BOUNDS FOR RUIN PROBABILITIES IN THE PRESENCE OF LARGE CLAIMS AND THEIR COMPARISON

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ABSTRACT

Upper and lower bounds of ruin probabilities for the S. Andersen model with large claims are proposed. The bounds are stated in terms of the corresponding ladder height distribution and have a reasonable accuracy, which is illustrated by numerical examples. Comparison with other known bounds is given.

1. INTRODUCTION

1.1 Motivation

The goal of this paper is to obtain two-sided bounds of ruin probabilities in the presence of large claims, compare them to other known bounds, and give numerical results. The actual inspiration of this work is the fact (that shocked the author but is perhaps well known to actuaries) that some of the known bounds as well as the famous asymptotic approximations of ruin probabilities may have huge errors. This does not contradict the mathematics behind the corresponding formulas, but it clearly shows that, in order to use any approximation, one should have their error bounds. This natural requirement has been understood in many engineering disciplines. Nobody will produce a mechanical device if admissible errors of the detail sizes are not given. In reliability and queueing, this fact was realized recently. Perhaps now it is time to do something similar in risk theory.

It should be noted that fathers of risk theory clearly understood the necessity of such bounds; as an example, let us mention the famous Lundberg inequality (see Grandell 1991), which gives a pessimistic bound of the ruin probability. The situation is similar for the theory of probability as a whole. To give one example, A. Lyapunov and A. Markov paid much attention to the convergence rate in the discovered limit theorems. Unfortunately these attempts were disregarded during the evolution of probability theory and have been renewed only recently (see Zolotarev 1983).

Returning to risk theory, we can mention only a few works devoted to the bounds of ruin probabilities. The following list, although incomplete, definitely contains the most important. The first group of works is concerned with risk models in the presence of small claims: Rossberg and Siegel (1974), Grigelionis (1993), Kalashnikov (1996), and Furrer and Schmidli (1994). The second group is concerned with risk models in the presence of large claims: Kalashnikov (1995, 1997), Lin (1994), Willmot (1994, 1996), and Willmot and Lin (1994). All these works are theoretical, and only Kalashnikov (1996, 1997) contain numerical results. One can find further references in the cited papers and in Kalashnikov and Konstantinidis (1996).

We concentrate on the problem of bounding ruin probabilities in the presence of large claims and confine ourselves to the S. Andersen risk model. The bounds are based on the approach we evolved for geometric sums (sums of independent, identically distributed random variables [i.i.d.r.v.'s] with a random number of summands having a geometric distribution); see Kalashnikov (1997). It is necessary to note that such bounding is a very delicate problem since the behavior of the distribution function of interest is quite different in different regions of its argument. This can be clarified with the help of well-known limiting results concerning the function (which serves as

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the ruin probability)

$$\Psi(x) = \sum_{k=0}^{\infty} q(1-q)^k(1-F_k(x)), \qquad (1.1)$$

where $0 < q < 1$, F stands for a distribution function (d.f.) of a random variable (r.v.) X, and F_k is the k-fold convolution of the d.f. F. Let $m_1 = \mathbf{E}X < \infty$. The following results are of interest to us.

1. *The Rényi limit theorem* (see Kalashnikov 1997):

$$\lim_{q\to 0} \Psi\left(\frac{xm_1}{q}\right) = e^{-x}. \qquad (1.2)$$

2. *The Cramér-Lundberg asymptotic approximation* (see Grandell 1991; Kalashnikov 1997): if there exists $\varepsilon_C > 0$ such that the *Cramér condition*

$$(1-q)\mathbf{E}\exp(\varepsilon_C X) = 1 \qquad (1.3)$$

holds, then

$$\Psi(x) \sim \Psi_{CL}(x) \equiv k_{CL}\exp(-\varepsilon_C x), \qquad (1.4)$$

where k_{CL} is *the Cramér-Lundberg constant,* its actual value being well known and found in Feller (1971) or Grandell (1991).

3. *The Embrechts-Veraverbeke asymptotic approximation* (see Embrechts and Veraverbeke 1982): if the d.f. F is *subexponential,* then

$$\Psi(x) \sim \Psi_{EV}(x) \equiv \frac{1-q}{q}(1-F(x)). \qquad (1.5)$$

Relation (1.2) gives us the opportunity to approximate and bound $\Psi(x)$ for *small* and *moderate* values of x (see Brown 1990; Kalashnikov 1993, 1997), whereas relations (1.4) and (1.5) approximate $\Psi(x)$ for *large* x (it remains unknown what ''large'' means and what the real accuracy of these approximations is).

One notices that $\varepsilon_C \sim q/m_1$ as $q \to 0$, and hence

$$\lim_{q\to 0} \Psi_{CL}\left(\frac{xm_1}{q}\right) = e^{-x};$$

that is, Ψ_{CL} satisfies (1.2). This explains why Ψ_{CL} approximates Ψ to a good accuracy even for moderate x.

