Skewness and Stock Option Prices

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Abstract

In the classical Black-Scholes model, the logarithm of the stock price has a normal distribution, which excludes skewness. In this paper we consider models that allow for skewness. We propose an option-pricing formula that contains a linear adjustment to the Black-Scholes formula. This approximation is derived in the shifted Poisson model, which is a complete market model in which the exact option price has some undesirable features. The same formula is obtained in some incomplete market models in which it is assumed that the price of an option is defined by the Esscher method. For a European call option, the adjustment for skewness can be positive or negative, depending on the strike price.

1. Introduction

We consider the problem of pricing a European option. Black and Scholes (1973) found an answer under the classical assumption that the stock price process is a geometric Brownian motion. Here we discuss answers under more general assumptions.

Let $S(t)$ be the price of a non-dividend-paying stock at time t. Often it is useful to reason in terms of $X(t)$, the accumulated rate of return, defined by the relation

$$S(t) = S(0)e^{X(t)},\ t \geq 0.$$

Throughout the paper we assume that $\{X(t)\}$ is a process with stationary and independent increments.

Mathematically, a European option is defined by a payoff function $\Pi(s) \geq 0$ and a maturity date τ. At time τ, the holder of the option will receive $\Pi(S(\tau))$. According to the fundamental theorem of asset pricing, the impossibility of arbitrage is essentially equivalent to the existence of a martingale measure having the same null sets as the original probability measure, so that the price of a contingent payment is calculated as the expected discounted payoff with respect to this measure. In the following we assume a constant risk-free force of interest, r. Thus the price of the option at time t is

$$e^{-r(\tau-t)}\mathrm{E}^*[\Pi(S(\tau))|S(t)],\ 0 \leq t < \tau. \tag{1}$$

The superscript asterisk indicates that the expectation is taken with respect to the equivalent martingale measure. This measure must be such that the pricing formula (1) is compatible with the observed price of the stock; that is, we require that

$$S(t) = e^{-r(\tau-t)}\mathrm{E}^*[S(\tau)|S(t)],\ 0 \leq t < \tau. \tag{2}$$

We consider only martingale measures under which the process $\{X(t)\}$ remains a process with stationary and independent increments. In this case, condition (2) is equivalent to the simpler condition that

$$e^{-r}\mathrm{E}^*[e^{X(1)}] = 1; \tag{3}$$

see Gerber and Shiu (1994). For notational simplicity, but without loss of generality, we consider an option with $\tau=1$ and compute its price at time $t=0$.

Let $M(z,t)=\mathrm{E}[e^{zX(t)}]$ denote the moment generating function of $X(t)$. We assume that $\{X(t)\}$ is a process with stationary and independent increments such that

$$M(z,t) = M(z)^t,\ t > 0,$$

where $M(z)=M(z, 1)$. Furthermore let μ, σ^2, and γ denote the first three cumulants per unit time of the process $\{X(t)\}$. Thus

$$\mathrm{E}[X(t)] = \mu t,$$
$$\mathrm{Var}[X(t)] = \sigma^2 t \tag{4}$$

and

$$\mathrm{E}[(X(t) - \mu t)^3] = \gamma t. \tag{5}$$

In the classical model, $\{X(t)\}$ is a Wiener process; that is, for given $t>0$, the random variable $X(t)$ has a normal distribution with mean μt, variance $\sigma^2 t$, and of course, $\gamma=0$. We are interested in alternative models in which $\gamma \neq 0$. A first idea is to consider a complete

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market model, in which the equivalent martingale measure and hence the price of the option are unique. Unfortunately, this price has some undesirable properties; therefore we propose to replace it by a linear approximation, which consists of the Black-Scholes price plus an adjustment that is proportional to γ.

Another approach is to examine some incomplete market models. Then, as in Gerber and Shiu (1994), we calculate a price of the option by the method of Esscher transforms. Finally, for small values of γ, we examine again the linear approximation for the price of an option.

The idea is as follows: let $\{Y(t)\}$ be a process with stationary and independent increments that is standardized in the sense that

$$\mathrm{E}[Y(t)] = 0,$$

$$\mathrm{Var}[Y(t)] = t$$

and

$$\mathrm{E}[Y(t)^3] = t.$$

Thus

$$\mathrm{E}[e^{zY(t)}] = e^{t\psi(z)}$$

where

$$\psi(z) = \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots. \quad (6)$$

Then we construct a three-parameter family of processes by defining

$$X(t) = kY(\lambda t) + \mu t,\ t \geq 0. \quad (7)$$

For given μ, σ^2 and γ, we set

$$k = \frac{\gamma}{\sigma^2},\ \lambda = \frac{\sigma^6}{\gamma^2}, \quad (8)$$

so that (4) and (5) are satisfied. We note that

$$\begin{aligned} M(z,t) &= \mathrm{E}[e^{kzY(\lambda t)}]\, e^{\mu t z} \\ &= \exp\,[\lambda t\psi(kz) + \mu tz] \quad (9) \\ &= \exp\,(\mu z + \frac{1}{2}\sigma^2 z^2 + \frac{1}{6}\gamma z^3 + \cdots). \quad (10) \end{aligned}$$

The classical model is obtained as the limit $\gamma \to 0$ for fixed values of μ and σ^2. Thus, the classical model is embedded in the richer family of models given by (7). We analyze the effect of the deviation from the classical model by expanding the option price in powers of γ. Our approach differs from the method of Jarrow and Rudd (1982); their idea is to match moments up to a certain order and to use a generalized Edgeworth expansion.

2. EXACT SOLUTION IN THE COMPLETE MARKET MODEL

As in Gerber and Shiu (1994) and in Chapter 10 of Boyle et al. (1997), we consider the *shifted Poisson model.* Here

$$X(t) = kN(\lambda t) - ct,$$

where k and c are constants and $\{N(t)\}$ is a Poisson process with parameter 1. To exclude arbitrage opportunities, we assume that $r+c$ and k have the same sign. For given μ, σ^2, and γ, we choose k and λ as in (8) and set

$$c = k\lambda - \mu = \frac{\sigma^4}{\gamma} - \mu.$$

In this model the equivalent martingale measure is unique. Under the martingale measure, the process $\{X(t)\}$ is of the same type, with unchanged values of k and c, but with λ replaced by λ^*, which is chosen such that (3) is satisfied. Thus λ^* is the solution of

$$\exp\,[-r + \lambda^*(e^k - 1) - c] = 1$$

which yields

$$\lambda^* = \frac{r + c}{e^k - 1}.$$

Consider an option with maturity date $\tau=1$. According to (1), its price at time 0 is

$$e^{-r}\mathrm{E}^*[\Pi(S(1))] = e^{-r}\sum_{j=0}^{\infty} e^{-\lambda^*}\frac{\lambda^{*j}}{j!}\,\Pi\,(S(0)e^{kj-c}). \quad (11)$$

As an example, we consider a European call option with exercise price K. Here $\Pi(s)=(s-K)_+$, where x_+ denotes the positive part of x; that is, $x_+=x$ if $x>0$ and $x_+=0$ if $x\leq 0$. Then the current price of the option is

$$e^{-r}\sum_{j=0}^{\infty} e^{-\lambda^*}\frac{\lambda^{*j}}{j!}\,(S(0)e^{kj-c} - K)_+. \quad (12)$$

For a numerical illustration, we set $S(0)=100$, and assume that $r=0.1$, $\mu=0.1$, and $\sigma=0.2$. The resulting option prices are displayed for different values of K in Table 1 (under the heading "shifted Poisson") for $\gamma=0.008$, which corresponds to a coefficient of skewness of 1. The exact option prices in this table are the same as in Tables 1 to 4 of Gerber and Shiu (1994), which also include prices for other maturities.

TABLE 1
CALL OPTION PRICES WITH $S = 100$, $\mu = 0.1$, $\sigma = 0.2$, $\gamma = 0.008$, AND $r = 0.1$

Exercise Price (K)	Black-Scholes	Rate of Change at $\gamma = 0$	Linear Approxi-mation	Shifted Poisson	Shifted Gamma	Shifted Inverse Gaussian
80	27.993	−61.638	27.500	27.613	27.624	27.640
85	23.864	−80.451	23.220	23.089	23.237	23.274
90	19.989	−90.464	19.265	18.565	19.174	19.214
95	16.439	−87.137	15.742	15.696	15.591	15.613
100	13.270	−69.422	12.714	13.005	12.547	12.544
105	10.515	−39.948	10.196	10.315	10.031	10.005
110	8.183	−3.868	8.152	7.624	7.989	7.949
115	6.258	32.886	6.522	6.418	6.352	6.306
120	4.708	65.177	5.230	5.383	5.050	5.006

TABLE 2
CALL OPTION PRICES WITH $S = 100$, $\mu = 0.1$, $\sigma = 0.2$, $\gamma = 0.001$, AND $r = 0.1$

Exercise Price (K)	Black-Scholes	Rate of Change at $\gamma = 0$	Linear Approxi-mation	Shifted Poisson	Shifted Gamma	Shifted Inverse Gaussian
80	27.993	−61.638	27.931	27.929	27.932	27.933
85	23.864	−80.451	23.783	23.784	23.782	23.784
90	19.989	−90.464	19.898	19.896	19.896	19.898
95	16.439	−87.137	16.352	16.344	16.349	16.351
100	13.270	−69.422	13.200	13.190	13.198	13.200
105	10.515	−39.948	10.475	10.468	10.473	10.475
110	8.183	−3.868	8.179	8.178	8.177	8.179
115	6.258	32.886	6.291	6.296	6.289	6.291
120	4.708	65.177	4.773	4.775	4.771	4.773

