SOCIETY OF ACTUARIES

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4G - Call for Paper Session: A Quantitative Perspective on ERM

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A Cost of Capital Approach to Extrapolating an Implied Volatility Surface

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Introduction

- AEGON Context: European based life insurer that needs to develop market consistent financial statements
- Basic idea: use observed market prices for hedgeable risk
  use cost of capital to price non-hedgeable risk
- Practical Problem: “Holes” in observed market data, e.g. long interest rates, implied volatilities
- Key Idea: Use cost of capital concepts developed for insurance liabilities to risk adjust P measure to get a new C measure
- Use C measure to fill the “holes” in market data
- Three Step Process (adapted from insurance models)
  - Start with Best Estimate model (P measure)
  - Consider risk of current period loss (Contagion Risk)
  - Consider losses from model revisions (Parameter Risk)

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Step 1: P measure

- For simplicity, assume standard lognormal model
- Key ideas not dependent on this assumption
- Real world process for stock price $dS = \mu S dt + \sigma S dz$
- Examples will assume $\mu = r + 4.0\%,\ \sigma = 15.0\%$
- Discount using appropriate swap/risk free rate r
- ***Assume*** analog of a mortality contagion event is a market jump $S \rightarrow JS$ e.g. $J = 60\%$

Step 2: The Responsible Speculator

- Speculator does not hedge
- Holds economic capital for a jump $V(t, JS) - V(t, S)$
- Computes Value by

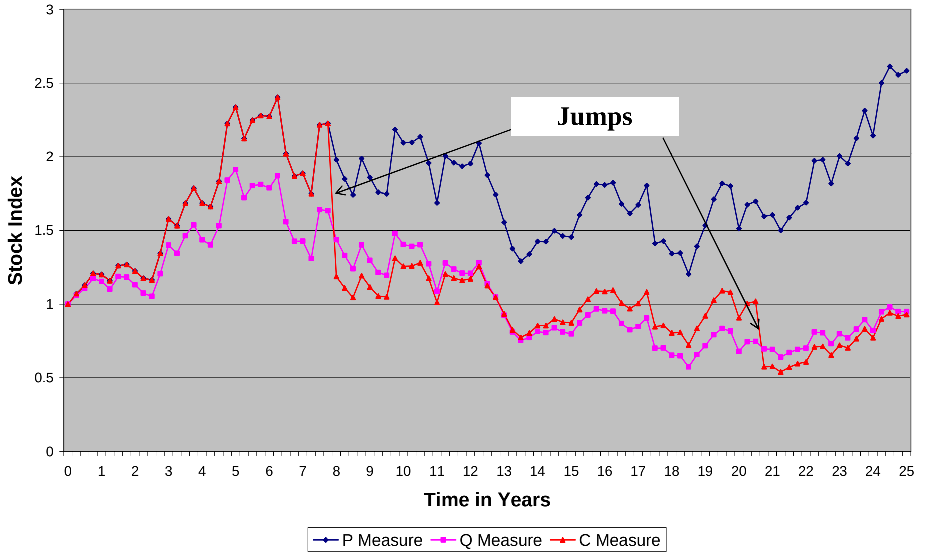
$$V(t,S) = E_P[PV\text{"}Cash\ Flows\text{"} + \pi\{V(t,JS) - V(t,S)\}]$$

- Mathematically equivalent to a 'C' measure calc

$$V(t,S) = E_C[PV\text{"}Cash\ Flows\text{"}]$$

- C measure is P measure augmented by a jump process $S \rightarrow JS$ where jump intensity equal to cost of capital rate π.
- Model prices stock process back to itself if and only if $\pi = (\mu\text{-}r)/(1\text{-}J)$ (makes sense from investor viewpoint)
- Example: $\mu = .08,\ r = .04,\ J = .6 \rightarrow \pi = .10$

Three Price Processes



Step 2: Responsible Hedger Interpretation

- Assume delta hedger holds capital for un-hedged loss if a jump occurs

$$"\textit{Cost of Capital}" = \underbrace{\pi}_{\text{Cost of Capital rate}} \left[\underbrace{\underbrace{V(t,JS)-V(t,S)}_{\text{Gross Loss}} - \underbrace{(J-1)S\frac{\partial V}{\partial S}}_{\text{Hedge}}}_{\text{Economic Capital}} \right]$$

- PV Calculation done in Q measure

$$V(t,S) = E_Q[PV"\textit{Cash Flows}"+\pi\{V(t,JS)-V(t,S)-(J-1)S\partial V/\partial S\}]$$

- Agrees with Speculator's C measure if $\pi = (\mu - r)/(1-J)$

Step 2: Summary – What does it mean?