In contrast to this, Ψ_{EV} does not satisfy (1.2). This explains why Ψ_{EV} has a bad accuracy for moderate values of x and, therefore, cannot be recommended for practical calculations. Note that the upper bounds proposed in Lin (1994), Willmot (1994, 1996), and Willmot and Lin (1994) satisfy neither (1.2) nor (1.5), and this is why they have relatively bad accuracy (see numerical results in Section 6).

In this paper we propose new bounds of ruin probabilities in the presence of large claims that definitely satisfy (1.2) and tend to satisfy (1.5) (we explain later what ''tend to satisfy'' means), and we give relevant numerical examples.

1.2 Ruin Probability and Geometric Sums

Consider an *S. Andersen risk process* (see Grandell 1991)

$$X(t) = x + ct - \sum_{i\le K(t)} Z_i,$$

defined in terms of the following values:

$x = X(0) \ge 0$—*initial capital*
$c > 0$—*constant premium rate*
$B(u) = \mathbf{P}(Z_i \le u)$—the common d.f. of *non-negative i.i.d. claim sizes* Z_i $(i \ge 1)$
$b_s = \mathbf{E}Z_1^s$ $(s > 0)$
$A(u) = \mathbf{P}(\theta_i \le u)$—the common d.f. of *i.i.d. interoccurrence times* θ_i $(i \ge 1)$
$a_s = \mathbf{E}\theta_1^s$ $(s > 0)$
$K(t) = \max\{n: \theta_1 + \cdots + \theta_n \le t\}$—the number of claims that occur within $[0, t]$.

Assume the *relative safety loading*

$$\kappa = \frac{ca_1}{b_1} - 1 \qquad (1.6)$$

is positive. Denote by

$$\Psi(x) = \mathbf{P}(\inf_{t>0} X(t) < 0 | X(0) = x) \qquad (1.7)$$

the probability of ultimate ruin for the indicated risk process.

It is well known that the ruin probability (1.7) can be represented in the form

$$\Psi(x) = \sum_{k=0}^{\infty} q(1-q)^k(1-F_k(x)), \qquad (1.8)$$

where F_k stands for the k-th convolution of the d.f. F, and F together with probability q are associated with the random walk

$$\sigma_n = \sum_{1\le k\le n} (Z_k - c\theta_k) \quad (n \ge 0) \qquad (1.9)$$

as follows. Let

$$L = \inf\{n: \sigma_n > 0\} \qquad (1.10)$$

be the *first strict ascending ladder epoch* of the random walk (σ_n) and σ_L be the *first strict ascending ladder height.* Then

$$q = \mathbf{P}(L = \infty) \quad (1.11)$$

and

$$F(u) = \mathbf{P}(\sigma_L \leq u | L < \infty) \quad (1.12)$$

(see Asmussen 1987; Feller 1971).

If

$$A(u) = 1 - \exp\left(\frac{-u}{a_1}\right), \quad (1.13)$$

then we call such a risk process *classical* and, in this case,

$$q = \frac{k}{k + 1} \quad (1.14)$$

and

$$F(x) = \frac{1}{b_1} \int_0^x (1 - B(u))\, du \quad (1.15)$$

(see Asmussen 1987; Feller 1971).

Let $(X_i)_{i \geq 1}$ be a sequence of non-negative i.i.d.r.v.'s having the common d.f. $F(x)$,

$$S_k = X_1 + \cdots + X_k, \quad k \geq 1 \quad (1.16)$$

be a *renewal process* governed by $(X_i)_{i \geq 1}$, and

$$N(x) = \inf\{k\colon S_k > x\} \quad (1.17)$$

be the number of the first renewal exceeding level x. Denote by ν a r.v. that is independent on $(X_i)_{i \geq 1}$ and has the geometric distribution

$$\mathbf{P}(\nu = k) = q(1 - q)^{k-1}, \quad k \geq 1, \quad (1.18)$$

and let

$$W(x) = \mathbf{P}(S_\nu \leq x) = \sum_{k=1}^{\infty} q(1 - q)^{k-1} F_k(x). \quad (1.19)$$

Then

$$\Psi(x) = \mathbf{E}(1 - q)^{N(x)} = (1 - q)(1 - W(x)) \quad (1.20)$$

(see Kalashnikov 1994b). Owing to (1.20), all bounds stated for $W(x)$ are immediately applicable to $\Psi(x)$.

Hereafter we can forget about ruin probability and deal with only the d.f. $W(x)$ of the *geometric sum* S_ν. But first we need to explain why we use probability q and the conditional d.f. F as the starting point of our research despite the fact that these characteristics are not actually known, in general, and they are not initial data. There are at least four different reasons for doing this.

