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R1-Research Paper Session: A Capital Allocation Based on a Solvency Exchange Option

Mary Hardy
Joseph Kim

Moderator/Reviewer
John Manistre

CAPITAL ALLOCATION METHODS

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Joseph Kim and Mary Hardy
University of Waterloo

Outline

- Economic Capital
- What is capital allocation, and why?
- Examples of allocation methods
- How do we choose between the allocation methods?
  - Fair Allocation Axioms
- A new method

Economic Capital

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- Capital required to ensure balance sheet solvency with some assurance
- Risk measure based eg
  - CTE
  - VaR
- Enterprise wide calculation
- This stage determines policyholder security
  - Lines of business can't independently fail

What is capital allocation?

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- Allocation of economic capital to individual business units
  - For example, lines of business
- Top Down
  - Determine enterprise wide economic capital, then allocate to business units
- Bottom Up
  - Determine stand-alone economic capital for each unit
  - Combine, and estimate diversification benefit

What is capital allocation?

- Consider a multi-line insurer with aggregate loss X from n lines;
- X_i is the loss from i^{th} line of business

$$X = \sum_{i=1}^{n} X_i \qquad u = \rho(X)$$

- u is the enterprise-wide risk capital, ρ is the risk measure

What is capital allocation?

- Capital allocation is a rule to break u down into n parts assigned to individual lines.
- i.e. to determine u_i, $i=1,2,...,n$, where u_i is the allocated capital for line i.
- Formally,

$$u_i = \rho(X_i \mid \Omega), \quad \Omega = \{X_1, X_2, ... X_n\}$$

Why allocate capital?

- Analyzing profitability by line
  - Risk Adjusted Return on Capital
- Pricing
- Setting management incentives and remuneration
- Analysis and strategic planning

Questions

- Top down or bottom up?
- How to account for diversification benefit?
- Can a single allocation method address all objectives?
- P-measure or Q-measure?
- Full or marginal allocation?
- Transfer pricing

Allocation approaches

- **Full Allocation**
  - Set $u=\rho(X)$ and allocate u_i to line i
  - Using an allocation rule

- **Marginal Allocation**

$$u_i = \rho\left(X_i \mid \Omega\right) = \rho\left(X\right) - \rho\left(X_{\setminus i}\right)$$

Allocation approaches

- **Incremental Marginal Allocation**

$$u_i = \rho\left(X_i \mid \Omega\right) = \lim_{\varepsilon \to 0} \frac{\rho(X + \varepsilon X_i) - \rho(X)}{\varepsilon}$$

$$= \left. \frac{\partial \rho(X + hX_i)}{\partial h} \right|_{h=0}$$

RAROC compatible for $0<h<\varepsilon_i$

$\mathrm{RAROC}(X_i|X) > \mathrm{RAROC}(X)$

$\Rightarrow \mathrm{RAROC}(X+hX_i) > \mathrm{RAROC}(X)$

Allocation approaches

- Valuing the insolvency put option
  - Myers and Read
  - Sherris
- We will come back to this...

Some Examples of Allocation Rules

- Using full allocation method
- Top Down
- The total enterprise-wide capital $u=\rho(X)$ decided *ex ante*

Relative Allocation

$$\rho(X_i \mid \Omega) = \frac{\rho(X_i)}{\rho(X_1) + \rho(X_2) + \ldots \rho(X_n)}$$

- $\rho(X_i)$ is the stand-alone risk measure
- Some attractive features, but
- No allowance for diversification benefit of i^{th} line

Covariance Allocation

$$\rho(X_i \mid \Omega) = \frac{Cov(X_i, X)}{V[X]} \times \rho(X)$$

- Diversification accommodated
- What if covariance < 0 ?

CTE Allocation

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- Where ρ(X) is the CTE...

$$\rho\left(X_i \mid \Omega\right) = E\left[X_i \mid X > Q_\alpha(X)\right]$$

- $Q_\alpha(X)$ is the α-quantile of X (not X_i)
- Focus on scenarios which are adverse for the whole enterprise.

Limitations

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- May generate negative capital allocation
- No link to assets; hard to identify loss of i^{th} line
  - We need to integrate capital and asset allocation
- Weak link to management performance
- → Fair allocation axioms

Fair Allocation Axioms

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- Denault (2001); Hesselager and Anderson (2002)
- Cf Coherent Risk Measures
- **Full Allocation**

$$\sum_{i=1}^{n} \rho(X_i \mid \Omega) = \rho\left(\sum_{i=1}^{n} X_i \mid \Omega\right) = \rho(X)$$

  - Total allocated capital equals the total enterprise wide economic capital.

Fair Allocation Axioms

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- **No undercut**
  - For any $H \subset \Omega$

$$\sum_{H} \rho\left(X_i \mid \Omega\right) \leq \rho\left(\sum_{H} X_i \mid H\right)$$

  - The total capital allocated to any set of lines is less than or equal to
  - The capital allocated by treating the lines as an independent business.