- speculators and hedgers agree on C measure if
  - They each hold capital for their respective un-hedged risk
  - They agree on the cost of capital $\pi = (\mu - r)/(1-J)$
- For long dated options, law of large numbers means C measure jumps add a margin to P measure volatility

$$\sigma_{imp}^2 \rightarrow \sigma^2 + 2\pi(J - 1 - \ln J),$$
$$= \sigma^2 + 2(\mu - r)(J - 1 - \ln J)/(1 - J).$$

- Example: $\sigma = .15$ $\mu = .08$, $r = .04$, $J = .6 \rightarrow \pi = .10$ then

$$\sigma_{imp}^2 \approx (.15)^2 + 2(.10)(.6 - 1 - \ln .6) = (.21)^2.$$

- Key issue is the assumed equity premium $\mu - r$

Step 3: Parameter Risk

- If $\sigma = 15\%$ new information might arrive that causes us to reevaluate that assumption
- E.g. Realized volatility of S&P 500 in 2008 was 41%
- Plausible new assumption

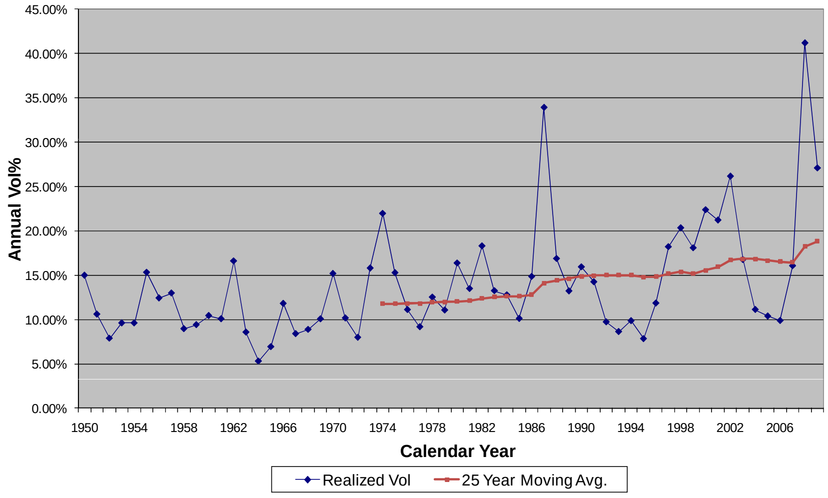
$$\hat{\sigma}_1^2 = \frac{24(.15)^2 + (.41)^2}{25} \approx (.17)^2$$
$$\approx (.15)^2 + (.08)^2$$

- This could happen more than once. In theory, need to consider an assumption shock hierarchy

$$\sigma^2 \rightarrow \hat{\sigma}_1^2 \rightarrow \ldots \rightarrow \hat{\sigma}_\infty^2$$

- At each level hold capital for transition to the next, cost of capital $\tilde{\pi}$ (different from π).

S&P 500 Total Return Realized Volatility



Step 3: Parameter Risk Summary

- Technical Solution: Augment C measure by allowing volatility parameter to randomly jump up the shock hierarchy.
  - We are modeling the assumption setting process
- Annual probability of a move is just assumed cost of capital $\tilde{\pi}$
- A number of reasonable short cuts available for practical implementation
- Example: Assume $\sigma = .15, \hat{\sigma} = .17, ..., \hat{\sigma}_\infty = .19, \tilde{\pi} = .04$

  Implied vol for 50 year options grows from 21% to 22%

  Ultimate implied vol is about 24% (more than 100 yrs)
- Issue not as material as contagion risk

Conclusion

- *C* measure can put a value on a long dated option using only *P* measure assumptions (Responsible Speculator) and some simple assumptions about parameter uncertainty
- Model does not assume delta hedging but is not inconsistent with hedging
- Key assumptions/issues are transparent
- Technicalities not onerous
- Consistent with CRO Forum approach to valuing non-hedgeable risk