(i) As we have already mentioned, the bounds, proposed below, are yielded by the methods elaborated for geometric sums. So representation (1.19) is convenient from a mathematical point of view.

(ii) It is well known that, for the classical risk model (which is very popular in actuarial research; see Dufresne and Gerber 1989, 1991; Grandell 1991; Kalashnikov 1997; Willmot 1994), both q and F have a simple explicit representation via initial data; see (1.14) and (1.15).

(iii) Although explicit formulas for q and F do not exist, in general, there are many works devoted to their approximations and bounds. Borovkov (1976, chap. 4) gave tight bounds for q and F that agree, in a way, with (1.14) and (1.15), and they are valid for the S. Andersen model. Miyazawa and Schmidt (1993) proved that (1.14) and (1.15) are true for rather general stationary point processes. Let us also mention Bertoin and Boney (1994), who investigated the local behavior of ladder height distributions.

(iv) A geometric sum as an abstract model appears in many applications, not only in studies of ruin probabilities for the S. Andersen model. Even in risk theory other models exist that can be investigated with the help of geometric sums. For instance, ruin probability in some diffusion models can also be reduced to the form (1.20) (see Dufresne and Gerber 1991). Other important fields of application of geometric sums are reliability and queueing (see Kalashnikov 1997).

1.3 Summary of the Results

The principal purpose of this paper is to propose a variety of lower and upper bounds of d.f. $W(x)$, discuss them, and show relevant numerical results.

The paper is structured as follows. Section 2 contains auxiliary notions (some classes of functions) helpful for stating basic results. In Section 3 we give two-sided bounds of W that are obtained by methods elaborated by the author and based on renewal arguments. These results are proved in Kalashnikov (1997), and their origin can be traced back to Kalashnikov (1993, 1994a, 1994b, 1995).

Section 4 contains lower bounds of W obtained by the so-called test function method (see Kalashnikov 1994a). Whereas these estimates are similar to those obtained by Willmot (1994) (some difference exists and is discussed), the method by which they are obtained is different. We also propose a trick allowing us to improve these estimates significantly compared with Willmot's estimates in the case where $F(x)$ is a decreasing failure rate (DFR) d.f.

In Section 5 we discuss briefly basic properties of the proposed bounds. Section 6 contains numerical results giving a vivid impression of the proposed bounds. In particular, we compare the estimates indicated above with bounds obtained by the numerical method of Dufresne and Gerber (1989) and an asymptotic formula by Embrechts and Veraverbeke (1982).

2. CLASSES OF FUNCTIONS

In order to formulate the results, we have to introduce classes of functions, in terms of which some of our estimates are stated. We also provide statements disclosing specific features of these classes. Throughout the paper, we consider only classes of *non-negative functions defined on* $[0, \infty)$. So, this constraint will not be repeated below.

Definition 1

Function G is said to belong to the class $\mathcal{G}^{ui}$ if G is convex and has a concave derivative $g \geq 0$, and $g(x) \to \infty$ as $x \to \infty$.

For instance, functions x^s $(1 < x \leq 2)$ and $x \ln(1 + x)$ belong to $\mathcal{G}^{ui}$. The following lemma gives a criterion of *uniform integrability* in terms of the class $\mathcal{G}^{ui}$ (see Meyer 1966).

Lemma 1

A family of real r.v.'s χ is uniformly integrable that is

$$\lim_{x\to\infty} \sup_{X\in\chi} \mathbf{E}(|X|; |X| > x) = 0$$

iff there exists a function $G \in \mathcal{G}^{ui}$ such that

$$\sup_{X\in\chi} \mathbf{E}G(|X|) < \infty.$$

Corollary 1

For any real r.v. X,

$$\mathbf{E}|X| < \infty \Leftrightarrow \mathbf{E}G(|X|) < \infty, \quad \text{for some } G \in \mathcal{G}^{ui}.$$

Definition 2

Function G is called belonging to the class $\mathcal{G}_s$ $(s \geq 0)$ if $G(x) = \exp(\Lambda(x))$, Λ is monotonically increasing from 0 to ∞, $\Lambda(x)/x$ is monotonically decreasing to 0, and

$$\lim_{x\to\infty} \frac{G(x)}{(1 + x)^s} = \infty.$$

Evidently $\mathcal{G}_s \subset \mathcal{G}_t$ if $s > t$. Such functions as $(1 + x)^{s+\varepsilon}$ $(\varepsilon > 0)$ and $(1 + x)^s \ln(e + x)$ belong to $\mathcal{G}_s$ $(s \geq 0)$. The following lemma discloses possible uses of the classes $\mathcal{G}_s$ and refines some statements from Kalashnikov (1997).