TABLE 3
CALL OPTION PRICES WITH $S = 100$, $\mu = 0.1$, $\sigma = 0.2$, $\gamma = 0.008$, AND $r = 0.05$

Exercise Price (K)	Black-Scholes	Rate of Change at $\gamma = 0$	Linear Approxi-mation	Shifted Poisson	Shifted Gamma	Shifted Inverse Gaussian
80	24.589	−120.130	23.628	23.902	23.920	23.947
85	20.469	−153.284	19.243	19.145	19.375	19.438
90	16.699	−172.800	15.317	14.389	15.294	15.366
95	13.346	−173.773	11.956	11.815	11.846	11.899
100	10.451	−156.350	9.200	9.475	9.061	9.085
105	8.021	−124.970	7.022	7.134	6.878	6.877
110	6.040	−86.317	5.350	4.793	5.200	5.183
115	4.467	−47.111	4.090	3.920	3.927	3.902
120	3.247	−12.549	3.147	3.216	2.967	2.940

TABLE 4
CALL OPTION PRICES WITH $S = 100$, $\mu = 0.1$, $\sigma = 0.2$, $\gamma = 0.001$, AND $r = 0.05$

Exercise Price (K)	Black-Scholes	Rate of Change at $\gamma = 0$	Linear Approximation	Shifted Poisson	Shifted Gamma	Shifted Inverse Gaussian
80	24.589	−120.130	24.439	24.467	24.471	24.473
85	20.469	−153.284	20.316	20.317	20.317	20.319
90	16.699	−172.800	16.527	16.524	16.526	16.528
95	13.346	−173.773	13.173	13.164	13.171	13.174
100	10.451	−156.350	10.294	10.282	10.292	10.295
105	8.021	−124.970	7.896	7.888	7.894	7.897
110	6.040	−86.317	5.954	5.952	5.951	5.954
115	4.467	−47.111	4.419	4.423	4.417	4.419
120	3.247	−12.549	3.235	3.234	3.232	3.234

Similarly, Table 2 shows the prices for $\gamma=0.001$. Tables 3 and 4 are analogous, except that here $r=0.05$.

Figure 1, which shows the option price for $K=100$ as a function of γ reveals a surprise: this function is not monotone and its derivative with respect to γ has numerous discontinuities. A glance at (12) shows that these discontinuities occur whenever

$$S(0)e^{kj-c} - K = 0 \text{ for some } j.$$

In Figure 1, $S(0)=K=100$. Hence the discontinuities occur at values of γ for which $kj-c=0$ for some j. Substituting

$$k = \frac{\gamma}{\sigma^2},\ c = \frac{\sigma^4}{\gamma} - \mu$$

we see that the critical values of γ are obtained as the solution of the quadratic equations

$$\frac{j}{\sigma^2}\gamma^2 + \mu\gamma - \sigma^4 = 0.$$

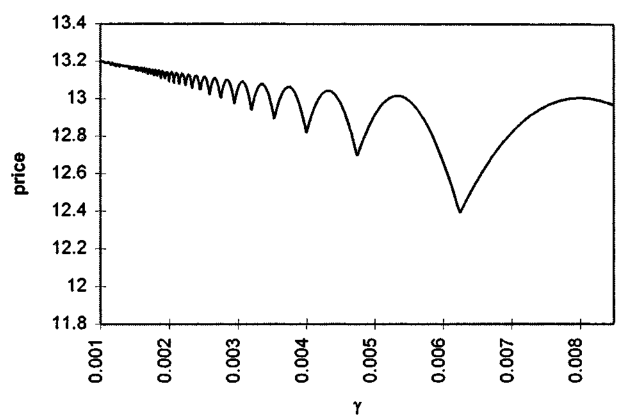
Hence, the discontinuities occur at the points

$$\gamma_j = \frac{-\mu + \sqrt{\mu^2 + 4j\sigma^2}}{2j/\sigma^2},\ j = 1, 2, \ldots.$$

We note that this is a monotone sequence that converges to 0.

All this shows that the exact solution has some unattractive properties. Furthermore, computational difficulties arise when γ is too close to 0. For these reasons, we propose a linear approximation to the option price, which is more appealing and can be used if γ is close to 0.

FIGURE 1
CALL OPTION PRICE AS A FUNCTION OF γ FOR $S = 100$, $K = 100$, $\mu = 0.1$, $\sigma = 0.2$, AND $r = 0.1$



3. LINEAR APPROXIMATION IN THE COMPLETE MARKET MODEL

3.1 Maclaurin Expansion of the Martingale Measure

The general idea is to expand the exact option price (11) in powers of γ (or, equivalently, in powers of k) and to replace it by a Maclaurin polynomial of a suitable degree; here we settle for the first-degree approximation. Instead of trying to do this directly, we first expand the martingale measure. Then we replace the martingale measure in (11) by its expansion to get the corresponding expansion of the price of the option.

It suffices to analyze the distribution of $X(1)$. Its cumulant generation function is

$$\ln E^*[e^{zX(1)}] = \lambda^*(e^{kz} - 1) - cz$$
$$= (r + c)\frac{e^{kz} - 1}{e^k - 1} - cz.$$

Noting that

$$\frac{e^{kz}-1}{e^k-1} = \frac{z + \frac{1}{2}kz^2 + \frac{1}{6}k^2z^3 + \cdots}{1 + \frac{1}{2}k + \frac{1}{6}k^2 + \cdots}$$

$$= \left(z + \frac{1}{2}kz^2 + \frac{1}{6}k^2z^3 + \cdots\right)\left(1 - \frac{1}{2}k + \frac{1}{12}k^2 + \cdots\right)$$

$$= z - \frac{1}{2}kz + \frac{1}{2}kz^2 + \frac{1}{12}k^2z - \frac{1}{4}k^2z^2 + \frac{1}{6}k^2z^3 + \cdots$$

we see that

$$\ln \mathrm{E}^*[e^{zX(1)}] = r\left(z - \frac{1}{2}kz + \frac{1}{2}kz^2 + \cdots\right) + c\left(-\frac{1}{2}kz + \frac{1}{2}kz^2 + \frac{1}{12}k^2z - \frac{1}{4}k^2z^2 + \frac{1}{6}k^2z^3 + \cdots\right).$$

Substituting $c=\sigma^2/k-\mu$, we find that

$$\ln E^*[e^{zX(1)}] = \mu^*z + \frac{1}{2}\sigma^2 z^2 + kP(z) + \cdots \quad (13)$$

where

$$\mu^* = r - \frac{1}{2}\sigma^2 \quad (14)$$

and

$$P(z) = \frac{1}{2}(r-\mu)(z^2 - z) + \sigma^2\left(\frac{1}{12}z - \frac{1}{4}z^2 + \frac{1}{6}z^3\right). \quad (15)$$

Exponentiating (13), we obtain the expansion of the moment generating function of $X(1)$:

$$E^*[e^{zX(1)}] = \exp\left[\mu^*z + \frac{1}{2}\sigma^2 z^2\right](1 + kP(z) + \cdots). \quad (16)$$

Now the problem is to invert this formula to find the corresponding approximation for the distribution of $X(1)$. First, we note that the leading term is the moment generating function of the normal distribution with mean μ^* and variance σ^2. Hence

$$\exp\left[\mu^*z + \frac{1}{2}\sigma^2 z^2\right] = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{zx}\,\phi\left(\frac{x-\mu^*}{\sigma}\right)dx,$$

where $\phi(\cdot)$ denotes the standard normal probability density function. If we multiply this equation by z, observe that $ze^{zx} = (e^{zx})'$ and perform the obvious integration by parts, we find that

$$z\exp\left[\mu^*z + \frac{1}{2}\sigma^2 z^2\right] = -\frac{1}{\sigma^2}\int_{-\infty}^{\infty} e^{zx}\phi'\left(\frac{x-\mu^*}{\sigma}\right)dx.$$

Repeating this, we see that

$$z^n \exp\left[\mu^*z + \frac{1}{2}\sigma^2 z^2\right] = (-1)^n\frac{1}{\sigma^{n+1}}\int_{-\infty}^{\infty} e^{zx}\phi^{(n)}\left(\frac{x-\mu^*}{\sigma}\right)dx,$$

for $n = 0, 1, 2, \ldots$.