Fair Allocation Axioms

- **Symmetry**

Consider X_i, X_j, and

If $\rho(X_i \mid H) = \rho(X_j \mid H) \ \forall \ H \supseteq \{X_i, X_j\}, H \subset \Omega, (H \neq \Omega)$

Then $\rho(X_i \mid \Omega) = \rho(X_j \mid \Omega)$

- If two lines have the same allocated capital for any subset of the firm, they must have the same allocated capital for the whole firm

Fair Allocation Axioms

- **Consistency**
  - For any $H \subset \Omega$, let $Y = \sum_H X_i$
  - We require

$$\rho(Y \mid \Omega) = \rho\left(\sum_H X_i \mid \{Y, \Omega - H\}\right)$$

  - so allocated capital is independent of the structure of the firm

Fair Allocation Axioms

- **Relative Allocation**
  - satisfies full allocation and symmetry
  - fails no-undercut and consistency
- **Covariance Allocation**
  - fails the no-undercut axiom
- **CTE Allocation**
  - satisfies all four axioms

Insolvency-based rules

- The risk measure approach uses the properties of $\rho()$
- The CTE approach focuses on the event that liabilities exceed the α-VaR
- The insolvency aproach focuses on the insolvency event.
  - Needs assets as well as liabilities

Sherris' Solvency Exchange Allocation

- Consider one-period problem.
- L = liability value at year end
- A=asset value at year end
- D=solvency option payoff=max$(L\text{-}A,0)$
- $V(D)$ is valued using option pricing
- Line i liability:

$$L_i - L_i\left(1 - A/L\right)^+ = \begin{cases} L_i & \text{if } A \geq L \\ \dfrac{L_i}{L} A & \text{if } A < L \end{cases}$$

Sherris' Solvency Exchange Allocation

- Let

$$D_i = L_i(1 - A/L)^+$$

  - With value=capital allocation...

$$V(D_i) = e^{-r} E_Q\left[L_i\left(1 - A/L\right)^+ \right]$$

$$D = \sum_i D_i \quad V(D) = \sum_i V(D_i)$$

Sherris' Allocation

- Focus on insolvency is consistent with function of economic capital
- No need to allocate assets, because assets are fungible
  - How to determine line performance?
  - Is Q-measure appropriate?
    - What should be done with the capital allocated?

A new allocation (Kim's)

- P_i is the premium for i^{th} line, $P=P_1+...+P_n$
- $u=\rho(X)$ is the total economic capital at t=0
- A=Assets at t=1
  - $A=Pe^{Rp}+ue^{Ru}$
  - R_p , R_u may be random
- L=liabilities at t=1 $\quad L=\sum_i L_i$

Proposed Allocation Method

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$$\left(L - ue^{Ru} - Pe^{Rp}\right)^+ = \left(L - A\right)^+ = \sum_i D_i$$

$$\Rightarrow E_P\left[\left(L - ue^{Ru} - Pe^{Rp}\right)^+\right] = E_P\left[\sum_i D_i\right]$$

$$= \sum_i E\left[D_i\right] = \sum_i E\left[L_i\left(1 - \frac{A}{L}\right)^+\right]$$

$$= \Pr\left[L > A\right] \sum_i E\left[L_i\left(1 - \frac{A}{L}\right) \middle| L > A\right]$$

Proposed Allocation Method

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- Similarly

$$E\left[\left(L - ue^{Ru} - Pe^{Rp}\right)^+\right]$$

$$= \Pr\left[L > A\right] E\left[\left(L - ue^{Ru} - Pe^{Rp}\right) \middle| L > A\right]$$

- Allocating P and L is straightforward – but u?
- Use the rules of insolvency – equal priority of all lines
- ie $L > A \Rightarrow A_i = \frac{L_i}{L} A$

Proposed Allocation Method

- So, capital allocated to line i is u_i where

$$E\left[L_i - u_i e^{Ru} - P_i e^{Rp} \middle| L > A\right] = E\left[L_i - \frac{L_i}{L} A \middle| L > A\right]$$

- That is, in the insolvency event, the net loss on each line (LHS) equals the payment shortfall (RHS)
- Then

$$u_i = \frac{E\left[\frac{L_i}{L} A - P_i e^{Rp} \middle| L > A\right]}{E\left[e^{Ru} \mid L > A\right]}$$

Proposed Allocation Method

- If the economic capital is invested risk free...

$$u_i = e^{-r} E_P\left[\frac{A}{L} L_i - P_i e^{Rp} \middle| L > A\right]$$

- The allocated capital is the discounted expected value of the assets less premiums **under the insolvency scenarios**
- Exploiting equal priority of assets on insolvency

General Features

- The allocation satisfies the full allocation, symmetry and consistency fair allocation axioms.
- It does **not** satisfy the no undercut axiom
- Independent of how u is determined
- Allows for some interesting decomposition...