Lemma 2

A family $(|X|^s))_{X\in\chi}$ is uniformly integrable iff there exists a function $G \in \mathcal{G}_s$ such that

$$\sup_{X\in\chi} \mathbf{E}G(|X|) < \infty.$$

Remark 1

If $s = 1$ in Lemma 2, then the indicated property is the uniform integrability; for $s = 0$, this property is usually called tightness.

Corollary 2

For any real r.v. X,

$$\mathbf{E}|X|^s < \infty \Leftrightarrow \mathbf{E}G(|X|) < \infty, \quad \text{for some } G \in \mathcal{G}_s \ (s \geq 0).$$

Definition 3

Function T is called belonging to the class $\mathcal{T}$ if

$$T(x + y) \leq T(x)T(y), \quad \text{for any } x, y \geq 0, \ T(0) = 1. \tag{2.1}$$

Any function $T(x) = \exp(\int_0^x \lambda(u)du)$, where λ is a non-negative and nonincreasing function, belongs to $\mathcal{T}$. Evidently $\mathcal{G}_s \subset \mathcal{T}$ $(s \geq 0)$.

3. TWO-SIDED BOUNDS

Consider the d.f. W from (1.19) and put $m_s = \mathbf{E}X^s$ $(s > 0)$ and $m(G) = \mathbf{E}G(X)$ (for an appropriate function G). Assume furthermore that

$$\mathbf{E}X = 1. \tag{3.1}$$

This assumption has the virtue of simplicity but the vice of being wrong and only leads to a norming of the argument. Denote also

$$q' = -\ln(1 - q). \tag{3.2}$$

The following theorem presents upper bounds of W.

Theorem 1

1. If $m_2 < \infty$, then

$$W(x) \leq 1 - \exp\left(-\frac{q'(x \vee (2m_2))^2}{x \vee (2m_2) - m_2} - q'(m_2 - 1)\right) - \frac{1 - F(x)}{q} K_1(q, x), \tag{3.3}$$

where

$$K_1(q,x) = \frac{q^2}{(q')^2} \left(1 + \exp(-q'y_1) - \frac{2(1 - \exp(-q'y_1))}{q'y_1}\right) \quad (3.4)$$

and

$$y_1 = \frac{x}{2} + \frac{m_2 - 1}{2}\left(1 - \sqrt{1 + \frac{2x}{m_2 - 1}}\right). \quad (3.5)$$

2. If $m(G) < \infty$ for a function $G \in \mathcal{G}^{ui}$, then

$$W(x) \le 1 - \frac{1}{1-q} \exp\left(-q'(x \vee x^*)\left(\frac{3m(G) \vee g(x/2)}{3m(G) \vee g(x/2) - m(G)}\right)^2\right) - \frac{1 - F(x)}{q} K_2(q, x), \quad (3.6)$$

where

$$x^* = \inf\left\{x: g\left(\frac{x}{2}\right) > 3m(G)\right\}, \quad (3.7)$$

$$K_2(q, x) = \frac{q^2}{(q')^2} \left(1 + \exp(-q'y_2) - \frac{2(1 - \exp(-q'y_2))}{q'y_2}\right) \quad (3.8)$$

and y_2 is the unique solution of the functional equation

$$y_2 m(G) + G(y_2) = G\left(\frac{x}{2}\right). \quad (3.9)$$

Proof

The idea of the proof is based on the truncation. Let us denote $Z_i = \min(X_i, b)$, for an appropriate $b > 0$. Then

$$1 - W(x) = \mathbf{P}\left(\sum_{i=1}^{\nu} Z_i > x\right) + \mathbf{P}\left(\sum_{i=1}^{\nu} X_i > x, \sum_{i=1}^{\nu} Z_i \le x\right). \quad (3.10)$$

Take $b = x/2$. Since the first term on the right-hand side of (3.10) is a distribution of a geometric sum of *bounded* (by $x/2$) summands, it can be estimated similarly to the Cramér case yielding the exponential term in (3.3) and (3.6). The bound for the second term in (3.10) is proportional to $1 - F(x)$, as can be seen from (3.3) and (3.6), owing to the fact that

$$\mathbf{P}\left(\sum_{i=1}^{\nu} X_i > x, \sum_{i=1}^{\nu} Z_i \le x\right) = \mathbf{P}\left(\sum_{i=1}^{\nu} X_i > x, \sum_{i=1}^{\nu} Z_i \le x, A_\nu\right),$$

where

A_ν = {there exists exactly one summand X_i, $1 \le i \le \nu$, such that $X_i > x/2$}.

The complete proof of the theorem can be found in Kalashnikov (1997), which is a refinement of the proof from Kalashnikov (1995).

We now proceed to state the result revealing lower bounds of W. These bounds are written in terms of an auxiliary function $G \in \mathcal{G}_s$ (for appropriate $s \ge 0$) such that $\mathbf{E}G(X) < \infty$. This means, in particular, that the quality of a bound may depend on the choice of G (and this is actually the case), and therefore this choice is a crucial step in the method. Let us introduce some variables necessary to state the result.