This shows that

$$z^n \exp\left[\mu^*z + \frac{1}{2}\sigma^2 z^2\right]$$

is the transform of the function

$$(-1)^n\frac{1}{\sigma^{n+1}}\phi^{(n)}\left(\frac{x-\mu^*}{\sigma}\right) = \frac{1}{\sigma}(-D)^n\,\phi\left(\frac{x-\mu^*}{\sigma}\right),$$

where D is the differentiation operator with respect to x. Now we are prepared to invert (16) term by term. The resulting expansion for the density of $X(1)$ is

$$f_0(x) + kf_1(x) + \cdots \quad (17)$$

with

$$f_0(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu^*}{\sigma}\right) \quad (18)$$

and

$$f_1(x) = \frac{1}{\sigma}P(-D)\,\phi\left(\frac{x-\mu^*}{\sigma}\right).$$

Not surprisingly, $f_0(x)$ is the martingale density of $X(1)$ in the classical Black-Scholes model. By substituting (15) we find that

$$f_1(x) = \frac{1}{2}(r-\mu)\left[\frac{1}{\sigma^3}\phi''\left(\frac{x-\mu^*}{\sigma}\right) + \frac{1}{\sigma^2}\phi'\left(\frac{x-\mu^*}{\sigma}\right)\right] - \frac{1}{12}\phi'\left(\frac{x-\mu^*}{\sigma}\right) - \frac{1}{4\sigma}\phi''\left(\frac{x-\mu^*}{\sigma}\right) - \frac{1}{6\sigma^2}\phi'''\left(\frac{x-\mu^*}{\sigma}\right). \quad (19)$$

Note that the exact distribution of $X(1)$, which is discrete, is approximated by a continuous distribution with density (17). Although this is not a probability density in the proper sense (it takes on negative values), it can be used successfully to approximate certain expectations.

3.2 Linear Approximation of the Option Price

The price of the European option is the discounted expectation of the payoff. If we replace the exact distribution of $X(1)$ by the linear approximation developed in the preceding section, we obtain the formula

$$e^{-r}\mathrm{E}^*[\Pi(S(0)e^{X(1)})] \simeq e^{-r}\int_{-\infty}^{\infty}\Pi(S(0)e^{x})\, f_0(x)dx + ke^{-r}\int_{-\infty}^{\infty}\Pi(S(0)e^{x})\, f_1(x)dx.$$

Here $k=\gamma/\sigma^2$, and $f_0(x)$ and $f_1(x)$ are given by (18) and (19). Note that

$$\frac{e^{-r}}{\sigma^2}\int_{-\infty}^{\infty}\Pi(S(0)e^{x})\, f_1(x)dx$$

can be interpreted as the rate of change of the option price with respect to γ at $\gamma=0$.

In general, these integrals have to be evaluated by numerical integration. In exceptional cases, closed-form solutions are available. Let us again consider the European call option with exercise price K. The linear approximation for its price is given by the expression

$$e^{-r}\int_{\kappa}^{\infty}(S(0)e^{x} - K)\, f_0(x)dx + ke^{-r}\int_{\kappa}^{\infty}(S(0)e^{x} - K)\, f_1(x)dx \qquad (20)$$

where $\kappa=\ln(K/S(0))$. To evaluate the first integral, we observe that

$$e^{x}\phi\left(\frac{x-\mu^*}{\sigma}\right) = \exp[\sigma^2/2 + \mu^*]\,\phi\left(\frac{x-\mu^*-\sigma^2}{\sigma}\right),$$

from which it follows that

$$\frac{1}{\sigma}\int_{\kappa}^{\infty}e^{x}\phi\left(\frac{x-\mu^*}{\sigma}\right)dx = \exp[\sigma^2/2+\mu^*]\left[1-\Phi\left(\frac{\kappa-\mu^*-\sigma^2}{\sigma}\right)\right] = e^{r}\left[1-\Phi\left(\frac{\kappa-r-\sigma^2/2}{\sigma}\right)\right], \qquad (21)$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function. Hence the leading term of (20) is

$$S(0)\left[1-\Phi\left(\frac{\kappa-r-\sigma^2/2}{\sigma}\right)\right] - e^{-r}K\left[1-\Phi\left(\frac{\kappa-r+\sigma^2/2}{\sigma}\right)\right].$$

This is of course the celebrated Black-Scholes formula. Thus the linear approximation (20) is the Black-Scholes price (first term) combined with a first-order adjustment for skewness (second term). To compute the latter, we have to evaluate integrals of the form

$$I_n = \frac{1}{\sigma}\int_{\kappa}^{\infty}e^{x}\phi^{(n)}\left(\frac{x-\mu^*}{\sigma}\right)dx.$$

Applying integration by parts, we obtain the recursive formula

$$I_n = -e^{\kappa}\phi^{(n-1)}\left(\frac{\kappa-\mu^*}{\sigma}\right) - \sigma I_{n-1},\ \text{for } n = 1, 2, 3, \ldots.$$

Note that I_0 is given by (21).

A numerical illustration is given in Tables 1–4, in which the linear approximations of the option prices are calculated according to Formula (20). The first-order effect of the skewness on the price of a call option is indicated in the third column of Tables 1–4 and documented more completely in Figures 2 and 3; they show by how much the Black-Scholes price must be adjusted per γ. We note that this adjustment can be positive or negative, depending on the exercise price K.

FIGURE 2

ADJUSTMENT OF THE CALL OPTION PRICE PER γ ($\boldsymbol{r}$ = 0.10)

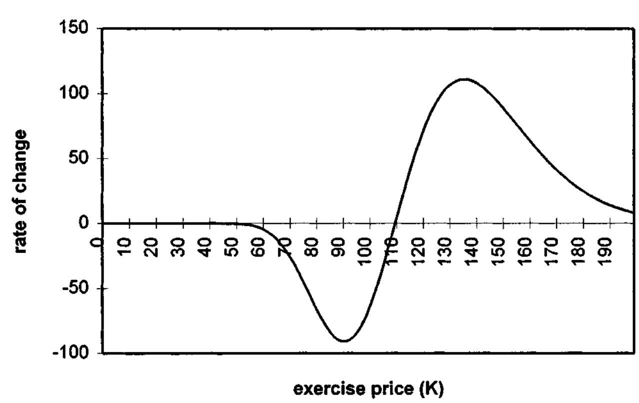
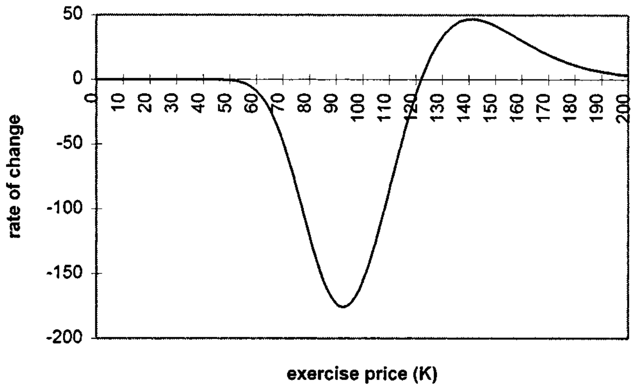


FIGURE 3

ADJUSTMENT OF THE CALL OPTION PRICE PER γ (r = 0.05)



4. EXACT SOLUTIONS IN INCOMPLETE MARKET MODELS

In an incomplete market model, the equivalent martingale measure is not unique, and a priori, it is not clear which martingale measure should be chosen to calculate the price of an option. Gerber and Shiu (1994) proposed that, to obtain a unique answer, the choice of the equivalent martingale measure could be limited to the family of *Esscher transforms*.

Under the Esscher transform corresponding to parameter h, $\{X(t)\}$ is defined as a process with stationary and independent increments whereby the new moment generating function of $X(1)$ is

$$\mathrm{E}\,[e^{zX(1)}; h] = \frac{M(z+h)}{M(h)}. \qquad (22)$$

The price of an option is then defined to be the discounted expectation of the payoff whereby the expectation is taken according to the Esscher transform of parameter h^*, where $h=h^*$ is determined so that (3) is satisfied; that is, it is the solution of

$$e^{-r}\,\mathrm{E}\,[e^{X(1)}; h^*] = 1. \qquad (23)$$

For a more thorough discussion of this method, refer to Gerber and Shiu (1994, 1996). In the following, we assume that the price of an option in an incomplete market model is determined by this method.

For a numerical illustration, we look at two examples. For both the process $\{X(t)\}$ is such that

$$X(t) = Z(t) - ct$$

where $\{Z(t)\}$ is a process with stationary and independent increments.

Example 1: The Shifted Gamma Process

Here it is assumed that $\{Z(t)\}$ is a gamma process with shape parameter α and scale parameter β. Then the moment generating function of $X(1)$ is

$$M(z) = \left(\frac{\beta}{\beta - z}\right)^{\alpha} e^{-cz},\ z < \beta. \qquad (24)$$

For given values of μ, σ, and γ, the three parameters are chosen so that (5) is satisfied; that is, we set

$$\alpha = \frac{4\sigma^6}{\gamma^2},\ \beta = \frac{2\sigma^2}{\gamma},\ c = \frac{2\sigma^4}{\gamma} - \mu.$$

From (22) and (24), it can be seen that the Esscher transform of $\{X(t)\}$ is again a shifted gamma process with unchanged values of α and c but β replaced by $\beta - h$. The martingale measure is obtained for

$$\beta - h^* = \left(1 - \exp\left[-\frac{c+r}{\alpha}\right]\right)^{-1}.$$

Example 2: The Shifted Inverse Gaussian Process

Here $\{Z(t)\}$ is an inverse Gaussian process, say with parameters a and b, so that the moment generating function of $X(1)$ is

$$M(z) = \exp\,[\alpha(\sqrt{b} - \sqrt{b-z}) - cz],\ z < b. \qquad (25)$$

For given values of μ, σ, and γ, the three parameters are chosen so that (5) is satisfied; that is, we set

$$a = 3\sqrt{\frac{6\sigma^{10}}{\gamma^3}},\ b = \frac{3\sigma^2}{2\gamma},\ c = \frac{3\sigma^4}{\gamma} - \mu.$$

From (22) and (25), it can be seen that the Esscher transform of $\{X(t)\}$ is again a shifted inverse Gaussian process with unchanged values of a and c but b replaced by $b-h$. The martingale measure is obtained for

$$b - h^* = \frac{(a^2 + c^2 + r^2 + 2rc)^2}{4a^2\,(r+c)^2}.$$

Both processes can be written in the standard form (7). For example, we can define the shifted gamma process by setting

$$Y(t) = G(t) - \mathrm{E}[G(t)],$$

where $\{G(t)\}$ is a gamma process with shape parameter $\alpha=4$ and scale parameter $\beta=2$. Tables 1–4 contain the option prices that result from the two models. Figures 4 and 5 show that the option prices as functions of γ, unlike in the shifted Poisson model, are almost linear in both cases. As in the complete market model, computational difficulties arise when γ is too close to 0.