First Decomposition

$$u_i = uE\left[\frac{L_i}{L}\middle| L > A\right] + e^{-r}E\left[e^{Rp}\left(\frac{L_i}{L}P - P_i\right)\middle| L > A\right]$$

- First term: allocation from liabilities
  - Always >0
  - Sum to u
- Second term: premiums and investment performance
  - May be <0; sum to 0

Second Decomposition

$$u_i = e^{-r}\left\{\left(E\left[L_i \mid L > A\right] - E[L_i]\right) + \left(E\left[P_i e^{Rp}\right] - E\left[P_i e^{Rp} \mid L > A\right]\right)\right.$$
$$\left. - \left(E\left[P_i e^{Rp}\right] - E[L_i]\right) - E\left[\frac{L_i}{L}(L - A)\middle| L > A\right]\right\}$$

- First term: extra liability in insolvency
- Second term: investment performance in insolvency
- Third term: reduction for premium adequacy
- Fourth term: reduction from limited liability

Third Decomposition

$$u_i = e^{-r}\left\{LM_i + IM_i + RM_i - E\left[{}^{L_i}\!/_{L}\,(L - A)\middle| L > A\right]\right\}$$

- LM_i= Line Manager's allocation
- IM_i = Investment Manager's allocation
- RM_i=Risk Manager's allocation
- Fourth term=reduction due to limited liability
  - Assumed unallocated for performance measure

Line Manager's Allocation

$$LM_i = E\left[L_i \mid L_i > \rho(L_i)\right] - P_i e^r$$

$$= \left(E\left[L_i \mid L_i > \rho(L_i)\right] - E\left[L_i\right]\right) - \left(P_i e^r - E\left[L_i\right]\right)$$

- First term is the stand-alone economic capital above the mean
- Second term is the risk loading
- LM might manage this allocation piece
- Generally positive...

Investment Manager's Allocation

$$IM_i = P_i e^r - E\left[P_i e^{Rp} \mid Pe^{Rp} < -\rho(-Pe^{Rp})\right]$$

- Second term is left tail risk for investments
- Sum over all lines under IM control for allocation under IM responsibility

Risk Manager's Allocation

$$RM_i = \left(E\left[L_i \mid L > A\right] - E\left[L_i \mid L_i > \rho(L_i)\right]\right)$$
$$+\left(E\left[Pe^{Rp} \mid Pe^{Rp} < -\rho(-Pe^{Rp})\right] - E\left[P_i e^{Rp} \mid L > A\right]\right)$$

- Reduction in capital due to diversification in liabs and investment
- Sum should be negative
- Some individual lines may generate positive values
- RM may minimize Σ_i RM_i

Numerical Example

- Insurer with 3 lines
  - Line 1: 1-Year GMMB with 90% gtee
    - $L_1=(90 - F_T)^+$
  - Line 2: 1-Year GMMB with 100% gtee
    - $L_2=(100 - F_T)^+$
  - Line 3: 1-Year EIA with 100% gtee
    - $L_3=(F_T - 100)^+$

Example

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- $100 invested in each line (F_0)
- F_t follows Geometric Brownian Motion
- Liabilities are un-hedged
- Monthly premiums 1% of fund
- u is assumed determined ex ante as 95% CTE =9.35

Example

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Line	GMMB 1	GMMB 2	EIA	Total
$V(L_i)$	3.3	8.5	2.9	14.7
$V(P_i)$	11.5	11.5	11.5	34.6
$V(P_i)$-$V(L_i)$	8.2	3.0	8.6	19.9
$V(D_i)$	0.185	0.265	0.001	0.451
EC 95% CTE				9.35
Cap+ Surplus				29.67

Example – comparison of allocation rules

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Allocation Rule	GMMB 1	GMMB 2	EIA	Total
Covariance	2.93	4.69	1.72	9.35
CTE	3.62	12.26	–6.54	9.35
Proposed	4.79	12.45	–7.90	9.35

Example – Third Decomposition

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	GMMB 1	GMMB 2	EIA
LM (Loss) LM (Prem load)	17.05 –10.18	22.89 –6.60	33.59 -6.26
IM	2.03	2.03	2.03
RM (liab) RM (assets)	–0.66 –0.44	–1.08 –0.44	–36.55 –0.44
Limited Liability	-3.00	-4.34	-0.25

The no-undercut axiom

- The proposed allocation rule fails the no-undercut axiom for risk concentrations.
- That is, comonotonic, and increasing Pr[L>A]
- But the no-undercut axiom assumes the assets belong to managers
- Managers are the agents of shareholders
- Structure of the firm is shareholder decision.

Conclusions

- Misallocating may overstate the returns on risky lines and understate returns on less risky lines
- Could lead to poor risk management strategy
- Proposed allocation allows for RAROC and can be used for management incentive through decomposition.