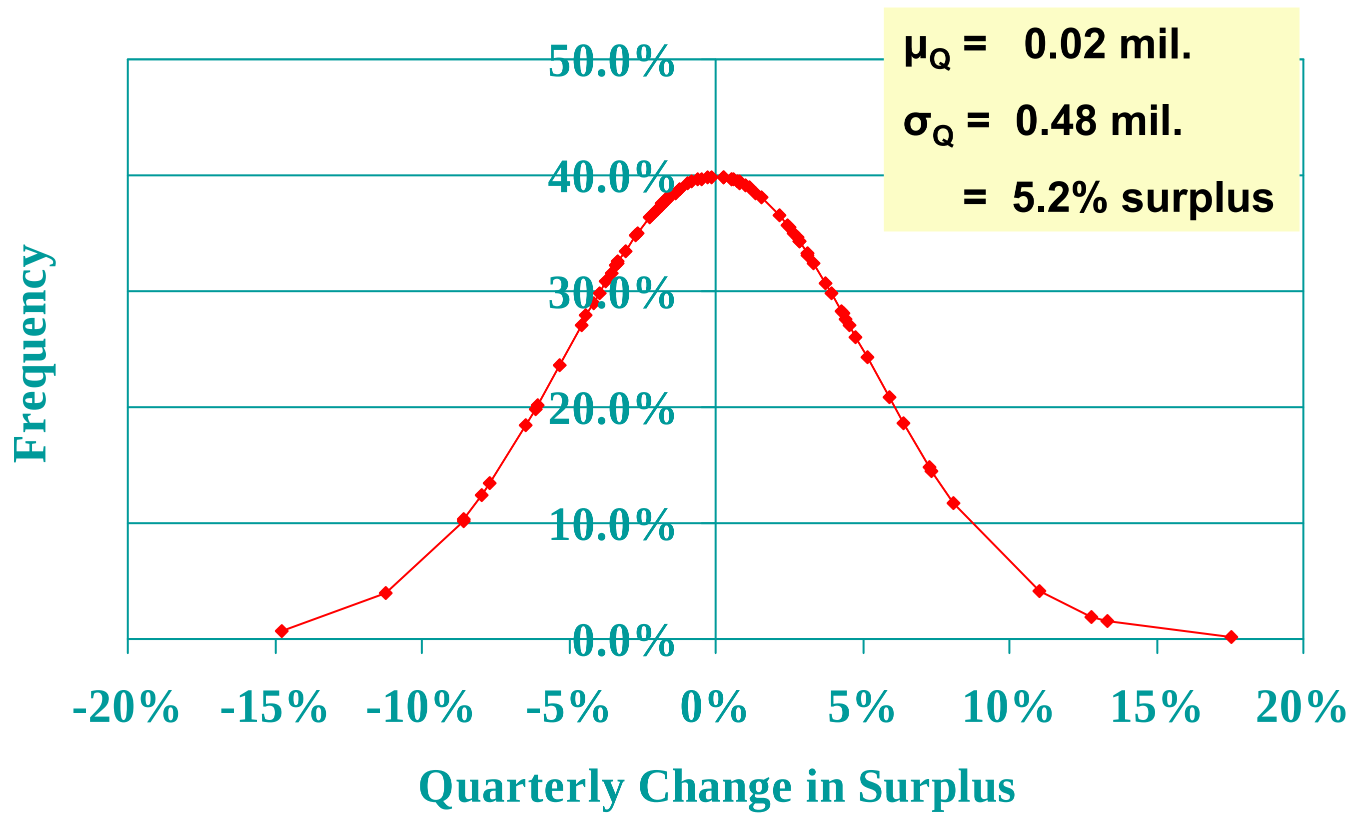
Generate Distribution of Surplus

- Historical monthly Treasury rates
- Calculate quarterly rate changes
  - Or changes over other periods of interest
- Estimate change in surplus:

$$\Delta S \cong -S(0)\sum_j D_j \Delta i_j$$

- Graph distribution of results

Actual Distribution of Surplus Changes
(Treasury rates - 3/77 to 4/98)



"Alternative" Standard Deviation Calculation

- Create Treasury covariance matrix ("**K**")
  - Based on M, Q, or Annual rate changes
  - Can model absolute or relative rate changes

- Standard deviation:

$$\sigma \sim S(0)\ [\mathbf{D}\ \mathbf{K}\ \mathbf{D}^{\mathbf{T}}]^{0.5}$$

- Here **D** = vector of Partial Durations
- Is the result too high? Manage it!