FIGURE 4
CALL OPTION PRICE FOR THE GAMMA PROCESS WITH S = 100, K = 100, μ = 0.1, σ = 0.2, and r = 0.1

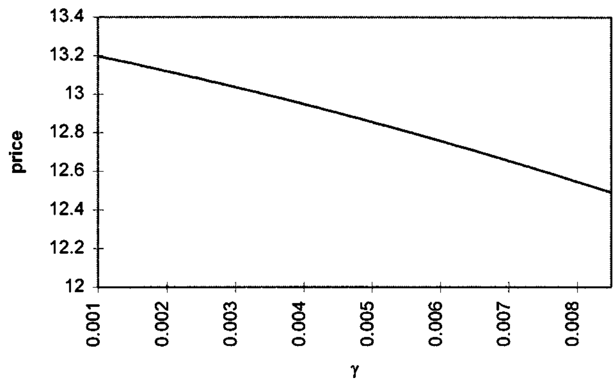
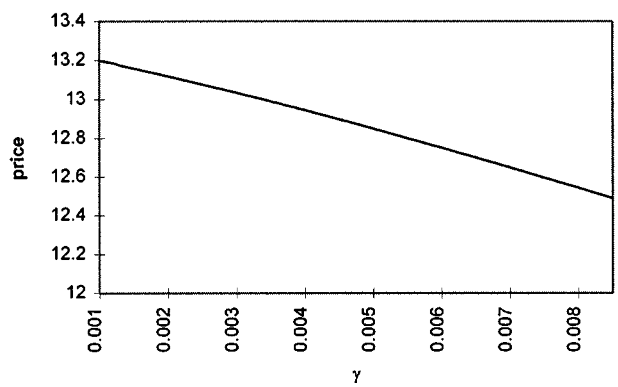


FIGURE 5
CALL OPTION PRICE FOR THE INVERSE GAUSSIAN PROCESS WITH S = 100, K = 100, μ = 0.1, σ = 0.2, and r = 0.1



The choice of the model is somewhat arbitrary, and one might wonder whether this choice has a significant impact on the price of the option. Looking at Tables 2 and 4, we get the impression that it is not the case for small values of γ. In the next section, we provide a mathematical explanation for this.

5. LINEAR APPROXIMATION IN INCOMPLETE MARKET MODELS

There is an infinite number of incomplete market models, and of course, the exact option price, calculated by the Esscher method, depends on the choice of the model. However, we show that in any given incomplete market model the linear approximation for the price of the option is identical to the approximation found in Section 3.2. Hence, this linear approximation is independent of the model chosen.

For the proof, we assume that the process $\{X(t)\}$ is of the form (7) with k and λ given by (8). According to (22) and (9), the cumulant generating function of $X(1)$ with respect to the Esscher transform of parameter h is

$$\begin{aligned}\ln \mathrm{E}\,[e^{zX(1)}; h] &= \ln M(z+h) - \ln M(h)\\ &= \lambda\,[\psi\,(k(z+h)) - \psi(kh)] + \mu z.\end{aligned}$$

Substituting (6) we get the expansion

$$\ln \mathrm{E}[e^{zX(1)}; h] = \frac{1}{2}\,\sigma^2\,(z^2 + 2hz) + \mu z + \frac{k}{6}\,\sigma^2\,(z^3 + 3hz^2 + 3h^2 z) + \cdots. \quad (26)$$

According to (23), $h=h^*$ is the solution of

$$\frac{1}{2}\,\sigma^2\,(1 + 2h^*) + \mu + \frac{k}{6}\,\sigma^2\,(1 + 3h^* + 3h^{*2}) + \cdots - r = 0. \quad (27)$$

We expand h^* in powers of k and set $h^*=a+kb+\cdots$. We substitute this in (27) and compare the coefficients of the powers of k. Comparison of the constant terms gives the equation

$$\frac{1}{2}\,\sigma^2 + \sigma^2 a + \mu - r = 0,$$

from which it follows that

$$a = \frac{r-\mu}{\sigma^2} - \frac{1}{2}. \quad (28)$$

Comparing the coefficients of k leads to

$$\sigma^2 b + \frac{1}{6}\,\sigma^2\,(1 + 3a + 3a^2) = 0,$$

or

$$b = -\frac{1}{6}\,(1 + 3a + 3a^2). \quad (29)$$

Replacing h by $h^*=a+kb+\cdots$ in (26), we obtain the expansion of the cumulant generating function of $X(1)$ under the martingale measure:

$$\ln \mathrm{E}\,[e^{zX(1)}; h^*] = \frac{1}{2}\,\sigma^2 z^2 + \sigma^2 az + \mu z + k\sigma^2 bz + \frac{k}{6}\,\sigma^2\,(z^3 + 3az^2 + 3a^2 z) + \cdots.$$

We note that because of (28) and (14), the coefficient of z in the leading term simplifies to μ^*. Furthermore, substituting for b according to (29) yields

$$\ln \mathrm{E}[e^{zX(1)}; h^*] = \mu^* z + \frac{1}{2}\sigma^2 z^2 + \frac{k}{6}\sigma^2 z^3 + \frac{k}{2}\sigma^2 a z^2 - \frac{k}{2}\sigma^2\left(a + \frac{1}{3}\right) z + \cdots. \qquad (30)$$

Substituting for a according to (28), we see that (30) is identical to (13). This implies that the linear approximation for the option price is indeed identical to the one found in Section 3.2.

6. CONCLUSION

To examine the effect of skewness, we propose a linear approximation for the price of a European option. For example, we see that skewness can have a positive or negative effect on the price of a European call option, depending on the exercise price. The approximation consists of the Black-Scholes price combined with an adjustment for skewness. It is remarkable that the approximation formula does not depend on the underlying model, as long as option prices are calculated by the Esscher method. By using the approximation, we can avoid computational difficulties that can arise when a model is close to the Black-Scholes model. Furthermore, the exact option price in the complete market model has some undesirable features that disappear when the linear approximation is used.

ACKNOWLEDGMENTS

The authors thank Elias Shiu and two anonymous referees for their helpful comments.

REFERENCES

BLACK, F., AND SCHOLES, M. 1973. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81: 637–55.

BOYLE, P., COX, S., DUFRESNE, D., GERBER, H., MUELLER, H., PANJER, H. (editor), PEDERSEN, H., PLISKA, S., SHERRIS, M., SHIU, E. AND TAN, K. 1997. *Financial Economics: with Applications in Insurance and Pensions*. Schaumburg, Ill.: Society of Actuaries, in press.

GERBER, H.U., AND SHIU, E.S.W. 1994. "Option Pricing by Esscher Transforms," *Transactions of the Society of Actuaries* XLVI:99–140.

GERBER, H.U., AND SHIU, E.S.W. 1996. "Actuarial Bridges to Dynamic Hedging and Option Pricing," *Insurance: Mathematics and Economics* 18:183–218.

JARROW, R., AND RUDD, A. 1982. "Approximate Option Valuation for Arbitrary Stochastic Processes," *Journal of Financial Economics* 10:347–69.

Discussions

TERENCE CHAN*

I congratulate the authors on a fine paper that provides an interesting new perspective on stock option prices when the price of the underlying security is not lognormal.

I wish to emphasize that the paper is primarily concerned with corrections to the Black-Scholes price arising from perturbations of a Wiener process, rather than perturbations of a symmetric process by a skewed one; the skewness as measured by the third cumulant, γ, is merely a means of developing a suitable asymptotic expansion. To see this, consider the process Y in (7): since $\lambda \to \infty$ as $\gamma \to 0$, the central limit theorem says that

$$\frac{Y(\lambda t)}{\sqrt{\lambda t}} \Rightarrow N(0, 1)$$

as $\gamma \to 0$. Since

$$\frac{Y(\lambda t)}{\sqrt{\lambda t}} = \frac{kY(\lambda t)}{\sigma\sqrt{t}},$$

we see that $kY(\lambda\cdot) \Rightarrow \sigma B(\cdot)$ as $\gamma \to 0$, where B is a standard Brownian motion. Therefore, processes of the form (7) are implicitly perturbations of Brownian motion with drift. Indeed, skewness need not enter into the discussion at all: let Y be any process with stationary independent increments satisfying $\mathbb{E}[Y(t)]=0$ and $\mathbb{E}[Y(t)^2]=t$. If we take

$$X(t) = \sigma\varepsilon Y(\varepsilon^{-2}t) + \mu t,$$

then the same procedure as used by the authors will result in exactly the same asymptotic expansions in ε. Because Y has stationary independent increments, the third cumulant of X is necessarily a linear function of ε, but there is no reason for Y to be skewed: the third cumulant of Y and hence X could be 0 (although in this case, the asymptotic expansions will not be the same as those obtained by the authors, but their method will still work).