As just mentioned, one of the components of the bounds is a function G such that $m(G) = \mathbf{E}G(X) < \infty$. By default, $G \in \mathcal{G}_0$, but recall that $\mathcal{G}_0$ is the largest class among $\mathcal{G}_s$ ($s \ge 0$) (see Definition 2). So if one imposes additional restrictions on F, then it is possible to take G from a narrower class. If one assumes that $m_2 < \infty$ (which is the case in the first part of the following theorem), then it is possible to choose $G \in \mathcal{G}_2$ in accordance with Lemma 2.

Another parameter of the bounds is the so-called *truncation parameter* (see Kalashnikov 1995, 1997), a positive number designated by a. Given $a > 0$ and $G \in \mathcal{G}_0$, define

$$\delta(a, G) = \mathbf{E}\left(G(X); X > \frac{a}{q}\right). \quad (3.11)$$

Since $M(G) < \infty$,

$$\lim_{a\to\infty} \delta(a, G) = 0. \quad (3.12)$$

The following quantity $M(a)$ is introduced to simplify the formulas:

$$M(a) = \frac{2}{a^2}(e^a - 1 - a). \quad (3.13)$$

One more real parameter involved in our estimates is denoted by θ. It takes values from (0, 1].

Actually, G, a, and θ can be chosen rather arbitrarily, and this leads to a variety of lower bounds. The problem is what is the best choice? No answer to this question is available. We discuss only some plausible arguments and propose numerical results. In obtaining numerical results, the aforementioned arguments are used as well as some skill and informal conjectures.

Let Λ be a function such that $G(x) = \exp(\Lambda(x))$ (see Definition 2). If $m_2 < \infty$, then define

$$\delta_1(x) = \frac{\Lambda(\theta x)}{\theta x}\left(1 + \frac{m_2 M(a)\Lambda(\theta x)}{2\theta x}\right). \quad (3.14)$$

If $m_2 = \infty$, then denote

$$\delta_2(x) = \frac{\Lambda(\theta x)}{\theta x}\left(1 + \frac{aM(a)(m(G) - 1)\Lambda(\theta x)}{qg(a/q)\theta x}\right), \quad (3.15)$$

where $G \in \mathcal{G}^{ui} \cap \mathcal{G}_0$ and g is the derivative of G. Note that both $\delta_1(x)$ and $\delta_2(x)$ tend to 0 when $x \to \infty$ (provided that G, a, and θ are fixed). Denote also

$$d_1 = d_1(a,G,x) = (1 - q)(1 + \delta(a, G))(1 + \delta_1(x)) \quad (3.16)$$

and

$$d_2 = d_2(a,G,x) = (1 - q)(1 + \delta(a, G))(1 + \delta_2(x)). \quad (3.17)$$

The properties of $\delta(a, G)$ and $\delta_i(x)$ ($i = 1, 2$) indicated above yield the following lemma.

Lemma 3

Given G, there exist $a^* > 0$, $d^* < 1$, and $y^* > 0$ such that $d(a^*, G, x) \le d^*$ for all $\theta x \ge y^*$.

The main result can be stated as follows.

Theorem 2

1. Let $m_2 < \infty$, $G \in \mathcal{G}_0$, and parameters a, θ, q, and x be such that $\Lambda(\theta x)/(\theta x) < q$ and $d_1 < 1$. Then

$$W(x) \ge 1 - \frac{1}{1 - q}\exp(-\varepsilon_1 x) - \frac{1 - F(\theta x)}{q} - \frac{qd_1}{1 - d_1} G^{-1/\theta}(\theta x), \quad (3.18)$$

where

$$\varepsilon_1 = \frac{1}{m_2 M(a)}\left(-1 + \sqrt{1 + 2qm_2 M(a)}\right). \quad (3.19)$$

2. Let $m_2 = \infty$, $G \in \mathcal{G}^{ui} \cap \mathcal{G}_0$, and parameters a, θ, q, and x be such that $\Lambda(\theta x)/(\theta x) < q$ and $d_2 < 1$. Then

$$W(x) \ge 1 - \frac{1}{1 - q}\exp(-\varepsilon_2 x) - \frac{1 - F(\theta x)}{q} - \frac{qd_2}{1 - d_2} G^{-1/\theta}(\theta x), \quad (3.20)$$

where

$$\varepsilon_2 = q\left(1 - (m(G) - 1)\frac{aM(a)}{g(a/q)}\right)_+. \quad (3.21)$$

