APPLICATION OF COHERENT RISK MEASURES TO CAPITAL REQUIREMENTS IN INSURANCE

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ABSTRACT

Risk measurements go hand in hand with setting of capital minima by companies as well as by regulators. We review the properties of coherent risk measures and examine their implications for capital requirement in insurance. We also comment on the specific risk-based capital computations.

1. INTRODUCTION

The author thanks the Society of Actuaries for the opportunity to present the application to insurance companies of the work on coherent risk measures, jointly done since early 1995 with F. Delbaen, J.-M. Eber, and D. Heath (see Artzner et al. 1997, 1998), at the initiative of the French Bank Société Générale.

Measurement of risk is a very important element in the prescription of capital requirements. The axiomatic approach chosen in Artzner et al. (1998) allows, even requires, us to search for similarities and differences between the banking and insurance industries. From an internal viewpoint risk measurement is also important for allocation of capital and performance evaluation.

We review in Section 2 the definitions given and the results already obtained in Artzner et al. (1998) for the coherent market-risk measurement systems, which we strongly advocate. We seek here to extend them to the case of insurance. Since we do not assume completeness of markets, there is indeed hope of introducing these definitions and results to the realm of insurance, where the notion of market value is not always present, especially when dealing with the liability side of the balance sheet.

Section 3 is also a review; it describes three risk measurement methods used by regulators and supervisors in financial markets. In any case a company subject to one type of regulation may want to use internally a coherent risk measurement system and benefit from it. We offer a possible solution to this dilemma.

Section 4 is devoted to the risk-based capital method (RBC) of the National Association of Insurance Commissioners (NAIC). It bears some similarities with two systems used in financial markets and for which we suggest improvements in Section 3. We examine in particular how RBC handles the aggregation or merger of risks and diversification effects.

2. REVIEW OF DEFINITION AND PROPERTIES OF COHERENT RISK MEASURES

Taking into account the features of insurance, we report, without proofs, on what has been done in Artzner et al. (1998), where we did the following:

1. Define ''acceptable'' future random values and provided a set of axioms about the set of acceptable future values.
2. Define the measure of risk of an unacceptable position *once a reference, ''prudent,'' investment instrument has been specified* as the minimum extra capital that, invested in the reference instrument, makes the future value of the modified position become acceptable.
3. State axioms on measures of risk and related them to the axioms on acceptance sets. We argued that these axioms should hold for any risk measure that is to be used to effectively regulate or manage risks. We called risk measures that satisfy the four axioms *coherent* (see Section 2.3).
4. Provide a general representation for all risk measures verifying the axioms in terms of ''generalized scenarios'' (see Section 2.4).

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5. Construct the most conservative coherent risk measure extending some measurement given on a special class of risks, taking care of the ''portfolio effect'' (see Section 2.5).
6. Explain how tasks (4) and (5) allow us to construct coherent measures for risks of the multiline type when simple line risks are already coherently measured (see Section 2.6).

2.1 Risk as the Random Variable: Future Value

Although several papers define risk in terms of *changes* in values between two dates, we argue that because risk is related to the variability of the *future value* of a position, due to market changes or more generally to uncertain events, it is better only to consider future values instead. Notice that there is no need for the initial costs of the components of the position to be determined from universally defined market prices (think of over-the-counter transactions). The principle of ''bygones are bygones'' leads to this ''future wealth'' approach.

The basic objects of our study are therefore the random variables on the set of states of nature at a future date, interpreted as possible future values of positions or portfolios currently held. A first, crude but crucial, measurement of the risk of a position is whether its future value belongs or does not belong to the subset of *acceptable risks*, as decided by supervisors such as

a. An *exchange's clearing firm*, which has to make good on the promises to all parties of transactions being securely completed (see Section 3.1 for the SPAN system)
b. A *regulator*, who takes into account the unfavorable states when allowing a risky position that may draw on the resources of the government, for example, as a guarantor of last resort
c. An investment *manager*, who knows that his or her firm has basically given to its traders an exit option in which the strike ''price'' consists in being fired in the event of big trading losses on one's position.

In each case there is a trade-off between severity of the risk measurement and level of activities in the supervised domain. The axioms and characterizations we provided do not single out a specific risk measure, and additional economic considerations have to play a role in the final choice of a measure.

Remark

When—as most of the time—no secondary market for insurance liabilities exists, one has to rely on actuarial valuation of reserves to define a net value or ''surplus'' (see Daykin, Pentikainen, and Pesonen 1994, pp. 11, 423).

For an *unacceptable risk* (that is, a position with an unacceptable future value) one remedy may be to alter the position. Another remedy is to look for some commonly accepted instruments that, added to the current position, make its future value acceptable to the regulator/supervisor. The current cost of getting enough of one or more of these instruments is a good candidate for a measure of risk of the initially unacceptable position.

For simplicity, we consider only one period of uncertainty $(0, T)$ between two dates 0 and T. This is a very severe restriction for relating theory to actual practice, but we want to show that many problems occur and many choices have to be made, even in this simple setting. One ''reference'' instrument is given, which carries one unit of date 0 into r units of date T. The default-free zero-coupon bonds with maturity at date T can be chosen as a particularly simple reference instrument. Other possible reference instruments are mentioned in Section 2.3, before the statement of Axiom T.

The period $(0, T)$ can be the period between hedging and rehedging, a fixed interval like two weeks, the period required to liquidate a position, or the length of coverage provided by an insurance contract.

We take the point of view of a firm subject to regulations and/or supervision in only one country but considered in Artzner et al. (1998) the question of currency risk. For the sake of simplicity, it is not addressed here.

A firm's initial portfolio consists of positions A (possibly within some institutional constraints, like the absence of short sales and a ''congruence'' between assets and liabilities). The position A provides a firm's value $A(T)$ at date T, and we call *risk* this *future, random* value.

2.2 The Set of Accepted Future Values and the Related Measure of Risk

We suppose that the set of all possible states of the world at the end of the period is known, but the probabilities of the various states occurring may be unknown or not subject to common agreement. When we deal with market risk, the state of the world might be described by a list of the prices of all securities, and we assume that the set of all possible such lists is known. Of course, this assumes that markets at date T are liquid; if they are not, more complicated models are required, in which we can distinguish the risks of a position and of a future net worth, since, with illiquid markets, the mapping from the former to the latter may not be linear.

Notation

In this paper we use the following notation.

a. We call Ω the set of states of nature and assume it is finite. Considering Ω as the set of outcomes of an experiment providing asset prices and claims sizes, we compute the final value of a position for each element of Ω. It is a random variable denoted by X. Its negative part, $\max(-X, 0)$, is denoted by X^-, and the supremum of X^- is denoted by $\|X^-\|$. The random variable identically equal to 1 is denoted by **1**. The indicator function of state ω is denoted by $\mathbf{1}_{\{\omega\}}$.
b. Let $\mathcal{G}$ be the set of all risks, that is, the set of all real valued functions on Ω. Since Ω is assumed to be finite, $\mathcal{G}$ can be identified with $\mathbb{R}^n$, where $n = card(\Omega)$.
c. We call $\mathcal{A}$ the set of final values accepted by the regulator.