It is also not so surprising that the leading term in the asymptotic expansions is the same whatever the

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distribution of Y. This is essentially because the coefficient of γ in the expansion of the logarithmic moment generating function of Y depends only on the third moment of Y_1 (which has been fixed at 1). If Y is taken to be a different process, the differences will emerge when one examines the higher-order corrections in γ.

Given that the asymptotic expansions studied in the paper arise from perturbations of Brownian motion of a certain form, it is natural to consider more explicit perturbations of the Wiener process in the classical Black-Scholes model. In the case of complete markets, the choice of processes that can be used as the logarithm of the stock price is severely restricted: the only possible candidates are those that have already been studied by the authors, namely, a Wiener process or a Poisson process. In incomplete markets, the range of possibilities is far wider. As an example, consider as the log stock price a process of the form

$$X(t) = \sigma B(t) + \varepsilon Y(t) + \mu t,$$

where Y is any process with stationary independent increments (necessarily discontinuous if it is not Wiener process) that has finite exponential moments, and B is a standard Brownian motion independent of Y. We might as well take $\mathbb{E}[Y(t)]=0$. Write the moment generating function of Y as

$$\mathbb{E}[e^{zY(t)}] = e^{t\psi_Y(z)}$$

where

$$\psi_Y(z) = \frac{\sigma_Y^2 z^2}{2} + \sum_{j=3}^{\infty} \frac{\kappa_j z^j}{j!}.$$

(Of course, $\sigma_Y^2=\mathbb{E}[Y_1^2]$ and κ_j are the higher-order cumulants of Y_1.) Then as is shown by the authors, the Esscher transform parameter h^* associated with the equivalent Esscher martingale measure is the solution to

$$\frac{\sigma^2}{2}(1 + 2h) + \mu - r + \psi_Y[\varepsilon(1 + h)] - \psi_Y(\varepsilon h) = 0.$$

Expanding h^* in powers of ε, writing $h^*=\sum_{i=0}^{\infty} \alpha_i\varepsilon^i$, and equating coefficients, we can obtain the following equations up to third order:

$$\frac{\sigma^2}{2}(1 + 2\alpha_0) + \mu - r = 0,$$

$$\sigma^2\alpha_1 = 0,$$

$$\sigma^2\alpha_2 + \frac{\sigma_Y^2}{2}(1 + 2\alpha_0) = 0,$$

$$\sigma^2\alpha_3 + \sigma_Y^2\alpha_1 + \frac{\kappa_3}{6}(1 + 3\alpha_0 + 3\alpha_0^2) = 0.$$

Under the Esscher martingale measure, the logarithmic moment generating function of X is given by

$$\psi^*(z) = \frac{\sigma^2}{2}(z^2 + 2h^*z) + \mu z + \psi_Y[\varepsilon(z + h^*)] - \psi_Y(\varepsilon h^*)$$

and substituting the expansion for h^* into the above, we obtain

$$\psi^*(z) = \frac{1}{2}\sigma^2 z^2 + \mu^* z + P_2(z)\sigma_Y^2\varepsilon^2 + P_3(z)\kappa_3\varepsilon^3 + \cdots,$$

where $\mu^*=r-\sigma^2/2$ is the same μ^* as defined in the paper and

$$P_2(z) = \frac{z^2 - z}{2},$$

$$P_3(z) = \frac{z^3}{6} + \left(\frac{r-b}{2\sigma^2} - \frac{1}{4}\right) z^2 - \left(\frac{r-b}{2\sigma^2} - \frac{1}{12}\right) z.$$

The asymptotic expansion for the density of X_1 is then

$$\sigma^{-1}\phi((x - \mu^*)/\sigma) + \varepsilon^2\sigma_Y^2 f_2(x) + \varepsilon^3\kappa_3 f_3(x),$$

where $f_n(x)=\sigma^{-1}P_n(-d/dx)\phi((x-\mu^*)/\sigma)$. Note that $\varepsilon^2\sigma_Y^2$ and $\varepsilon^3\kappa_3$ are the contributions to the variance and the 3rd cumulant of X, respectively, as a result of perturbing B by a process Y.

MICHEL JACQUES*

The authors consider the pricing of options when the underlying price process does not follow the usual geometric Brownian motion process. They consider the case in which the underlying accumulated rate of return exhibits skewness. For a broad range of models, they provide an exact price formula that is numerically unstable for some values of the parameters. They then proceed to develop a clever approximation for the price formula that is usable for all values of the parameters. The authors should be congratulated for that nice piece of work.

The authors mention that the exact price formula (12) for the shifted Poisson model presents numerical difficulties when the skewness parameter γ is close to 0. I tried to use Formula (12) when the volatility parameter σ is larger than in the numerical examples of the paper. It seems that the formula is just as delicate to use for large volatilities as it is for small skewness. This point is important because option prices for large volatilities often behave similarly to option

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prices for long maturities. The latter are relevant to actuaries because the options embedded in insurance contracts are usually long-lived.

The approximation developed by the authors is based on an expansion in $k=\gamma/\sigma^2$. Hence, increasing σ has the same effect on k as decreasing γ. The question is then: Is the linear approximation price Formula (20) a good approximation for the exact price when volatility is large?

If the exact price formula is not computationally usable, we cannot directly test for the error contained in the approximation. The essence of arbitrage-free pricing in a complete market is the existence of the replicating strategy. The option price is not the whole story; one also has to be able to hedge the risk. I think a nice way of studying the quality of the approximation would be to generate a large number of trajectories of the underlying price process. Then by using the replicating strategy, it could be checked at maturity if the strategy built from the approximate formula indeed replicates the intrinsic value of the option for most of the trajectories. It is not obvious to me what the replicating strategy is for the shifted Poisson model. However, although $\{S(t)\}$ is a discrete state process, in the limit where γ is small or σ large, the jumps are small and it makes sense to test for the derivative. This usual delta could then be used as a benchmark.

It is easy to compute the (approximate) delta of the option by simply differentiating the approximation Formula (20) with respect to $S(0)$. Due to algebraic simplifications, the delta turns out to be, using the authors notations for the linear approximation:

$$\Phi\left(\frac{r+0.5\sigma^2-\kappa}{\sigma}\right)+ke^{-r}\left[\left(\frac{r-\mu}{2\sigma}-\frac{\sigma}{12}\right)I_1 + \left(\frac{r-\mu}{2\sigma^2}-\frac{1}{4}\right)I_2-\frac{1}{6\sigma}I_3\right],$$

whereas the exact value of the delta is

$$e^{-r}\sum_{j=0}^{\infty}e^{-\lambda^*}\frac{\lambda^{*j}}{j!}e^{kj-c}\,1_{\{S(0)e^{kj-c}-K>0\}}.$$

For the numerical values given in Table 1 of the paper (p. 52), the relative error between the approximate price and the exact price (columns 4 and 5) and the relative error between the approximate delta and the exact delta are given in the following table.

TABLE 1
RELATIVE ERRORS ON
LINEAR APPROXIMATION

K	Price	Delta
80	0.004	0.021
85	0.006	0.064
90	0.038	0.132
95	0.003	0.169
100	0.022	0.020
105	0.012	0.133
110	0.069	0.277
115	0.016	0.318
120	0.028	0.084

The table shows that the relative error on the prices is of the order of a few percents at worse, whereas the error on the delta can be as high as 30%. The conclusion is that the linear approximation should be used carefully if one is interested in hedging.

HEINZ H. MÜLLER*

The authors analyze the pricing of an European option on a non-dividend-paying stock. The price of the stock at the time t is given by

$$S(t) = S(0)e^{X(t)},\ t \geq 0$$

As in the Black-Scholes (1973) model, the increments of the process $\{X(t)\}$ are stationary and independent. However, in contrast to Black-Scholes, where the increments of $\{X(t)\}$ are normally distributed, the authors want to admit skewed distributions as well. They show that for any μ, σ^2, and γ, there exists a shifted Poisson model such that

$$\begin{aligned} E[X(t)] &= \mu t,\\ \text{Var}\ [X(t)] &= \sigma^2 t, \\ E[(X(t)-\mu t)^3] &= \gamma t. \end{aligned} \tag{1}$$

Due to market completeness, the option prices are unique. The authors derive an explicit pricing formula. This result is a very nice generalization of the Black-Scholes model.

Nevertheless, the authors are worried by the fact that the option price is neither monotone nor differentiable in the skewness parameter γ. Therefore, they derive an approximate option-pricing formula, which

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is linear in γ. To be more specific, the formula is of the type

$$c(\gamma) = c_{BS} + \gamma \cdot \text{const.}, \tag{2}$$

where c_{BS} denotes the Black-Scholes option price. This formula looks very nice and is definitely useful for many purposes. However, as the authors point out, the corresponding risk-neutral probability density for $X(t)$ takes on negative values. Therefore the risk-neutral probability density $f(S, t)$ for $S(t)$ takes on negative values as well. This leads to arbitrage opportunities, and the option-pricing formula is no longer convex in the strike price. To see this, look at the price $c(K, t)$ of an European option with strike price K expiring in t. Then under general conditions (Breeden and Litzenberger 1978), the risk-neutral probability density is given by

$$f(S, t) = \text{const.} \cdot \frac{\partial^2}{\partial K^2} c(K, t)|_{K=S}, \tag{3}$$

and it becomes obvious that $c(K, t)$ is convex in K if and only if $f(S, t) \geq 0 \forall S$. Of course for $\gamma = 0$, one obtains the Black-Scholes model, and these drawbacks disappear. However, for large values of γ, the phenomenon may be disturbing.