Proof

In obtaining lower bounds we use the equality (3.10) from the preceding theorem but put there $b = a/q$, where $a > 0$ is a free truncation parameter. Again, we can estimate the first term in (3.10) by an exponential function; see the corresponding exponent on the right-hand side of (3.18) and (3.20). The other terms in (3.18) and (3.20) appear owing to rather sophisticated estimates of the probability

$$\mathbf{P}\left(\sum_{i=1}^{\nu} X_i > x, \sum_{i=1}^{\nu} Z_i \le x\right)$$

using the arguments similar to those proposed in Fuk and Nagaev (1971). See Kalashnikov (1997, subsection 4.3.2) for the details. □

Note that ε_1 and ε_2 are both equivalent to q when $q \to 0$. Hence, the exponential summands in (3.18) and (3.20) can be regarded as analogues of Lundberg's exponential bound for ruin probability. But there are additional summands in the bounds from Theorem 2. The term $(1 - F(\theta x))/q$ is the correct asymptotic expression of $1 - W$ (for at least subexponential distributions F; see Embrechts and Veraverbeke 1982), but unfortunately for θx, not for x. If one takes $\theta = 1$, then the last terms in both (3.18) and (3.20) dominate $(1 - F(x))/q$ for large x. So the parameter θ is introduced to equalize the last two summands in the expressions of the bounds. Even if one takes $\theta = \theta(x) \to 1$ when $x \to \infty$, this is still insufficient for getting a proper asymptotic $(1 - F(x))/q$. It is noteworthy that we assume neither subexponentiality nor any other property of F except $m_1 = 1$. One can see that the larger G, the larger θ can be chosen, which yields a better estimate. But this conclusion is true only asymptotically, for large x. If x is finite, then different variants are possible, but numerical calculations show that the best variant is $\theta = 1$. Note that under $\theta \ge \theta^* > 0$ and fixed G, Lemma 3 yields the existence of a^* and $x^* > 0$ such that $d_i < 1$ ($i = 1, 2$) for all $x > x^*$.

The bounds given in Theorem 2 have several degrees of freedom to be used for their improvement. More specifically, one can choose θ, a, and G to attain the best result. The most informal procedure is the choice of function G. In some aspects, this resembles the choice of the Lyapunov function in stability theory. Estimates from Theorem 2 allow us to reveal the behavior of the bounds qualitatively. But it is difficult to obtain quantitative results purely analytically. Luckily inspection of the estimates assures us that they are stated in a form ready for computer calculations. Thus, we split the problem, leaving the study of qualitative properties and obtaining general formulas to mathematics and numerical calculations to a computer.

4. LOWER BOUNDS BY TEST FUNCTIONS

In this section we propose another method for obtaining lower bounds of W based on so-called test functions. Actually the test function method can be traced back to Lyapunov's works on stability. Recently it was generalized to study a variety of properties of stochastic models (continuity, recurrence, transience, etc.); see Kalashnikov (1994a) and references therein.

The following theorem contains a lower bound expressed in terms of a function $T \in \mathcal{T}$ that can be called a *test function*. Note that the function G in Theorem 2 plays approximately the same role. As in Theorem 2, the quality of the bound in the following theorem may depend heavily on the choice of the test function T, as we will see in Section 6.

Theorem 3

If $T \in \mathcal{T}$ and

$$(1 - q)\mathbf{E}T(X) \leq 1, \qquad (4.1)$$

then

$$W(x) \geq \frac{\mathbf{E}(T(x) - T(X))_+}{T(x)}. \qquad (4.2)$$

Proof

Let $x > 0$ and $(S_k)_{k\geq 1}$ and $N(x)$ be defined as in (1.16) and (1.17), respectively. Consider a homogeneous Markov chain $\xi_n = (n, S_n)$ $(n \geq 0)$, where we put $S_0 = 0$. The r.v. $N(x)$ is a stopping time for this Markov chain, and $\mathbf{E}N(x) < \infty$ for all $x > 0$ (see Kalashnikov 1994b). Denote by $\mathbf{A}$ the *generator* of the Markov chain $(\xi_n)_{n\geq 0}$, that is, the operator mapping a real function V defined on the state space of the Markov chain into another function (denoted as $\mathbf{A}V$) in accordance with the rule

$$\mathbf{A}f(\xi) = \mathbf{E}(f(\xi_{n+1}) - f(\xi_n)|\xi_n = \xi). \qquad (4.3)$$

Take

$$V(\xi_n) = (1 - q)^{n-1}T(S_n), \quad n \geq 0. \qquad (4.4)$$

Relations (2.1) and (4.1) together with (4.3) immediately yield that

$$\mathbf{A}V(\xi_n) \leq 0, \quad n \geq 0. \qquad (4.5)$$

In other words, the sequence $\{V(\xi_n)\}_{n\geq 0}$ forms a supmartingale, and since $N(x)$ is a stopping time for (ξ_n) and $\mathbf{E}N(x) < \infty$, we have

$$\mathbf{E}V(\xi_{N(x)}) \leq \mathbf{E}V(\xi_1) = \mathbf{E}T(X_1). \qquad (4.6)$$