In Artzner et al. (1998), we first stated axioms for acceptance sets and then translated them into axioms on risk measures, reported below.

Sets of acceptable future values are the primitive objects considered to describe acceptance or rejection of a risk. We present here how, *given* some ''reference instrument,'' there is a natural way to define a measure of risk by describing how close or how far from acceptance a position is.

Definition 2.1

A measure of risk is a mapping from $\mathcal{G}$ into $\mathbb{R}$

In Section 3 we speak of a *model-dependent* measure of risk when an explicit probability on Ω is used to construct it (see, for example, Sections 3.1 and 3.3) and of a *model-free* measure otherwise (see, for example, Section 3.2). Model-free measures can still be used in the case where only risks of positions are considered.

When positive, the number $\rho(X)$ assigned by the measure ρ to the risk X is interpreted (see Definition 2.2) as the minimum extra cash the agent has to *add* to the risky position X *and* to invest ''prudently,'' that is, in the reference instrument, to be allowed to proceed with his plans. If it is negative, the cash amount $-\rho(X)$ can be withdrawn from the position or received as restitution as in the case of organized markets for financial futures.

Remark 1

It may be surprising that we define a measure of risk on the *whole* of $\mathcal{G}$. Why, in particular, should we consider a risk, a final net worth, like the constant **−1**? No one would or could willingly enter into a deal that for sure entails a negative of final net value equal to 1.

Let us provide three answers.

a. We want to extend the accounting procedures dealing with future certain bad events (like loss in inventories, degradation—wear and tear—of physical plant) into measurement procedures for future uncertain bad events.
b. Actual measurements used in practice seem to be defined only for risks where *both* states with positive and states with negative final net worth exist. Section 2.5 shows that, under well-defined conditions, they can be extended without ambiguity to measurements for *all* functions in $\mathcal{G}$.
c. Multiperiod models may naturally introduce at some intermediate date the prospect of such final values.

Remark 2

It has been pointed out that describing risk ''by a single number'' involves a great loss of information. However, the actual decision about taking a risk or allowing one to take it, is fundamentally binary, of the ''yes or no'' type, and we claimed at the beginning of Section 2.1 that this is the actual origin of risk measurement.

Remark 3

The expression ''cash'' deserves some discussion in the case of a publicly traded company. It refers to an increase in equity. The amount $\rho(X)$ may, for example, be used to lower the amount of debt in the balance sheet of the company.

We define a correspondence between acceptance sets and measures of risk.

Definition 2.2

Risk measure associated to an acceptance set: given the total rate of return r on a reference instrument, the risk measure associated to the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{G}$ to $\mathbb{R}$ denoted by $\rho_{\mathcal{A},r}$ and defined by

$$\rho_{\mathcal{A},r}(X) = \inf\{m \mid m \cdot r + X \in \mathcal{A}\}.$$

Definition 2.3

Acceptance set associated to a risk measure: the acceptance set associated to a risk measure ρ *is the set denoted by* $\mathcal{A}_\rho$ *and defined by*

$$\mathcal{A}_\rho = \{X \in \mathcal{G} | \rho(X) \leq 0\}.$$

2.3 Axioms for Coherent Risk Measures

We consider now several possible properties for a risk measure ρ defined on $\mathcal{G}$. The first requirement ensures that the risk measure is stated in the same units as the final value, except for the use of the reference instrument. This particular asset is modeled as having the initial price 1 and a strictly positive price r (or *total return*) in any state of nature at date T. It is the regulator's responsibility to accept for r possible random values as well as values smaller than 1.

Axiom T means that adding (respectively subtracting) the sure initial amount α to the initial position and investing it in the reference instrument simply decreases (respectively increases) the risk measure by α.

Axiom T

Translation invariance: for all $X \in \mathcal{G}$ *and all real numbers* α, $\rho(X + \rho \cdot r) = \rho(X) - \alpha$.

Axiom S

Subadditivity: for all X_1 *and* $X_2 \in \mathcal{G}$, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

We contend that this property, which could be stated in the succinct form ‘‘a merger does not create extra risk,’’ is a natural and essential requirement:

a. If an *exchange's* risk measure were to fail to satisfy this property, then, for example, an individual wishing to take the risk $X_1 + X_2$ may open two accounts, one for the risk X_1 and the other for the risk X_2, incurring the smaller margin requirement of $\rho(X_1) + \rho(X_2)$, a matter of concern for the *exchange*.
b. If a *firm* were forced to meet a requirement of extra capital that did not satisfy this property, the firm might be motivated to break up into two separately incorporated affiliates, a matter of concern for the *regulator*.
c. Bankruptcy risk inclines *society* to require less capital from a group without ‘‘firewalls’’ between various business units than it requires when one unit is protected from liability attached to failure of another unit.
d. Suppose that two *desks* in a firm compute, in a decentralized way, the measures $\rho(X_1)$ and $\rho(X_2)$ of the risks they have taken. If the function ρ is subadditive, the *supervisor* of the two desks can count on the fact that $\rho(X_1) + \rho(X_2)$ is a feasible guarantee relative to the global risk $X_1 + X_2$. If he or she has an amount m of cash available for their joint business, the supervisor knows that imposing limits m_1 and m_2 with $m = m_1 + m_2$ allows the firm to decentralize the cash constraint into two cash constraints, one per desk. Similarly, the firm can allocate its capital among managers.

Axiom PH

Positive homogeneity: for all $\lambda \geq 0$ *and all* $X \in \mathcal{G}$, $\rho(\lambda X) = \lambda\rho(X)$.

Remark 1

If position size directly influences risk (for example, if positions are large enough that the time required to liquidate them depends on their sizes), then we should consider the consequences of lack of liquidity when computing the future value of a *position*. With this in mind, Axioms S and PH about mappings from *random variables* into the reals remain reasonable.

Remark 2

Axiom S implies that $\rho(nX) \leq n\rho(X)$ for $n = 1, 2, \ldots$. In Axiom PH we have imposed the reverse inequality (and require equality for all positive λ) to model what a government or an exchange might impose in a situation in which no netting or diversification occurs, in particular because the government does not prevent many firms all to take the same position.

Remark 3

Axioms T and PH imply that for each α, $\rho(\alpha \cdot (-r)) = \alpha$.

Axiom M

Monotonicity: for all X *and* $Y \in \mathcal{G}$ *with* $X \leq Y$, $\rho(Y) \leq \rho(X)$.