Finally, the authors look at incomplete market models with the Esscher transform as an exact pricing method. It is shown that the approximate option-pricing formula developed for complete market models can still be used.

To summarize, the article contains a highly interesting exact pricing method and a very useful approximate method for options on stocks with skewed logarithmic returns. Especially for small values of the skewness parameter γ, both methods can be considered as very appropriate extensions of the Black-Scholes formula.

Perhaps it is of some interest to compare the Gerber and Landry approach with other models admitting skewness. Shimko (1993) and Rubinstein (1994) infer the risk-neutral probability from simultaneously observed market prices for European options with the same expiration date but different strike prices. Whereas Rubinstein uses binomial trees, Shimko's method is essentially based on Formula (3). Given observed market prices for liquid European calls with strike prices $K_1, K_2, \ldots, K_n \in [K, \mathbf{K}]$, Shimko proposes to use an interpolation method to calculate

$$c(K, t), K \in [K, \mathbf{K}].$$

Afterwards Formula (3) is used for the calculation of the risk-neutral density

$$f(S, t), S \in [K, \mathbf{K}].$$

For the tails $S < K$, respectively, $S > \mathbf{K}$, a matching technique with lognormal distributions is applied. Such a method is very powerful if there are liquid European options for a wide range of strike prices. In these cases Shimko's method admits skewness and other deviations from the Black-Scholes model. For less liquid markets, however, the Gerber and Landry approach is obviously more appropriate.

REFERENCES

BLACK, F., AND SCHOLES, M. 1973. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81: 637–55.

BREEDEN, D., AND LITZENBERGER, R. 1978. "Prices of State-Contingent Claims Implicit in Option Prices," *Journal of Business* 51:621–52.

RUBINSTEIN, M. 1994. "Implied Binomial Trees," *The Journal of Finance* 69, no. 3:771–818.

SHIMKO, D. 1993. "Bounds of Probability," *Risk* 6, no. 3:33–37.

GÉRARD PAFUMI*

The authors are to be congratulated for their paper, which generalizes the Black-Scholes formula to the case in which the logarithm of the stock price has a skewed distribution.

From a practical point of view, the only parameter in the Black-Scholes formula that cannot be observed directly is the volatility. By using the historical standard deviation to estimate the volatility, we assume that the past variability of the stock's returns is invariant through time. It is not obvious that volatility is constant for long periods of time and that the historical volatility is independent of the time series from which it is calculated. It is therefore difficult to measure directly the volatility in practice.

However, option prices are quoted in the market. An alternative concept, *implied volatility*, consists of estimating the volatility of stock returns implicitly reflected in current option prices. A call option price increases monotonically with volatility, so there is a one-to-one correspondence between the volatility and the option price. The idea is to invert the Black-Scholes formula from the currently observed price of a call option. In this way we obtain the market's opinion of the value of the volatility over the remaining life of the option. This method was originally proposed

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by Latané and Rendleman (1976). The implied volatility derived from several options written on the same stock will generally not be equal.

Now the problem is to take a suitable weighted average of the individual implied volatilities. One can think about taking the arithmetic average or even weighting each option's implied volatility according to its degree of price elasticity with respect to the volatility. Here we mention Beckers' empirical study (1981) of stock returns' future variability estimates. He suggests the use of only one call option price, the one whose price is most sensitive to σ. We measure the sensitivity of an option with respect to σ by the partial derivative of its price with respect to σ, that is,

$$\frac{\partial C}{\partial \sigma} = S(0)\sqrt{\frac{T}{2\pi}} \exp\left(-\frac{1}{2}\frac{(rT + \ln(S(0)/K) + \frac{1}{2}\sigma^2 T)^2}{\sigma^2 T}\right).$$

This expression is maximal for

$$\tilde{K} = S(0) \exp\left[rT + \frac{1}{2}\sigma^2 T\right].$$

Hence the call option whose strike price is the nearest to $\tilde{K}$ will be chosen.

The parameters in the linear approximation given by Formula (20) in the authors' paper that cannot be observed directly are the first three cumulants per unit time of the stock price. Now the idea is to invert this approximation formula by observing the current call option's price. In this way we obtain the market's opinion of the value for the drift, the volatility, and the third cumulant per unit time over the remaining life of the option. To do this, we apply the following algorithm. Choose arbitrary initial values for μ and γ, say μ_0 and γ_0 (a good idea is to choose historical estimates). Then compute σ_1, the value of σ that makes the approximation formula meet exactly the last observed price. Now use σ_1 and γ_0 to compute μ_1, the value of μ that makes the approximation formula meet exactly the last observed price. Repeat these steps to obtain γ_1, σ_2, μ_2, γ_2, σ_3, μ_3, γ_3, ... until convergence is observed.

In practice, for both formulas and for a particular choice of $S(0)$ and T, the largest difference is obtained for the more out-of-the-money call option. Both for the Black-Scholes formula and the linear approximation, we remark that the mean absolute spread decreases significantly when implied parameters are used. The following table shows the prices obtained for a call option on stocks of the Swiss Bank Corporation on August 18, 1994, using implied parameters. We found an annual implied σ of 0.29321 for the Black-Scholes formula. For the linear approximation formula, we obtained the following implied annual parameters: $\mu=-0.18889$, $\sigma=0.29122$, $\gamma=0.00043496$.

More details and examples can be found in Pafumi (1997).

CALL OPTION (ON SWISS BANK CORPORATION) PRICES WITH $S(0) = 374$, $T = 64$ DAYS ON AUGUST 18, 1994

Exercise Price (K)	Observed Settlement Prices	Black-Scholes Prices, Using Implied σ	Linear Approximation, Using Implied μ, σ and γ
350	33.00	33.963	33.903
360	26.50	27.381	27.339
370	21.00	21.649	21.633
380	15.00	16.783	16.794
390	11.50	12.756	12.793
400	8.50	9.506	9.565
425	3.40	4.191	4.273

REFERENCES

BECKERS, S. 1981. "Standard Deviations in Option Prices as Predictors of Future Stock Price Variability," *Journal of Banking and Finance* 5:363–82.

LATANÉ, HENRY A., AND RENDELMAN, RICHARD J. 1976. "Standard Deviations of Stock Price Ratios Implied in Option Prices," *Journal of Finance* 31:369–82.

PAFUMI, GÉRARD. 1997. "A Study of a Family of Equivalent Martingale Measures to Price an Option with an Application to the Swiss Market," *Working Paper 97.05*. University of Lausanne, HEC.

ELIAS S.W. SHIU* AND SERENA TIONG†

Dr. Gerber and Mr. Landry are to be congratulated for this elegant paper. It is observed in Section 5 that, if option prices are calculated by the Esscher method, the linear approximation in incomplete market models does not depend on the model (and hence is the same as in the complete market model). This is probably not true in general; in this sense the Esscher method is quite special.

In the paper there are two results derived by repeated applications of integration by parts. Here we

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present an operational calculus proof for them. First, we derive the formula

$$P(z)\exp(\mu z + \frac{1}{2}\sigma^2 z^2)$$

$$= \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{zx}\, P(-D_x)\phi\left(\frac{x-\mu}{\sigma}\right)\, dx, \qquad \text{(D.1)}$$

which is used to invert (16) in the paper. Here, D_x is the differentiation operator with respect to x; in the paper, D_x is simply denoted as D. Consider the moment generating function formula for the normal distribution:

$$\exp\left(\mu z + \frac{1}{2}\sigma^2 z^2\right) = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{zx}\, \phi\left(\frac{x-\mu}{\sigma}\right)\, dx. \qquad \text{(D.2)}$$

Let D_μ be the differentiation operator with respect to μ; then

$$D_\mu \phi\left(\frac{x-\mu}{\sigma}\right) = -D_x \phi\left(\frac{x-\mu}{\sigma}\right).$$

Differentiating both sides of (D.2) n times with respect to μ, we have

$$z^n \exp\left(\mu z + \frac{1}{2}\sigma^2 z^2\right) = \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{zx}\, D_\mu^n\, \phi\left(\frac{x-\mu}{\sigma}\right)\, dx$$

$$= \frac{1}{\sigma}\int_{-\infty}^{\infty} e^{zx}\, (-D_x)^n\, \phi\left(\frac{x-\mu}{\sigma}\right)\, dx,$$

from which (D.1) follows.