Now,

$$\mathbf{E}V(\xi_{N(x)}) = \mathbf{E}(T(S_{N(x)}); N(x) = 1) + \mathbf{E}((1 - q)^{N(x)-1}T(S_{N(x)}); N(x) > 1). \qquad (4.7)$$

Evidently $\{N(x) = 1\} = \{X_1 > x\}$ and $S_1 = X_1$. Using the fact that

$$W(x) = 1 - \mathbf{E}(1 - q)^{N(x)-1}$$

(see (1.20), we arrive at

$$\begin{aligned}\mathbf{E}(T(X_1); X_1 \leq x) &\geq \mathbf{E}((1 - q)^{N(x)-1}T(S_{N(x)}); N(x) > 1)\\ &\geq T(x)\, \mathbf{E}((1 - q)^{N(x)-1}; N(x) > 1)\\ &= T(x)(1 - W(x) - \mathbf{P}(X_1 > x)),\end{aligned}$$

which yields the assertion of the theorem. □

Remark 2

Inequality (4.1) serves as an analogue of the Cramér condition in the presence of heavy tails; see (1.3).

Remark 3

The right-hand side of (4.2) is equal to 0 at $x = 0$. Note that such an equality was attained in Willmot (1994) only for NWU (*New Worse than Used*) d.f.'s F.

Remark 4

Willmot (1994) considered not only geometric sums but also random sums consisting of ν summands, where ν is an r.v. having the distribution

$$p_k = \mathbf{P}(\nu = k), \quad k \geq 1,$$

under the restriction

$$\frac{p_{k+1} + p_{k+2} + \cdots}{p_k + p_{k+1} + \cdots} \leq 1 - q, \quad k \geq 1$$

(for some $0 < q < 1$), but this restriction allows us to consider this case exactly in the same way as for geometric sums.

The following result is a corollary of Theorem 3.

Corollary 3

Under the assumptions of Theorem 3,

$$W(x) \geq 1 - \frac{1}{(1-q)T(x)}. \quad (4.8)$$

Proof

Since

$$\mathbf{E}(T(x) - T(X))_+ \geq T(x) - \mathbf{E}T(X) \geq T(x) - (1-q)^{-1},$$

the result immediately follows from (4.2). □

Further results can be obtained if one imposes additional restrictions on d.f. F. We are interested in the following two cases.

1. F is a NWU d.f., that is (see Stoyan 1983),

$$1 - F(x+y) \geq (1 - F(x))(1 - F(y)), \quad \text{for any } x \geq 0 \text{ and } y \geq 0. \quad (4.9)$$

2. F is a DFR d.f., that is, F has the form (see Stoyan 1983)

$$F(x) = 1 - \exp\left(-\int_0^x \lambda_F(u)\,du\right), \quad (4.10)$$

where the failure rate function λ_F is nonincreasing.

Any DFR d.f. is automatically NWU, but not vice versa, in general.

Corollary 4

If, under the assumptions of Theorem 3, F is NWU, then

$$W(x) \geq F(x) - \frac{\mathbf{E}(T(X);\, X \leq x)}{\mathbf{E}T(x+X)}. \quad (4.11)$$

Proof

Since F is NWU,

$$\mathbf{P}(S_{N(x)} > u | N(x)) \geq \mathbf{P}(x + X > u), \quad u \geq 0,$$

where X is distributed as F and independent of $N(x)$. Therefore,

$$\mathbf{E}((1-q)^{N(x)-1} T(S_{N(x)});\, N(x) > 1) \geq \mathbf{E}T(x+X)\mathbf{E}((1-q)^{N(x)-1};\, N(x) > 1).$$

Plugging this into (4.7), we arrive at the desired bound with the help of (4.6). □

We limit ourselves to this result.

In numerical calculations, we prefer employing estimate (4.8) as the simplest one, even in particular cases of NWU or DFR distributions.

We give two examples.

Example 1 NWU distribution

Let F be a NWU d.f., $F(x) < 1$ for all $x > 0$. Take

$$T(x) = (1 - F(x))^{-p}, \quad 0 < p < 1. \quad (4.12)$$

By (4.9), $T \in \mathcal{T}$. Since $\mathbf{E}T(X) = (1-p)^{-1}$, condition (4.1) holds for any $p \leq q$. Then general estimate (4.2), applied to $p = q$, yields

$$W(x) \geq \frac{1}{1-q}\,(1 - (1 - F(x))^q - qF(x)).$$

If we use a cruder estimate (4.8), then we arrive at

$$W(x) \geq 1 - \frac{(1 - F(x))^q}{1-q}. \quad (4.13)$$

Note that Willmot (1994) obtained a better estimate for this case, namely,

$$W(x) \geq 1 - (1 - F(x))^q. \quad (4.14)$$

Example 2 DFR distribution

Let F be a DFR d.f. with the failure rate function $\lambda_F(x) \to 0$ as $x \to \infty$. The simplest estimate (4.8) can be improved in this case as compared to (4.13) or to Willmot's estimate. Take an arbitrary $x^* > 0$ and let $b = \lambda_F(x^*)$,