Remark

Axiom M rules out the risk measure defined by $\rho(X) = -\mathbf{E}_{\mathbb{P}}[X] + \alpha \cdot \sigma_{\mathbb{P}}(X)$, where $\alpha > 0$ and $\sigma_{\mathbb{P}}$ denotes the standard deviation operator, computed under the probability $\mathbb{P}$. Axiom S rules out the ‘‘semivariance’’-type risk measure defined by $\rho(X) = -\mathbf{E}_{\mathbb{P}}[X] + \sigma_{\mathbb{P}}((X - \mathbf{E}_{\mathbb{P}}[X])^-)$.

Axiom R

Relevance: for all $X \in \mathcal{G}$ *with* $X \leq 0$ and $X \neq 0$, $\rho(X) > 0$.

Remark

This axiom is clearly necessary, but not sufficient, to prevent concentration of risks to remain undetected.

The following choice of required properties defines coherent risk measures.

Definition 2.4

Coherence: a risk measure satisfying the four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity is called coherent.

Axiom C

Conservatism: for all $X \in \mathcal{G}$, $\rho(\mathbf{X}) = \rho(\mathbf{X}^-)$.

Axiom C says that the risk measure depends only on the possibility of states of nature with a negative final value; states with final *positive* value are irrelevant. This reflects very strict attitudes and very risk-adverse utility functions on the part of a manager or of a regulator.

Remark

Professor Hans Bühlmann kindly provided us with references to works by Hattendorff (1868), Kanner (1867), and Wittstein (1867) he mentioned in his Göttingen presentation (1995). These authors considered, in the case of insurance risks, possible losses *only* (see the quotation in Remark 2 following the proof of Proposition 2.1). For example, risk for a company providing annuities was linked to the random *excess* number of survivors over the expected number given by the life table.

Our axioms for coherence are not restrictive enough to specify a unique risk measure. They instead characterize a large class of risk measures. The choice of precisely which measure to use (from this class) should presumably be made on the basis of additional economic considerations.

2.4 Representation of Coherent Risk Measures by Scenarios

This subsection provides a representation of coherent risk measures by showing that *any* coherent risk measure arises as the supremum of the expected negative of final value, for some collection of ''generalized scenarios'' or probability measures on states of the world. Section 3.1 provides a practical example: the Chicago Mercantile Exchange's SPAN system.

Proposition 2.1

Given the total return r on a reference investment, a risk measure ρ is coherent if and only if there exists a family $\mathcal{P}$ *of probability measures on the set of states of nature, such that*

$$\rho(X) = \sup\{\mathbf{E}_{\mathbb{P}}[-X/r] \mid \mathbb{P} \in \mathcal{P}\}.$$

Remark 1

We note that ρ can also be seen as an insurance premium principle. In that case, denoting by $\mathbb{Q}$ the physical measure, we find that the condition $\mathbb{Q} \in \mathcal{P}$ (or in the convex hull of this set) is of great importance. This condition is translated as follows: for all $X \leq 0$ we have $\mathbf{E}_{\mathbb{Q}}[-X/r] \leq \rho(X)$.

Remark 2

The more scenarios considered, the more conservative (that is, the larger) is the risk measure obtained.

Proof of Proposition 2.1

1. We thank a referee of Artzner et al. (1998) for pointing out that the mathematical content of Proposition 2 is already in Huber (1981). We therefore simply identify the terms in Proposition 2 (chapter 10 of Huber) with these of our terminology of risks and risk measure.
2. The sets Ω and $\mathcal{M}$ of Huber (1981) are our set Ω and the set of probabilities on Ω. Given a risk measure ρ we associate to it the functional E^* by $E^*(X) = \rho(-r \cdot X)$. Axiom M for ρ is equivalent to Property (2.7) of Huber for E^*, Axioms PH and T together are equivalent to Property (2.8) for E^*, and Axiom S is Property (2.9).
3. The ''if'' part of Proposition 2.1 is obvious. The ''only if'' part results from the ''representability'' of E^*, since Proposition 2 of Huber (1981) states that

$$\rho(X) = E^*(-X/r) = \sup\{\mathbf{E}_{\mathbb{P}}[-X/r] \mid \mathbb{P} \in \mathcal{P}_\rho\},$$

where $\mathcal{P}_\rho$ is defined as the set

$$\{\mathbb{P} \in \mathcal{M} \mid \text{ for all } X \in \mathcal{G}\text{: } \mathbf{E}_{\mathbb{P}}[X] \leq \mathbf{E}^*(X) = \rho(-r \cdot X)\}$$
$$= \{\mathbb{P} \in \mathcal{M} \mid \text{ for all } Y \in \mathcal{G}\text{: } \mathbf{E}_{\mathbb{P}}[-Y/r] \leq \rho(Y)\}.$$

□

Remark 1

Model risk can be taken into account by including into the set $\mathcal{P}$ a family of distributions for the future prices, possibly arising from other models.

Remark 2

The papers by Hattendorff, Kanner, and Wittstein mentioned above give an example of a risk measure used in life insurance, namely, the ''mittlere Risico'' constructed out of one scenario, related to the life table used by a company. It is defined as the mathematical expectation of the *positive part* of the loss, as ''die Summe aller möglichen Verluste, jeden multipliciert in die Wahrscheinlichkeit seines Eintretens,'' that is, the sum of all possible losses, each being multiplied by the probability of its occurrence. This procedure defines a risk measure satisfying Axioms S, PH, and M.

Remark 3

It is important to distinguish between a point-mass scenario and a simulation trial: the first is chosen by the investor or the supervisor, whereas the second is chosen randomly according to a *distribution* that has been prescribed beforehand.

Conclusion

The result in Proposition 2.1 completely explains the occurrence of a first type of actual risk measurement: the SPAN method, based on scenarios, described in Section 3.1. Any coherent risk measure therefore appears as given by a ''worst-case method'' in a framework of generalized scenarios. The systematic use of scenarios instead of value-at-risk measurement (see Section 3.1) *supplemented* by stress scenarios is the key to coherence and its advantages.

At this point we emphasize that scenarios should be announced to all traders within the firm (by the manager) or to all firms (by the regulator). In the first case, we notice that decentralization of risk management within the firm is available only *after* these announcements.

2.5 Construction of Coherent Risk Measures by Extension of Certain Risk Measurements

Section 3.2 describes a measure of risks built as follows. Margin requirements are given for certain basic portfolios considered as ''standard risks.'' Combinations of those risks are then used to ''support'' other risks for which required capital is bound above, using the margins required for the ''supporting'' standard risks. It is shown that some implicit scenarios are involved, and it is interesting to address the question of extending some risk measurement into a coherent risk measure.