Second, we derive a formula for

$$I_n(\mu) = \frac{1}{\sigma}\int_{\kappa}^{\infty} e^{x}\, \phi^{(n)}\left(\frac{x-\mu}{\sigma}\right)\, dx. \qquad \text{(D.3)}$$

Obviously,

$$I_n(\mu) = (-\sigma D_\mu)^n\, I_0(\mu).$$

By (21) of the paper,

$$I_0(\mu) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)\left[1 - \Phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right]. \qquad \text{(D.4)}$$

Applying $(-\sigma D_\mu)^n$ to both sides of (D.4) and using the *exponential shift formula*

$$p(D_\mu)[e^{z\mu} f(\mu)] = e^{z\mu}\, p(z + D_\mu)\, f(\mu), \qquad \text{(D.5)}$$

we have

$$I_n(\mu) = \exp\left(\mu + \frac{1}{2}\sigma^2\right)(-\sigma)^n\, (1 + D_\mu)^n$$

$$\left[1 - \Phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right] \qquad \text{(D.6)}$$

$$= \exp\left(\mu + \frac{1}{2}\sigma^2\right)(-\sigma)^n$$

$$\sum_{k=0}^{n}\binom{n}{k} D_\mu^k\left[1 - \Phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right]$$

$$= \exp\left(\mu + \frac{1}{2}\sigma^2\right)\left\{(-\sigma)^n\left[1 - \Phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right]\right.$$

$$\left. - \sum_{k=1}^{n}\binom{n}{k}(-\sigma)^{n-k}\, \phi^{(k-1)}\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right\}. \qquad \text{(D.7)}$$

To obtain the recursion formula in Section 3.2, we substitute the identity

$$(1 + D_\mu)^n = 1 + \sum_{k=0}^{n-1}(1 + D_\mu)^k\, D_\mu$$

in (D.6) to obtain

$$I_n(\mu) = (-\sigma)^n\, I_0(\mu) - (-\sigma)^{n-1}\exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$\sum_{k=0}^{n-1}(1 + D_\mu)^k\, \phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right). \qquad \text{(D.8)}$$

It follows from the exponential shift formula that

$$\exp\left(\mu + \frac{1}{2}\sigma^2\right)(1 + D_\mu)^k\, \phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)$$

$$= D_\mu^k\left[\exp\left(\mu + \frac{1}{2}\sigma^2\right)\phi\left(\frac{\kappa-\mu-\sigma^2}{\sigma}\right)\right]$$

$$= D_\mu^k\left[e^{\kappa}\, \phi\left(\frac{\kappa-\mu}{\sigma}\right)\right]$$

$$= (-\sigma)^{-k}\, e^{\kappa}\, \phi^{(k)}\left(\frac{\kappa-\mu}{\sigma}\right).$$

Hence

$$I_n(\mu) = (-\sigma)^n\, I_0(\mu)$$

$$- e^{\kappa}\sum_{k=0}^{n-1}(-\sigma)^{n-1-k}\, \phi^{(k)}\left(\frac{\kappa-\mu}{\sigma}\right), \qquad \text{(D.9)}$$

which is simpler than (D.7). Finally, it follows from (D.9) that

$$I_n(\mu) = -\left[e^{\kappa}\, \phi^{(n-1)}\left(\frac{\kappa-\mu}{\sigma}\right) + \sigma\, I_{n-1}(\mu)\right],$$

which is the recursion formula given in Section 3.2.

Authors' Reply

HANS U. GERBER AND BRUNO LANDRY

Dr. Chan points out two alternative expansions in terms of power of ε. The second model is particularly interesting because it also provides very transparent results. Perhaps an advantage of our model is that γ can be interpreted more easily than ε.

Dr. Jacques proposes an interesting simulation method to monitor the quality of the linear approximation in the complete market model. Replicating strategies for the shifted Poisson model are discussed in Gerber and Shiu (1996) and Chapter 10 of Boyle et al. (1997).

We agree with Dr. Müller's remark that the linear approximation cannot be used universally because it would imply negative probability masses, which in turn would lead to arbitrage opportunities. However, one merit of the linear approximation is that it shows the first-order effect of skewness on an option price.

Formula (3) in Dr. Müller's discussion, which is the basis of Shimko's technique, is known to actuaries in a different form. The second derivative of the stop-loss premium with respect to the deductible is the probability density function of the aggregate claims random variable. See exercises 14.10 and 14.11 of Bowers et al. (1997).

Mr. Pafumi addresses an issue of great practical importance. Instead of estimating the parameters from historical data, it might be indeed better to use only recent option prices from which the underlying implied parameter values are determined. Mr. Pafumi shows how this can be done when three parameters have to be estimated.

Dr. Shiu and Ms. Tiong provide a clever and intriguing application of operational calculus. At the beginning of their discussion, they conjecture that in an imcomplete market model the linear approximation depends on the choice of the method for determining an equivalent martingale measure. This is in fact true and can be illustrated by means of an example. We suppose that $\gamma > 0$ and that

$$X(t) = Z(t) - ct$$

where $\{Z(t)\}$ is a compound Poisson process, with Poisson parameter λ and exponential jump amount distribution with mean $1/\beta$. For given values of μ, σ^2, and γ, we determine λ, β, and c so that Equation (5) of the paper is satisfied. This leads to

$$\lambda = \frac{9}{2}\frac{\sigma^6}{\gamma^2},\ \beta = 3\frac{\sigma^2}{\gamma},\ \text{and } c = \frac{3}{2}\frac{\sigma^4}{\gamma} - \mu.$$

An equivalent martingale measure can now be constructed as follows: we replace λ and β by $\lambda^*>0$ and $\beta^*>0$, such that condition (3) of the paper is satisfied, that is

$$\frac{\lambda^*}{\beta^* - 1} = c + r. \tag{R1}$$

There is an infinite number of possibilities, but three seem to be particularly "natural."

Method 1

The Esscher method, where both λ and β are replaced by new values λ^* and β^*, as described in Section 4 of the paper.

Method 2

Let $\beta^*=\beta$ (unchanged) and λ^* according to (R1), that is, set

$$\lambda^* = (\beta - 1)(c + r).$$

Method 3

Let $\lambda^*=\lambda$ (unchanged) and β^* according to (R1), that is, we set

$$\beta^* = 1 + \frac{\lambda}{c + r}.$$

In all three cases, ln $E^*[e^{zX(1)}]$ can be calculated and expanded explicitly. Each time we obtain a linear approximation of the form (13), where the polynomial $P(z)$ varies from method to method. In Method 1 it is of course given by Formula (15) of the paper, which can also be written as

$$P(z) = \left[\frac{1}{2}(\mu - r) + \frac{\sigma^2}{12}\right] z + \frac{1}{2}(\mu^* - \mu)z^2 + \frac{\sigma^2}{6} z^3.$$

For Method 2, we find that

$$P(z) = \frac{1}{3}(\mu - r)z + \frac{1}{3}(\mu^* - \mu)z^2 + \frac{\sigma^2}{6} z^3,$$

and for Method 3 the result is that

$$P(z) = \left[\frac{2}{3}(\mu - r) + \frac{\sigma^2}{6}\right] z + \frac{2}{3}(\mu^* - \mu)z^2 + \frac{\sigma^2}{6} z^3.$$

We note that the three polynomials follow a similar pattern, but only Method 1 produces a linear approximation that is identical to the one found in the complete market model.

We are grateful to the discussants for their comments. They provide valuable additional insight and may very well stimulate further research.

REFERENCES

BOWERS, N., GERBER, H.U., HICKMAN, J., JONES, D., AND NESBITT, C. 1997. *Actuarial Mathematics*, 2nd ed. Schaumburg, Ill. Society of Actuaries.

BOYLE, P., COX, S., DUFRESNE, D., GERBER, H., MÜLLER, H., PANJER, H. (EDITOR), PEDERSEN, H., PLISKA, S., SHERRIS, M., SHIU, E., AND TAN, K. 1997. *Financial Economics: with Applications in Insurance and Pensions*. Schaumburg, Ill.: Society of Actuaries, in press.

GERBER, H.U., AND SHIU, E.S.W. 1996. "Actuarial Bridges to Dynamic Hedging and Option Pricing," *Insurance: Mathematics and Economics* 18:183–218.

Additional discussions on this paper will be accepted until January 1, 1998. The authors reserve the right to reply to any discussion. See the Table of Contents page for detailed instructions on the preparations of discussions.

Most realistic utility functions (those with decreasing absolute risk aversion and constant relative risk aversion) do likewise

The reasonableness of this result can be demonstrated by some examples. With regard to investing equal amounts in common stock and risk-free Treasuries, Mr Creswell states, "Clearly . . . the actual risk premium will be half of what it would have been with all funds invested in common stock rather than one quarter." This conclusion is true only if the risk premium is proportional to standard deviation (that is, risk is *defined* as standard deviation).

But let us explore the implications of this by considering the more general case of investing portion p of one's wealth in a risky portfolio with expected return, μ_R, and standard deviation, σ_R, and $(1-p)$ in risk-free securities with expected return, R_{RF}. A risk adjustment proportional to standard deviation would result in a certainty equivalent rate, R_{CE}, of

$$R_{CE} = p\ \mu_R + (1-p)\ R_{RF} - k\ ap\sigma_R \quad \text{(R.1)}$$

where k is the constant of proportionality and a is the degree of risk aversion. This equation is linear with respect to p, and if the value of a was calculated from μ_R, σ_R, R_{RF}, that is,

$$a = \frac{\mu_R - R_{RF}}{k\sigma_R} \quad \text{(R.2)}$$

then Formula (R.1) reduces to

$$R_{CE} = R_{RF}.$$

In which case the investor is indifferent to the mix of investments (the value of p) This is not consistent with normal risk-averse behavior, which seeks to diversify investments (between, say, a common stock portfolio and a money market fund)

If the value of a is higher than that given by Formula (R.2), then Formula (R.1) indicates that the investor would prefer that *all* investments be in risk-free investments ($p=0$); if the value of a is lower, then the investor would prefer *all* investments be in the risky portfolio ($p=1$) Again, this does not conform to normal risk-averse behavior, which would prefer a weighted mix.