Stochastic Immunization
(Minimize risk subject to practical constraints)

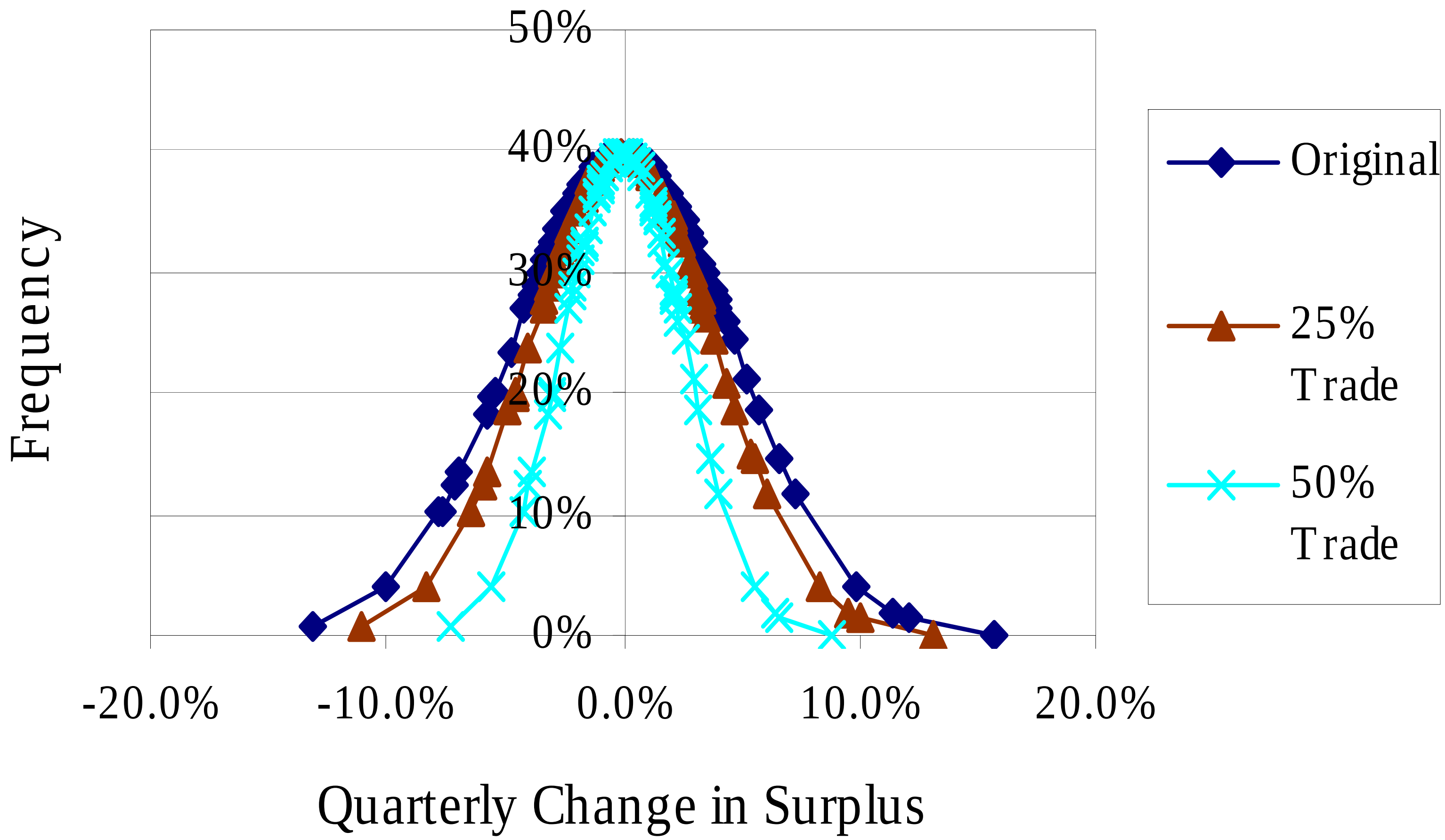
- Define risk tolerance (e.g., 95% confidence)
- Determine variance-minimizing trade(s) using constrained quadratic optimization
- Possible constraints:
  - Trade size (volume)
  - Available assets/derivatives
  - Cash-neutrality in cash markets
  - Duration-type constraints
  - Expected return target