Economic Scenario Generators

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Scenario Generators and Derivatives Pricing

- Many financial models aimed at pricing derivatives
  - Essentially interpolate/extrapolate prices
  - Parameters selected to match prices
- Scenario generators need realistic scenarios and paths based on history
  - Percentages of scenarios with various properties should correspond to history
  - Paths of changes should as well

No Arbitrage

- Basis of derivatives pricing
- Important for scenario generators
  - Scenarios with arbitrage are possible but disappear quickly
  - Including in model would distort strategic decision making

Model Complexity

- As more historical properties are studied, the models get more complex to capture them
- Stochastic volatility a dividing line
  - Some options need to recognize
  - Realistic paths would have this
    - But not so clearly important for insurers
  - Models with it more complex, slower
    - No longer closed-form yield curves

Review Time Series

- AR1 process
- $r_{i+1} = r_i + a(m - r_i) + b\varepsilon_{i+1}$
- ε_{i+1} usually standard normal
- $r_{i+1} = r_i(1 - a) + am + b\varepsilon_{i+1}$
- k^{th} autocorrelation is $(1 - a)^k$ – gets small fast unless a is close to zero
  - Not always realistic, but multi-factor processes can fix this
- Brownian motion is continuous version
  - But simulated with small steps of AR1 process

Multifactor Brownian Motion

- Double mean reverting AR1 processes
- $r_{i+1} - r_i = \Delta r_{i+1} = a(m_i - r_i) + b\varepsilon_{i+1}$
- $m_{i+1} - m_i = \Delta m_{i+1} = p(m - m_i) + q\eta_{i+1}$
- ε_{i+1} and η_{i+1} are iid standard normal
- Double mean reverting Brownian motion process
- $dr(t) = a[m(t) - r(t)]dt + bdZ(t)$
- $dm(t) = p[m - m(t)]dt + qdW(t)$
- Z and W are standard Brownian motions

Fitting Multifactor Models by Simulated Method of Moments

- Choose starting values of the parameters
- Simulate a long series of target variable
- Measure statistics of the simulated series
- Called generalized moments
- Calculate weighted sum of errors of the generalized moments from target values
- Use an optimization routine to iterate the parameters to minimize the errors
- Main problem is judgment selection of statistical properties to match

MODELS FOR INTEREST RATES

Short-rate Models

- Short-rate models use multi-factor Brownian motion for shortest duration interest rate
  - In practice often 3-month rate due to data
- No-arbitrage theory used to produce whole yield curve from short rate and market price of risk
  - Basically this theory says that a future payment should be discounted along all possible paths of the short-rate process *increased for risk* and the mean taken
  - Same market price of risk has to be used for all maturities

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Vasicek Model

- dr(t) = [b – ar(t)]dt + sdZ
- Another way to write mean reverting
- Parameters a, b and s can be estimated by MLE from incremental observations
- Price at t of a bond paying 1 at T is P(t,T)
- P(t,T) = $E_t\left[\exp\left(-\int_t^T r(x)dx\right)\right]$ where the expectation is under the risk-neutral process, which can be taken to be
- dr(t) = [b – (a – λs)r(t)]dt + sdW

Risk Neutral Process

- $dr(t) = [b - (a - \lambda s)r(t)]dt + sdW$
- Some models add λs instead of $\lambda sr(t)$
  - Either way makes future rates higher
  - Factor of s comes from arbitrage theory
- Taking $k = a - \lambda s$, $q = b/k$ gives
- $dr(t) = k[q - r(t)]dt + sdW$ as risk-neutral process
- Bond price is closed form:
- $P(t,T) = A(t,T)\exp[-B(t,T)r(t)]$
- $B(t,T) = [1 - \exp(-k(T - t))]/k$
- $A(t,T) = \exp\left\{(q-\tfrac{1}{2}(s/k)^2)[B(t,T)-T+t] - \tfrac{1}{4}(s^2/k)B(t,T)^2\right\}$
- Can estimate λ to fit yield curve