Definition 2.5

Supports of a risk: given a set $\mathcal{Y}$ of functions on Ω, we consider a family, indexed by $\mathcal{Y}$, of nonnegative numbers $\mu = (\mu_Y)_{Y\in\mathcal{Y}}$, all of them but a finite number being zero, and we say that the couple (μ, γ), where γ is a real number, ''supports'' X, for $X \in \mathcal{G}$, provided

$$X \geq \sum_{Y\in\mathcal{Y}} \mu_Y Y + \gamma \cdot r.$$

The set of all such (μ, γ) that support X is denoted by $S_{\mathcal{Y}}(X)$.

The idea is now to use these ''supports,'' made of ''standard risks,'' to bound from above the possible extensions of a function Ψ defined on a subset of $\mathcal{G}$. A consistency condition is required to avoid supports leading to infinitely negative values.

Condition 2.1

Given a set $\mathcal{Y}$ of functions on Ω, and a function Ψ: $\mathcal{Y} \to \mathbb{R}$, we say that Ψ fulfills Condition 2.1 if for each support (μ, γ) of 0, we have the inequality $\Sigma_{Y\in\mathcal{Y}}\mu_Y\Psi(Y) - \gamma \geq 0$.

Proposition 2.2

Given a set $\mathcal{Y}$ of functions on Ω and a function $\Psi:\mathcal{Y} \to \mathbb{R}$, the equality

$$\rho_\Psi(X) = \inf_{(\mu,\gamma)\in S_{\mathcal{Y}}(X)} \sum_{Y\in\mathcal{Y}} \mu_Y \Psi(Y) - \gamma$$

defines a coherent risk measure ρ_Ψ, if and only if Ψ fulfills Condition 2.1. If so, ρ_Ψ is the largest coherent measure ρ such that $\rho \leq \Psi$ on $\mathcal{Y}$.

Remark

As opposed to the case of scenario-based measures of risks, the fewer initial standard risks are considered, the more conservative is the coherent risk measure obtained. This is similar to what happens with the SEC rules described in Section 3.2, where too many scenarios—and, dually, too few standard risks—are considered.

Condition 2.1 allows us to consider the function ρ_Ψ in particular on the set $\mathcal{Y}$, the set of prespecified risks. There it is clearly bounded above by the original function Ψ. An extra consistency condition will prove helpful in figuring out whether ρ_Ψ is actually equal to Ψ on $\mathcal{Y}$.

Condition 2.2

Given a set $\mathcal{Y}$ *of functions on* Ω *and a function* $\Psi: \mathcal{Y} \to \mathbb{R}$, *we say that Condition 2.2 is satisfied by* Ψ *if for each element* $Z \in \mathcal{Y}$ *and each support* (μ, γ) *of* Z *we have* $\Psi(Z) \leq \Sigma_{Y \in \mathcal{Y}} \mu_Y \Psi(Y) - \gamma$.

Remark

It is an easy exercise to prove that Condition 2.2 implies Condition 2.1.

Proposition 2.3

Given a set $\mathcal{Y}$ *of functions on* Ω *and a function* $\Psi: \mathcal{Y} \to \mathbb{R}_+$ *satisfying Condition 2.2, the coherent risk measure* ρ_Ψ *is the largest possible extension of the function* Ψ *to a coherent risk measure.*

Propositions 2.2 and 2.3 applied to $(\mathcal{Y}, \Psi) = (\mathcal{G}, \rho)$ provide a statement similar to Proposition 2.1 about representation of coherent risk measures.

Proposition 2.4

A risk measure ρ *is coherent if and only if it is of the form* ρ_Ψ *for some* Ψ *fulfilling Condition 2.1.*

Remark

It can be shown that for a coherent risk measure ρ built as a ρ_Ψ, the set of probabilities

$$\mathcal{P}_\psi = \{\mathbb{P} \mid \text{ for all } X \in \mathcal{G}: \mathbf{E}_\mathbb{P}[-X/r] \leq \Psi(X)\}$$

is nonempty and verifies the property

$$\rho(X) = \sup\{\mathbf{E}_\mathbb{P}[-X/r] \mid \mathbb{P} \in \mathcal{P}_\Psi\}.$$

2.6 A Simple Example of Measure for Composite Risks

We presented in Artzner et al. (1998) an application of combining the results of Sections 2.4 and 2.5. Given two measures already used on different lines of (risky) business, this application constructs the *most* conservative risk measure of contracts involving the two sources of uncertainty and equal to the former measures on the original contracts with only one source of uncertainty. It therefore provides a conservative *coherent* tool for risk management. Multiline aggregated combined risk optimization tools (see Shimpi 1998) would call for such a combined measure of risks. We give below the simplest example possible.

Let Ω be $\Omega_1 \times \Omega_2$, where $\Omega_1 = \{good, bad\}$ and $\Omega_2 = \{high, low\}$. Let $\mathcal{P}_1 = \{(0.5, 0.5), (0.1, 0.9)\}$ and $\mathcal{P}_2 = \{(0.6, 0.4), (0.2, 0.8)\}$ be two sets of scenarios on Ω_1 and Ω_2 respectively. Let ρ_i be the measure of risks from the i-th line of business (that is, random variables on Ω_i), which is defined by the scenarios in $\mathcal{P}_i$. We define a measure ρ for multiline risks involving uncertainty from both Ω_1 and Ω_2 (that is, random variables on the product space Ω) by using any scenario on Ω that, if "restricted" to Ω_i, provides a scenario already considered, that is, a scenario belonging to $\mathcal{P}_i$. For this purpose we use the set $\mathcal{P}$ of scenarios, that is, probabilities, (p, q, r, s) on $\Omega = \{(good, high), (bad, high), (good, low), (bad, low)\}$, and impose the following restrictions on its marginals: $(p + r, q + s) \in \mathcal{P}_1$ and $(p + q, r + s) \in \mathcal{P}_2$. The general results in Artzner et al. (1998) prove that ρ is the most conservative risk measure compatible with the two initial risk measures used for the individual lines of business.

3. REVIEW OF THREE METHODS OF MEASURING RISK USED IN FINANCIAL MARKETS

In this section we briefly report on the (simplified) description given in Artzner et al. (1998) and the effects on risk management of three currently used methods of measuring market risk:

a. SPAN (see SPAN 1995), developed by the Chicago Mercantile Exchange
b. The Securities Exchange Commission rules used by the National Association of Securities Dealers (see National Association of Securities Dealers 1996 and Board of Governers 1994 [hereafter NASD and Fed, respectively]), similar to rules used by the Pacific Exchange and the Chicago Board of Options Exchange
c. The quantile-based value-at-risk (VaR) method (see Basle Committee 1996; Dowd 1998; Derivatives Policy Group 1995).

We examine the relationship of these three methods with the abstract approach provided in Section 2, and we suggest improvements to the NASD-Fed and VaR methods. In particular, we use the representation results to suggest a specific coherent measure called tail conditional expectation (TailVaR). Tail conditional expectation is, under some assumptions, the least expensive among the risk measures that are coherent and acceptable by regulators since being more conservative than the VaR measurement.