The variance-proportionate risk adjustment given by my paper's Formula (VI.2), on the other hand, results in

$$R_{CE} = p\ \mu_R + (1-p)\ R_{RF} - \frac{1}{2}\, a\ (p\ \sigma_R)^2, \quad \text{(R.3)}$$

which is quadratic with respect to p with a maximum at

$$p_{max} = \frac{\mu_R - R_{RF}}{a\sigma_R^2}. \quad \text{(R.4)}$$

This does conform to normal risk-averse behavior, because a mix of assets is preferred over all-or-nothing When the value of a is calculated from μ_R, σ_R, R_{RF}, that is,

$$a = 2\ \frac{\mu_R - R_{RF}}{\sigma_R^2}$$

then

$$p_{max} = \frac{1}{2}.$$

For higher or lower values of a, p_{max} moves towards (but does not equal) 0% or 100%, respectively This again conforms to normal risk-averse behavior, because a weighted mix of assets is preferred over all-or-nothing

For example, if μ_R=13.8%, σ_R=16.9%, and R_{RF}=5.7%, using Formula (R.4), maximum R_{CE}'s occur at p=50% for the internally consistent value of a of 5.7, at p=35% for a=8, and at p=70% for a=4 These appear to be reasonable asset mix choices.

"Skewness and Stock Option Prices," Hans Gerber and Bruno Landry, July 1997

KENNETH O. KORTANEK* AND V. G. MEDVEDEV†

In a manner that students and quantitative finance professionals will find clear and illuminating, Gerber and Landry develop the condition required for a change of probability distribution to an equivalent martingale distribution in a context slightly more general that the Black and Scholes Brownian motion environment The clear intent is to include distributions,

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which, unlike the normal distribution, have third moment nonzero When the martingale distribution is unique, it has been convenient simply to term the associated market as *complete*.

The three classical models in Gerber and Shiu (1994, Sec. 3) are all complete, namely, the Weiner, the shifted Poisson, and the multiplicative binomial random walk processes The point of departure in this paper is general processes $\{X(t)\}$ with the first three moments specified,

$$E[X(t)] = \mu t,$$
$$\text{Var}\ [X(t)] = \sigma^2 t,$$

and

$$E[(X(t) - \mu t)^3] = \gamma t \qquad (1)$$

The Complete Market Case

In this case the shifted Poisson, $X(t)=kN(\lambda t)-ct$, is the distribution used, and students and members of the Society will relish making the connection to the textbook (Panjer et al. 1998, Sec 10.3.15–10.3.17) for a review of how the parameters k, λ, c are selected to deliver the prespecified moments of the original $X(t)$. There is a simple argument there (Ibid., Sec. 10.3) for showing that k and $c+r$ must have the same sign. Otherwise, no one would invest in the riskless asset

Members should also focus on the clearly presented equation that is needed for obtaining the equivalent martingale *Poisson* distribution, namely, determining the unique λ^* that defines the risk-neutral distribution This equation is identical to what the Esscher transform produces [Gerber and Shiu 1994, (3.2.6)], with δ there being r here. We are by now familiar with how to price an option in this case with quite general payoff, that is, use the special Poisson distribution in the expectation of the given payoff For example, if the payoff is a call option and $S(0)$ is the current price of the underlying asset while K is the strike price and r is the risk-free force of interest, then the current price of the option is determined by:

$$e^{-r-\lambda^*} \sum_{j=0}^{\infty} \frac{\lambda^{*j}}{j!} (S(0)e^{kj-c} - K)^+ \qquad (2)$$

This is an interesting function that is continuous and differentiable except at a denumerable number of points There has been a significant amount of work in numerical analysis on speeding up the rate of convergence of infinite series, and these techniques could be considered for computing the sum in (2). The authors nevertheless indicate that "the exact solution has some unattractive properties," which provides a motivation for constructing a linear approximation to the option price, (2).

The approximation is achieved by using familiar methods of the calculus or a next level of analysis, such as Taylor series-type expansions. Because of the martingale property, only the probability density of $X(1)$ need be approximated, and the authors turn to algebraic manipulations of the moment generating function, another familiar concept to most members of the Society. The resulting approximation begins with terms that involve the standard normal density function and its derivatives, so a connection is made to the Black-Scholes martingale density. The connection is nicely illustrated in tabular form showing the required adjustment of the Black-Scholes price per unit stemming from the nonzero third moment parameter, γ

It would appear that the authors will necessarily need to address how the possible negative values of their approximation will not be a demurrer to other applications. The authors may also need to elaborate on their claim that the approximation "can be used successfully to approximate certain expectations."

It would be interesting to compare the accuracy obtained through truncations of the underlying infinite series density appearing in (2) with the accuracy that the authors have obtained by means of their linear approximation The accuracy of the linear approximations is in general rather good Note, however, in Table 1 for $K=110$, $r=0.1$, and $\gamma=0.008$ that the error relative to the shifted Poisson is 6.9%, while in Table 3 it is 11.6%, where $r=0.5$. All the linear approximations are excellent for very small values of γ, but this feature is characteristic for any Maclaurin-Taylor series expansion

There have been other series expansion approaches for the pricing of contingent claims, such as European calls and puts We mention the particular approach of Abken, Madan, and Ramamurtie (1996) building on the earlier work of Madan and Milne (1994). In this approach the series expansions derive from employing a Hermite polynomial basis for contingent claims modelled as a separable Hilbert space, which is a fundamental function space in mathematical analysis The approach combines functional analysis with follow-up empirical work using data on Eurodollar Future Options in Abken et al (1996) and data on S&P 500 options in Madan and Milne (1994) Basically, after representing the risk-neutral density with respect to the Hermite basis, a truncation is employed in order to construct the statistical structural equations upon which the estimation procedures, such as least squares, are based

It would be very interesting to learn whether one could develop statistical estimation procedures for finite truncations of the series expansion of the option price, (2)

Incomplete Markets

In the earlier sections of the paper, the *Esscher* transform has not arisen because the martingale-determined equation has a unique solution, in this case determined by λ^*. But this situation changes when an *incomplete* market is encountered, for example, in the earlier paper of Gerber and Shiu (1994, Sec. 4), which includes the important example distributions of the shifted Gamma and shifted inverse Gaussian processes That paper and the published discussions of it provide important insights about how to respond to the nonuniqueness that characterizes incomplete markets

In the present paper the authors achieve a fruitful "universality" of their approximation formula for incomplete market models, provided that option prices are calculated by the Esscher method This result adds a further strong justification for using the Esscher approach in incomplete markets

In other non-binomial discrete stock price distributions, a similar equation on martingale probabilities arises without having a unique solution, see Chan and van der Hoek [1996, (2 5)], where the Esscher as well as other approaches are applied In particular, it is shown in Chan and van der Hoek that the Esscher transform distribution delivers an option price close to one determined by the minimum relative entropy distribution when the time steps of the underlying distribution, here $X(1)$, are sufficiently small

It would be interesting to investigate what kinds of approximations would be forthcoming from other approaches to incomplete markets, for example, the entropy approach; see also Buchen and Kelly (1996)

The Gerber and Landry paper opens up new areas of both applied and basic research in nonstandard financial markets

References

ABKEN, P A , MADAN, D B , AND RAMAMURTIE, S 1996 "Estimation of Risk-Neutral and Statistical Densities by Hermite Polynomial Approximations: With an Application to Eurodollar Futures Options," *Technical Report Working Paper 96-5*. Atlanta, Ga : Federal Reserve Bank of Atlanta

PANJER, H.H ED 1998. *Financial Economics: with Applications in Investments, Insurance and Pensions*. Schaumburg, Ill : Society of Actuaries, in press

BUCHEN, P W , AND KELLY, M 1996. "The Maximum Entropy Distribution of an Asset Inferred from Option Prices," *Journal of Financial and Quantitative Analysis* 31:143–59

CHAN, T , AND VAN DER HOEK, J 1996 "Pricing and Hedging Contingent Claims in Incomplete Markets: Discrete Time Models," Technical Report. Adelaide, Australia: Department of Mathematics, The University of Adelaide

GERBER, H U , AND SHIU, E S W 1994 "Option Pricing by Esscher Transforms," *Transactions of the Society of Actuaries* 46:99–140

MADAN, D B , AND MILNE, F 1994. "Contingent Claims Valued and Hedged by Pricing and Investing in a Basis," *Mathematical Finance* 4:223–45

AUTHORS' REPLY

HANS U. GERBER AND BRUNO LANDRY

We thank Dr. Kortanek and Dr Medvedev for their discussion, which captures well the ideas of our paper Since the publication of the paper, the linear approximation and other formulas have been used and tested with real data; see Pafumi (1997). We would also like to point out papers by Chan (1997) and Eberlein and Keller (1995), which contain additional aspects of the method of Esscher transforms

References

CHAN, T. 1997. "Pricing Contingent Claims on Stocks Driven by Lévy Processes," Technical Report. Edinburgh, Scotland: Heriot-Watt University

EBERLEIN, E , AND KELLER, U 1995. "Hyperbolic Distributions in Finance," *Bernoulli* 1:281–99

PAFUMI, G 1997 "A Study of a Family of Equivalent Martingale Measures to Price an Option with an Application to the Swiss Market," *Bulletin of the Swiss Association of Actuaries* no. 2: 159–94

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Angus Macdonald is to be congratulated for recognizing the importance of articulating a broader

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