CIR Model

- $dr(t) = [b - ar(t)]dt + sr(t)^{0.5}dZ$
- Volatility increases with interest rate
- If r(t) is zero, only have drift, upward
- Vasicek could have negative rates
- $P(t,T) = E_t\left[\exp\left(-\int_t^T r(x)dx\right)\right]$ where the expectation is under the risk-neutral process, which is now
- $dr(t) = [b - (a - \lambda s)r(t)]dt + sr(t)^{0.5}dW$

Advantages of CIR

- No negative rates
- Volatility historically is higher with higher rates
- Distribution of rates not normal but positively skewed, which again is realistic

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CIR Bond Price

- $dr(t) = [b - (a - \lambda s)r(t)]dt + s r(t)^{0.5} dW$
- Taking $k = a - \lambda s$, $q = b/k$ gives
- $dr(t) = k[q - r(t)]dt + s r(t)^{0.5} dW$ as risk-neutral process
- Bond price is still closed form:
- $P(t,T) = A(t,T)\exp[-B(t,T)r(t)]$

$$h = \sqrt{k^2 + 2s^2} \qquad C(t,T) = 2h + (k+h)\left(e^{h(T-t)} - 1\right)$$

$$A(t,T) = \left[\frac{2he^{(k+h)(T-t)/2}}{C(t,T)}\right]^{2kq/s^2} \qquad B(t,T) = \left[\frac{2(e^{h(T-t)} - 1)}{C(t,T)}\right]$$

Yield Rates

- Continuously compounded
  - $R(t,T) = -\log[P(t,T)]/(T-t)$
  - $1 = P(t,T)\exp[R(t,T)(T-t)]$
- Simply compounded
  - $L(t,T) = [1-P(t,T)]/P(t,T)(T-t)$
  - $1 = P(t,T)[1+L(t,T)(T-t)]$
- Annually compounded
  - $1 = P(t,T)[1+Y(t,T)]^{(T-t)}$

Problems with Single Factor Models

- Bond price and so yield curve is deterministic function of short-rate
  - Not realistic
- Simulated scenarios will not capture range of possible yield curves
  - Also market price of risk probably evolves with time
- Although constant for all T at time t
  - Could model that by estimating it over time

Sum of Independent CIRs

- Adding up a few independent CIRs can get around problem of deterministic yield curves
  - Can add continuously compounded yields
- But same short-rate can be sum of different partial short rates, so different shapes possible
  - Can have pretty good statistical properties
- Lack of stochastic volatility makes scenarios less realistic
  - But analytic tractability gives motivation to ignore that, and probably ok for us

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Three-Factor CIR

- $r = r_1+r_2+r_3$

  $dr_1(t) = [b_1 - a_1 r_1(t)]dt + s_1 r_1(t)^{0.5} dZ_1$

  $dr_2(t) = [b_2 - a_2 r_2(t)]dt + s_2 r_2(t)^{0.5} dZ_2$

  $dr_3(t) = [b_3 - a_3 r_3(t)]dt + s_3 r_3(t)^{0.5} dZ_3$
- $P_i(t,T)$ from CIR formula
- $P(t,T) = P_1(t,T)P_2(t,T)P_3(t,T)$

  $R(t,T) = -\log[P(t,T)]/(T - t) =$

  $R_1(t,T) + R_2(t,T) + R_3(t,T)$

Extend to Risky Bonds

- Make treasury short rate r and spread s to risky short rate linear functions of the same three factors:
  - $r_t = g_0 + g_1x_t + g_2y_t + g_3z_t$
  - $s_t = h_0 + h_1x_t + h_2y_t + h_3z_t$
  - Then yield curve formula works for r and r+s
- Can do in steps to ever riskier bonds

How Does It Fit?

- "An evaluation of multi-factor CIR models using LIBOR, swap rates, and cap and swaption prices," Jagannathan, Kaplin, Sun. Journal of Econometrics 116 (2003)
- Finds that 3 factor fits shapes of yield curves, etc. well – much better than 2 factor, but still does not price all options well – particularly those sensitive to stochastic volatility

Fitting

- Structural parameters a’s, b’s, s’s can be fit with historical data and simulated method of moments
- Current values of partial short rates and market prices of risk can be calibrated to current yield curve
- Should be done historically to get processes for market prices of risk which are also stochastic

Summary

- Arbitrage-free models used for derivatives pricing can be used as scenario generators
- Want to match historical properties of series
- Simulated method of moments can do this
- Closed-form yield curves probably good enough