3.1 An Organized Exchange's Rules: The SPAN Computations

It is surprising that most studies on market-risk measurement concentrate on the systems used by regulators (see Sections 3.2 and 3.3) and ignore the (coherent) systems in use on several organized exchanges. The development of alternative risk transfers in insurance is also an incentive for studying these systems.

To illustrate the SPAN margin system (see SPAN 1995) of the Chicago Mercantile Exchange, we consider how the initial margin is calculated for a simple portfolio consisting of units of a futures contract and of several puts and calls with a common expiration date on this futures contract. The SPAN margin for such a portfolio is computed as follows. First, 14 ''scenarios'' are considered. Each scenario is specified by an up or down move of volatility combined with no move, or an up move, or a down move of the futures price by 1/3, 2/3, or 3/3 of a specified ''range.'' Next, two additional scenarios relate to ''extreme'' up or down moves of the futures price. The measure of risk is the *maximum* loss incurred, using the full loss for the first 14 scenarios and only 35% of the loss for the last two ''extreme'' scenarios. A specified model, typically the Black model, is used to generate the corresponding prices for the options under each scenario.

The calculation can be viewed as producing the maximum of the expected loss under each of 16 probability measures. For the first 14 scenarios the probability measures are point masses at each of the 14 points in the space Ω of securities prices. The cases of extreme moves correspond to taking the convex combination (0.35, 0.65) of the losses at the ''extreme move'' point under study and at the ''no move at all'' point (that is, prices remain the same). We shall call these probability measures ''generalized scenarios.''

The account of the investor holding a portfolio is required to have sufficient current net worth to support the maximum expected loss. If it does not, then extra cash is required as a margin call, in an amount equal to the ''measure of risk'' involved. This is completely in line with our interpretation of Definition 2.2.

The following definition generalizes the SPAN computation and presents it in our framework.

Definition 3.1

The risk measure defined by a nonempty set $\mathcal{P}$ of probability measures or ''generalized scenarios'' on the space Ω and the total return r on a reference instrument is the function $\rho_{\mathcal{P}}$ on $\mathcal{G}$ defined by

$$\rho_{\mathcal{P}}(X) = \sup\{\mathbf{E}_{\mathbb{P}}[-X/r] \mid \mathbb{P} \in \mathcal{P}\}.$$

The scenario-based measures from Definition 3.1 are coherent risk measures.

Proposition 3.1

Given the total return r on a reference instrument and the nonempty set $\mathcal{P}$ of probability measures, or ''generalized scenarios,'' on the set Ω of states of the world, the risk measure $\rho_{\mathcal{P}}$ of Definition 3.1 is a coherent risk measure. It satisfies the relevance axiom if and only if the union of the supports of the probabilities $\mathbb{P} \in \mathcal{P}$ is equal to the set Ω.

Proof of Proposition 3.1

Axioms PH and M ensure that a coherent risk measure satisfies Axiom R if and only if the negative of each indicator function $\mathbf{1}_{\{\omega\}}$ has a (strictly) positive risk measure. This is equivalent to the fact that any state belongs to at least one of the supports of the probabilities found in the set $\mathcal{P}$. □

Section 2.4 had shown that *each* coherent risk measure is obtained by way of scenarios.

3.2 Some Model-Free Measures of Risks: The SEC Rules

The second example of a risk measure used in practice is found in the rules of the Securities and Exchange Commission and the National Association of Securities Dealers. Their approach is to consider portfolios as formal *lists* of securities and impose ''margin'' requirements on *them*, in contrast to the SPAN approach, which takes the random *variables*—gains and losses of the portfolios of securities—as basic objects to measure. In the terminology of the Basle Committee (1996) we have here something similar to a ''standardized measurement method'' that does not include a model of distribution of prices but may involve scenarios, even too many of them. We shall see later that the risk-based capital computation has some features of this method.

Certain spread positions on some underlying instrument, like a long call and a short call on the same stock but with higher exercise price (both calls having the same maturity date), are described in NASD (1996), p. 8133, where SEC rule 15c3-1a,(11) requires no margin (no ''deduction''). No justification is given for this specification. We will see that a few other portfolios should have been free of margin requirements.

Let A be a portfolio consisting of two long calls with strike 10, two short calls with strike 20, three short calls with strike 30, four long calls with strike 40, and one short call with strike 50. For simplicity assume all calls are European and exercise dates are equal to the end of the holding period. A simple graph shows that the final value of this position is never below -10, which should entail a margin deposit of *at most* 10.

Under the SEC method, the position A is represented or ''decomposed'' as a portfolio of long call spreads. No margin is required for a spread if the strike of the long side is less than the strike of the short side. A margin of $K - H$ is required for the spread consisting of a long call with strike K and a short call with strike H, when $H \leq K$. The margin resulting from a representation or ''decomposition'' is the sum of the margins attached to each call spread. The investor is presumably able to choose the best possible representation. A simple linear programming computation shows that 30 is the resulting minimum, that is, much more than the negative of the worst possible future value of the position. We therefore have to search for a strange feature of this ''list the elements of the portfolio and apply coefficients'' type of procedure.

We formalize the special role played by the call spreads, which we call ''standard risks,'' and the natural margin requirements on them, following the lines of Rudd and Schroeder (1982, Section 4; see also Cox and Rubinstein 1985, pp. 107–109). Given some underlying security, we denote by C_K the European call with exercise price K and exercise date equal to the end of the holding period and by $S_{H,K}$ the spread portfolio consisting of ''one long C_H, one short C_K,'' which we denote by $C_H - C_K$. These spreads are ''standard risks'' for which a simple rule of margin requirement is given. They are then used to ''support'' general portfolios of calls and provide conservative capital requirements.

We describe the extra capital requirement for a portfolio A consisting of a_H calls C_H, $H \in \mathcal{H}$, $\mathcal{H}$ being a finite set of strikes. For simplicity we assume that $\Sigma_H\, a_H = 0$, that is, we have no net long or short position. The exchange allows us to compute the margin for such a portfolio A by solving the linear programming problem

$$\inf_{n_{H,K}} \sum_{H,K,H\neq K} n_{H,K}(H - K)^+$$

under the conditions that

$$\text{for all } H, K, H \neq K, \text{ we have } n_{H,K} \geq 0 \quad \text{and}$$

$$A = \sum_{H,K,H\neq K} n_{H,K} S_{H,K}.$$

This program provides the holder of portfolio A with the cheapest decomposition ensuring that *each* spread showing in it has a nonnegative net worth at date T.