Example: Stochastic Immunization

- Risk minimizing trade with no constraints
- Trading set: (1, 5, & 10 year swaps)
- Optimal swap trade:
  - 1 yr swap = $ +43.5 million
  - 5 yr swap = $ -83.5 million
  - 10 yr swap = $ +48.4 million
- Adjusted surplus partial durations:

(4.03, -38.56, 39.84) ⇨ (0.00, 0.00, 0.00)

Surplus Distribution w/ Min. Swaps
(Treasury rates - 3/77 to 4/98)



Practical Considerations

- Data integrity
- Standardized valuation process(es)
- Counterparty exposure
- FAS 133 effects on financials
- Unrealistic expectations
  - Some assets may be illiquid
  - Time delay in data analysis

Technical Appendix

Immunization Risk

A. Variance

- Relative Risk Function:

  $P(\mathbf{i}+\Delta\mathbf{i})/P(\mathbf{i}) \cong R(\Delta\mathbf{i})$

- $Var[R(\Delta\mathbf{i})]=\mathbf{DKD}^T$

  where:

  $R(\Delta\mathbf{i}) \equiv 1- \mathbf{D}\cdot\Delta\mathbf{i}$

  K = Covariance Matrix

  D = Total Duration Vector

Immunization Risk

B. Outlier Risk

- Using Cauchy-Schwarz:

$$\left|R(\Delta i) - E[R(\Delta i)]\right|^2 \leq \left|D\right|^2 \left|\Delta i - E(\Delta i)\right|^2$$

- Where:

$$\left|D\right|^2 = DID^T$$

$$I \equiv IdentityMatrix$$

Immunization Risk

C. General Risk Measure Model

- Combining Risk Measures:

$$RM(w) = DK_w D^T$$

- Where:

$$K_w \equiv wK + (1-w)I$$

$$0 \le w \le 1$$

Immunization Constraints

- A. Expected Yield Curve Return
  - For: $E[R(\Delta \mathbf{i})] = 1 + r'$
  - Need: $\mathbf{D} \bullet \mathbf{E} = - r'$

    Where: $\mathbf{E} \equiv E(\Delta \mathbf{i})$
- B. Directional Duration Constraints
  - For: $D_N = r_N$
  - Need: $\mathbf{D} \bullet \mathbf{N} = r_N$

    Where: **N** = direction vector(s) of interest

Immunization Constraints

C. Asset Trading Set

- Given: n Assets

  With PD Vectors: $\mathbf{D}_1, \mathbf{D}_2, \ldots.\mathbf{D}_n$

- Constraints for Cash Neutral Trade:

$$\mathbf{D} \bullet \mathbf{N}_k = \mathbf{D(i)} \bullet \mathbf{N}_k \equiv r_k$$

where **D(i)** = initial Total Duration vector

and where $\{\mathbf{N}_k\}$ span Null Space of $\mathbf{A}^T$:

$$A^T \equiv \begin{pmatrix} D_1 - D_n \\ D_2 - D_n \\ \bullet \\ D_{n-1} - D_n \end{pmatrix}$$

Stochastic Immunization

The Problem Statement

- Minimize: $\mathbf{D\,K\,D}^{\mathbf{T}}$
- Subject to: $\mathbf{D \bullet N}_{\mathbf{k}} = r_k$, k=1, …,p
- Constraints In Matrix Form:

$$\mathbf{D\,B} = \mathbf{r}^T$$

- Where: $\mathbf{B} = (\,\mathbf{N}_{\mathbf{1}}, \mathbf{N}_{\mathbf{2}},\ldots,\mathbf{N}_{\mathbf{p}}\,)$

$$\mathbf{r} = (\,r_1, r_2,\ldots,r_p\,)$$

Stochastic Immunization

The Solution

- Total Duration Vector: $\mathbf{D_0^T = K_w^{-1}B\,(B^T K_w^{-1} B)^{-1} r}$
- Risk: $RM_0(w) = \mathbf{D_0 K_w D_0^T}$

  $= \mathbf{r^T (B^T K_w^{-1} B)^{-1} r}$
- Asset Trade: $\mathbf{A\,a'} = P(\mathbf{i})\,[\mathbf{D_0} - \mathbf{D(i)}]^T$,

  $\mathbf{a'} = (a_1, \ldots\ldots a_{n-1})$