It is shown in Artzner et al. (1998) that taking the dual of the problem above exactly explains why too much capital had been required for portfolio A above. The measurement method indeed implies scenarios of call prices against which the portfolio A is ''tested.'' Some of these scenarios do not fulfill elementary, model-independent, no-arbitrage conditions like the convexity of the call price with respect to the exercise price (see Theorem 8.4 p. 262, in Merton 1973). To avoid these difficulties, it would have been necessary to declare, from the outset, as margin free the ''butterfly'' portfolio $B(H, K) = C_H - 2C_{H+K} + C_{H+2K}$. In this case, the margin on $A = 2B(10, 10) + 2B(20, 10) - B(30, 10)$ would have been at most the margin on $-B(30, 10)$, that is, 10, as intuitively required.

In conclusion, by considering ''too few'' margin-free standard risks and, implicitly, ''too many'' scenarios, in particular impossible ones, the SEC method provides a risk measure that is too strict for some portfolios. This is a *warning* about stress-testing as well as about margining rules dealing only with notional values in a model-free setting.

3.3 Some Model-Dependent Rules Based on Quantiles: Value at Risk

The last example of measures of risk used in practice is the VaR measure. We consider its origin to be in actuarial risk theory, although, as one can expect from the remark at the beginning of Section 2.1, value at risk in finance deals with market values, while ruin probabilities consider surplus.

Value at risk is usually defined in terms of net wins or P/L and therefore ignores the difference between money at one date and money at a different date, which, for small time periods and a single currency, may be acceptable. It uses quantiles, which requires us to pay attention to discontinuities and *intervals* of quantile numbers.

Definition 3.2

Quantiles: given $\alpha \in\ ]0, 1[$ *the number* q *is an* α*-quantile of the random variable* X *under the probability distribution* $\mathbb{P}$ *if one of the three equivalent properties below is satisfied:*

a. $\mathbb{P}[X \le q] \ge \alpha \ge \mathbb{P}[X < q]$
b. $\mathbb{P}[X \le q] \ge \alpha$ *and* $\mathbb{P}[X \ge q] \ge 1 - \alpha$
c. $F_X(q) \ge \alpha$ *and* $F_X(q^-) \le \alpha$ *with* $F_X(q_-) = \lim_{x \to q, x<q} F(x)$, *where* F_X *is the cumulative distribution function of* X.

Remark

The set of such α-quantiles is a closed interval. Since Ω is finite, there is a finite left- (respectively right-) end point q_α^- (respectively q_α^+) that satisfies $q_\alpha^- = \inf\{x \mid \mathbb{P}[X \le x] \ge \alpha\}$ (equivalently $\sup\{x \mid \mathbb{P}[X \le x] < \alpha\}$) (respectively $q_\alpha^+ = \inf\{x \mid \mathbb{P}[X \le x] > \alpha\}$). With the exception of at most countably many α, the equality $q_\alpha^- = q_\alpha^+$ holds. The quantile q_α^- is the number $F^{\leftarrow}(\alpha) = \inf\{x \mid \mathbb{P}\{X \le x\} \ge \alpha\}$ defined in Embrechts, Klüppelberg, and Mikosch (1997, Definition 3.3.5).

We formally define VaR as follows.

Definition 3.3

Value-at-risk measurement: given $\alpha \in]0, 1[$ *and a reference instrument r the value at risk* VaR_α *at level* α *of the final net worth X with distribution* $\mathbb{P}$ *is the negative of the quantile* q_α^+ *of* X/r, *that is,*

$$VaR_\alpha(X) = -\inf\{x \mid \mathbb{P}[X \le x \cdot r] > \alpha\}.$$

Remark 1

Notice that what we are using to define VaR_α is really the amount of additional capital that a VaR_α-type calculation entails.

Remark 2

We have what is called an ''internal'' model in Basle Committee (1996), and it is not clear whether the (estimated) physical probability or a ''well-chosen'' subjective probability should be used.

Remark 3

Casualty actuaries working in actuarial risk theory have been involved, since the beginning of this century, in computation and use of quantiles. The choice of ''initial capital'' controls the probability of ruin at date T. Loosely speaking, ''ruin'' is defined in (retrospective) terms by the negativity, at date T, of the *surplus,* defined *there* (see Bowers et al. 1997, p. 399) to be

$$Y = \text{capital at date } 0 + \text{premium received}$$
$$- \text{claims paid (from date 0 to date } T).$$

Imposing an upper bound $1 - \alpha$ on the probability of Y being negative determines the initial capital via a quantile calculation (see the survey article by Hans Bühlmann, ''Tendencies of Development in Risk Theory,'' in Society of Actuaries 1989, pp. 499–522). Another meaning and use of *surplus* for risk measurement have already been mentioned in the first remark in Section 2.1.

We show now that, while satisfying Axioms T, PH, and M, VaR_α *fails* to satisfy the subadditivity property. Consider, as an example, the following two digital options on a stock, with the same exercise date T, the end of the holding period. The first option denoted by A (initial price u) pays 1,000 if the value of the stock at time T is more than a given U, and nothing otherwise, while the second option denoted by B (initial price l) pays 1,000 if the value of the stock at T is less than L (with $L < U$), and nothing otherwise.

Choosing L and U such that $\mathbb{P}\{S_T < L\} = \mathbb{P}\{S_T > U\} = 0.008$ we look for the 1% values at risk of the future net worths of positions taken by two traders writing respectively two options A and two options B. They are $-2u$ and $-2l$ respectively (r supposed to be 1). By contrast, the positive number $1{,}000 - l - u$ is the 1% value at risk of the future net worth of the position taken by a trader writing $A + B$. This implies that the set of acceptable net worths (in the sense of Definition 2.3 applied to the value-at-risk measure) is not convex. Notice that this feature is even worse than the nonsubadditivity of the measurement.

Remark 1

Some sort of superior form of subadditivity is desired in a statement like ''In estimating requirements with respect to each category of assets, we attempted to set a level which would provide between 90 percent and 92 percent coverage of default losses in that category. It is our expectation that the aggregation of all C-1 requirements for all asset categories, then, results in total capital for the larger, diversified portfolio covering up to 95 percent or more of default losses'' (see Webb and Lilly 1994, p. 134; this book is used here as a compendium of various documents on risk-based capital). This may translate in this paper's notations, for example, as $VaR_{0.08}(X) + VaR_{0.08}(Y) \ge VaR_{0.05}(X + Y)$. This example of digital options, admittedly a very special one, shows that much caution has to be exercised in order to obtain this kind of relation when aggregating risks.

Remark 2

We note that if quantiles are computed under a distribution for which all prices are jointly normally distributed, then the quantiles do satisfy subadditivity as long as probabilities of exceedance are smaller than 0.5. Indeed, $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$ for each pair (X, Y) of random variables. Since for a normal random variable X we have

$$VaR_\alpha(X) = -(\mathbf{E}_{\mathbb{P}}[X] + \Phi^{-1}(\alpha) \cdot \sigma_{\mathbb{P}}(X)),$$

with Φ the cumulative standard normal distribution, and since $\Phi^{-1}(0.5) = 0$, the proof of subadditivity follows. These are very special circumstances for the surplus in non-life insurance, under which the ''capital at risk'' is a measure that possesses the subadditivity property. This is judged by the U.K. Government Actuary as of ''general interest'' in the context of companies mergers (see Daykin et al. 1994, pp. 157, 168), quite in the spirit of comments following Axiom S in Section 2.3. These special circumstances also allow us to study in parallel the two relationships of risk and capital and risk and return, as shown in Schnieper (1997). With nonnormal distributions for returns, it is no longer possible to gain direct information on *return* to be expected by knowing the *risk* accepted without going into the computation of an amount of required capital.

Remark 3

Several works on quantile-based measures (see Dowd 1998; Risk 1996) consider mainly the *computational* and *statistical* problems they raise, without first considering the implications of this method of measuring risks.

Remark 4

We do not know of organized exchanges using value at risk as the basis of risk measurement for margin requirements.

Remark 5

In quantile-based methods, even after the announcements of individual limits, a problem preventing decentralized risk management remains: as the example of digital options showed, two operators ignorant of each other's actions may well each comply with their individual quantile limits, and yet no automatic procedure provides for an interesting upper bound for the measure of the joint risk due to their actions.

Another bad feature of the value-at-risk measurement is its inability to recognize an undue *concentration* of risks. A remarkably simple example concerning credit risk is due to Claudio Albanese (see Albanese 1997). Assume that the base rate of interest is zero, and that the spreads on all corporate bonds is 2%, while these bonds default, independently from company to company, with a (physical) probability of 1%. If an amount of 1,000,000 borrowed at the base rate is invested in the bonds of a single company, the 5% value at risk of the resulting position is negative, namely, $-20{,}000$, and there is ''no risk.''

If, in order to diversify, the whole amount is invested equally into bonds of 100 different companies, the following happens in terms of value at risk. Since the probability of at least two companies defaulting is greater than 0.18, it follows that the portfolio of bonds leads to a negative future net worth with a probability greater than 0.05: *diversification* of the original portfolio has *increased* the measure of risk, while the ''piling up'' of risky bonds issued by the same company had remained undetected. We should not rely on such a ''measure.''

Value at risk also *fails* to encourage a reasonable *allocation* of risks among firms, as can be seen from the following simple example. Let Ω consist of three states ω_1, ω_2, ω_3 with respective probabilities 0.94, 0.03, 0.03. Let two firms have the respective future values Y and Z, where $Y(\omega_1) = Z(\omega_1) \geq 0$, $Y(\omega_2) = Z(\omega_3) = -120$, $Y(\omega_3) = Z(\omega_2) = -80$. An extra capital of 80 for each of them would make the allocation $(Y + 80, Z + 80)$ acceptable to a regulator who uses the 5% value-at-risk measure.

If the firms are risk averse, they both prefer the allocation $(X + 80, X + 80)$, where $X(\omega_1) = Y(\omega_1) = Z(\omega_1)$, $X(\omega_2) = X(\omega_3) = -100$. This allocation can be reached by a risk exchange and does not change the total loss incurred by society in each of the bad states ω_2 and ω_3, but it is not accepted by the regulator.

In conclusion, the basic reasons to reject the value-at-risk measure of risks are the following:

a. Value at risk does not behave nicely with respect to addition of risks, even independent ones, creating severe aggregation problems.
b. The use of value at risk does not encourage, and sometimes prohibits, diversification, because value at risk does not take into account the *economic consequences* of the events, the probabilities of which it controls. While possibly adapted to the description of the limited liability of shareholders, this feature seems inappropriate for dealing with the negative side of the profit-and-loss curve and the concern of regulators for the interests of policyholders.

3.4 An Insurance-Motivated Improvement of the VaR Measure of Risk: The ''TailVaR'' or ''Worst Conditional Expectation''

Casualty actuaries have long been working to compute a pure premium for policies with a *deductible*, using the conditional average of the claim size, *given* that the claim exceeds the deductible (see Hogg and Klugman 1984 and Klugman et al. 1998). In the same manner, reinsurance treaties have involved the conditional distribution of a claim for a policy (or of the total claim for a portfolio of policies), *given* that it is above the ceding insurer's retention level.

To tackle the question of ''how bad is bad,'' which is *not* addressed by the value-at-risk measurement, some actuaries (see Albrecht 1993; Embrechts 1995) have first identified the deductible (or retention level) with the quantile used in the field of financial risk measurement. We state below that one of the suggested methods gets us close to *coherent* risk measures (for proofs see Artzner et al. 1998).

Considering the ''lower partial moment'' or expectation of the ''shortfall,'' the presentation in Albrecht (1993) would translate, with our paper's notations, into measuring a risk X by the number $\mathbf{E}_{\mathbb{P}}[\min(0, -VaR_\alpha(X) - X)]$.

The presentations in Bassi, Embrechts, and Kafetzaki (1998) and Embrechts (1995) use instead the *conditional* expectation of the shortfall, given that it is positive. The quoted texts (see also Embrechts et al. 1997, Definition 3.4.6, as well as the methods indicated there to *estimate* the whole conditional distribution) present the terminology ''mean excess function.'' We suggest the term *tail conditional expectation*, because we do not consider the excess but the whole of the variable X.

Definition 3.4

Tail conditional expectation (*TailVaR*): *given a base probability measure* $\mathbb{P}$ *on* Ω, *a total return* r *on a reference instrument, and a level* α, *the tail conditional expectation is the measure of risk defined by*

$$\mathrm{TCE}_\alpha(X) = -\mathbf{E}_{\mathbb{P}}[X/r \mid X/r \leq -VaR_\alpha(X)].$$

Definition 3.5

Worst conditional expectation: *given a base probability measure* $\mathbb{P}$ *on* Ω, *a total return* r *on a reference instrument, and a level* α, *the worst conditional expectation is the coherent measure of risk defined by*

$$WCE_\alpha(X) = -\inf\{\mathbf{E}_{\mathbb{P}}[X/r \mid A] \mid \mathbb{P}[A] > \alpha\}.$$

Remark 1

TCE_α has been suggested as a possible ingredient of reinsurance treaties (see Amsler 1991).

Remark 2

Although VaR and TailVaR are, for normal distributions, very close for small values of α, this is not at all the case for heavy-tailed distributions (see Embrechts et al. 1997).

Proposition 3.2

We have the inequality $TCE_\alpha \leq WCE_\alpha$.

Albanese (1997) makes numerical studies of portfolios built out of a collection of risky bonds. He looks for a coherent measure that dominates the value-at-risk measurement and gets close to it on a specific bond portfolio.

We interpret and generalize this search as the problem of a firm constrained by the supervisors along the lines of the quantile risk measurement. Nevertheless, the firm wishes at the same time to operate on a coherent basis, at the lowest possible cost. Proposition 3.5 provides circumstances in which the firm's problem has a clear-cut solution.

Proposition 3.3

For each risk X *one has the equality*

$$VaR_\alpha(X) = \inf\{\rho(X) \mid \rho \text{ coherent and } \rho \geq VaR_\alpha\}.$$

Definition 3.1 and Proposition 3.1 allow one to address a question by C. Petitmengin, Société Générale, about the coherence of the TCE_α measure.

Proposition 3.4

Assume that the base probability $\mathbb{P}$ *on* Ω *is uniform. If* X *is a risk such that no two values of the discounted risk* $Y = X/r$ *in different states are ever equal, then* $TCE_\alpha(X) = WCE_\alpha(X)$.

Proposition 3.5

Assume that the base probability $\mathbb{P}$ *on* Ω *is uniform. If a coherent risk measure* ρ *depends only on the distribution of the discounted risk and is greater than the risk measure* VaR_α, *then it is greater than the* WCE_α *(coherent) risk measure.*

Remark

It is worth noticing that if a value at risk at the 5% level is estimated by the maximum of the 500 worst of 10,000 simulated future net values, the *average* of these 500 numbers provide a simple estimate of

TailVaR. Applying this remark can only improve the interest of a study like Boyle and Hardy (1997).

4. THE RISK-BASED CAPITAL REQUIREMENTS IN INSURANCE CONSIDERED FROM THE COHERENT RISK MEASURES VIEWPOINT

As shown in Kastelijn and Remmerswaal (1986) and *Sigma* (1995), a number of risk-measurement methods exist in insurance. This section examines the RBC requirements of the NAIC, while keeping in mind the ideas, remarks, and results reviewed in Sections 2 and 3.

It is impossible to give an axiomatic description of the prescribed *formulas* for RBC. These formulas are based on the values in the statutory annual statements (see Webb and Lilly 1994; Cummins, Harrington, and Niehaus 1995), and this connects the RBC method to the ''standard'' models of the SEC method, where no account is taken of *current* market conditions for assets. On the other hand, there is a permanent reference to probability of solvency in the motivations for the formulas (see Webb and Lilly 1994, pp. 9, 11, 12, 21, 44, and 134), which connects RBC with the value-at-risk measurement and some underlying probability distribution for future asset prices and claims developments.

Before proceeding, note that, next to the concerns of regulators and firms about the solvency of the insurance business, there is also an industry-oriented explanation to the capital requirements. As mentioned in Boyle (1989) and Webb and Lilly (1994, p. 11), another possible solution to the solvency problem—namely, an industry-supported guarantee fund—would create a moral hazard problem, since, at least in the short term, a firm may have an incentive to ''free ride'' on the fund by taking very risky business, investments, or both.

The acceptability of a portfolio X (see Section 2.2) is defined by checking whether its total adjusted capital, $TAC(X)$, is at least twice its risk-based capital, $RBC(X)$. The measure of risk of X therefore is $RBC(X) - \frac{1}{2}TAC(X)$ (see Definition 2.2). It is clear that the way RBC handles the aggregation or merger of risks as well as diversification effects is very important, since the computation of final RBC is made by adding or square-rooting sums of squares of intermediate terms related to various business lines.

It would be interesting, from the economic viewpoint, to study the restriction emphasized in Webb and Lilly (1994, pp. 13, 55, 111) concerning the inability of RBC to rank companies.

4.1 General Remarks on RBC

RBC requirements differ from the solvency requirements in the European Community by the fact that RBC considers, albeit crudely, asset value risk (see Daykin et al. 1994, p. 403, section 14.6 (f); Farny 1997). This is done by a set of coefficients, resulting from past experiences and simulations. They do not take directly into account *current* market conditions for assets and interest rates. These conditions may also influence liabilities like in the simple case of participatory policies (see Köhler 1998) or in the case of catastrophe bonds, where claims payments and coupon payment liabilities are linked (see Schmock 1997).

RBC has only one period under scrutiny: the accounting year (like our Section 2.1), but the ''going concern'' approach requires the notion of surplus quoted in Daykin et al. (1994, pp. 11 and 423), rather than the one in Bowers et al. (1997, p. 399), where no liabilities are left over.

4.2 The ''Standard Model'' Aspect of Risk-Based Capital and Its Consequences

Although a distribution of year-end values could be assigned to many of the assets, fixed coefficients are applied to current assets, in the manner of the ''standard models'' in Basle Committee (1996).

Therefore, currently existing formulas will have to be redesigned over time, for example, because new types of assets or liabilities are being issued (see, for example, catastrophe bonds in *New York Times* 1997; Schmock 1997).

Note that the C-1 risk for asset formulas refers to bonds only via the default risk, not via the interest rates risk. The latter shows up only in the mismatching risk. This relies on assumptions like the use of ''buy and hold'' strategies or duration matching, which are internal to the firm and possibly subject to changes.

To incorporate diversification effects the RBC formula relies on ''square rooting'' (see Webb and Lilly 1994, p. 44), by analogy with the uses of variances and covariances with normal distributions. Some caution may be in order, since no probabilistic model is assumed (see Straumann 1998).

4.3 The ''Value-at-Risk'' Aspect of Risk-Based Capital and Its Consequences

The search for a level of security (see Webb and Lilly 1994, p. 12) common to all companies may well be a favor to companies with heavy-tailed surplus. No clear, simple link is found between coefficients used for charging assets and the effective distribution of the surplus.

We have raised questions in Section 3.3 (Remark 1 after the digital options example) about obtaining the superior form of subadditivity.

Liabilities are evaluated by actuarial reserves; therefore they do not take into account *market* rates of interest, and the probabilities used are not adjusted for risk.

The concentration factor applied to the nominal values of bonds (see Webb and Lilly 1994, p. 32) addresses the problem of piling up all defaults into the same small probability event. We doubt that a factor of 2 is sufficient in view of the potential disaster when a very big proportion of assets is loans to the same risky party.

The treatment of interest rate risk raises the question of the pertinence of the method of duration matching.

While very different timescales play a role in RBC, such as those related to claim reserves and to the risk of long-term mortality changes, bonds of various maturities are all treated in the same way.

4.4 Conclusion

To conclude, we will quote the section on Combination of Risks in Webb and Lilly (1994, p. 44): ''The general approach to risk-based capital taken by the task force has been to estimate the expected loss an insurer would suffer in the face of a catastrophic financial event. The size of that expected loss represents the risk-based capital required to deal with that loss.'' We see there the longing for—and the promise of—times when TailVaR or other coherent risk measures will be used.

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