$$\lambda^*(x) = \lambda_F(x) \wedge b,$$

and

$$T^*(x) = \exp\left(p\int_0^x \lambda^*(u)\,du\right), \quad (4.15)$$

where $p > q$ is taken to satisfy the condition

$$(1-q)\mathbf{E}T^*(X) \leq 1. \quad (4.16)$$

It is clear that one can choose a constant $p > q$ in such a way that $p \to 1$ as $b \to 0$ (this implies, in particular, that it is sufficient to require $\lambda_F(x)$ to be monotonically decreasing to 0 at the limit). We infer from (4.8) that

$$W(x) \geq 1 - \frac{c^*(1 - F(x))^p}{1-q}, \quad (4.17)$$

where

$$c^* = \exp\left(p \int_0^{x \wedge x^*} (\lambda_F(u) - b)\, du \right).$$

As $p > q$ and even $p \to 1$, the right-hand side of (4.17) converges to 1 faster than in (4.13).

5. DISCUSSION

Now we will review some properties of the above estimates, starting with lower bounds from Theorem 3. We have seen that straightforward application of Theorem 3 to NWU distributions leads to estimates having the tail $(1 - F(x))^q/(1 - q)$. Comparing this with the correct asymptotic tail $(1 - F(x))/q$ (see (1.5)), one can assert only that the proposed estimate should work badly for large x. Furthermore, this bound does not satisfy (1.2). So it should work badly for moderate x too. In Example 2 we showed how to improve the estimate (4.8). Since p can be taken as close to 1 as necessary, there is hope that (4.17) approximates W to a good accuracy. If we choose b in Example 2 in such a way that $pb = q$, then the right-hand side of (4.17) resembles the Lundberg estimate.

It is instructive to review the estimates from Theorems 1 and 2. We will discuss first the upper bounds from Theorem 1. In the two cases (1) and (2) the tail $1 - W(x)$ is decomposed into two summands. The first summand is exponential, whereas the second one has the form $K_i(x, q)(1 - F(x))/q$. Note that $K_i(x, q) \to 0$ when $x \to 0$. Therefore, the exponential term prevails if x is not too large. If $x \to \infty$, then the second term becomes dominant, and its form is similar to the correct asymptotic formula. It can be easily seen that

$$\lim_{x\to\infty} K_i(x, q) = \left(\frac{q}{q'}\right)^2, \quad i = 1, 2,$$

and this value is close to 1 when q is small. Such a structure of the bound is very intuitive. For small x the behavior of $W(x)$ does not depend on the tail of F, and, therefore, $1 - W(x)$ should decay exponentially. Moreover, the exponential term in both (3.3) and (3.6) has the parameter that is equivalent to q (if $q \to 0$). The same property holds for the parameters of exponential terms from Theorem 2. When x is large, upper estimates (3.3) and (3.6) are proportional to the tail of F.

Now turn to the lower bounds from Theorem 2. In this case, $1 - W(x)$ consists of three summands. Properties of the exponential summand are similar to properties of the exponential term from Theorem 1. The same is true for the second summand $(1 - F(\theta x))/q$, and we have discussed the role of θ. The presence of the third term can partly be explained by the fact that F is not subexponential in general.

In our opinion, the estimates from Theorem 2 are preferable to those from Theorem 3. In fact, it is possible to choose

$$G(x) = T(x) = \exp(\Lambda(x))$$

with appropriate Λ. However, in Theorem 3, the additional condition (4.1) should be satisfied.

6. NUMERICAL RESULTS

We focus now on a simple example that can be investigated easily, but it gives a good idea of the real properties of the bounds discussed and their accuracy. Corresponding calculations will only be sketched.

Let

$$F(x) = 1 - (1 + \beta x)^{-\alpha}, \quad \alpha > 1$$

be a translated Pareto distribution. The inequality $\alpha > 1$ provides $\mathbf{E}X < \infty$. If, in addition, $\beta = 1/(\alpha - 1)$, then $\mathbf{E}X = 1$. If $\alpha > 2$, then

$$m_2 = \frac{2(\alpha - 1)}{\alpha - 2} < \infty.$$

Evidently F is a subexponential and DFR d.f., for which

$$\lim_{\alpha\to\infty} F(x) = 1 - \exp(-x).$$

The density f of F has the form

$$f(x) = \frac{\alpha\beta}{(1 + \beta x)^{\alpha+1}},$$

and the failure rate function is

$$\lambda_F(x) = \frac{\alpha\beta}{1 + \beta x}.$$

In numerical calculations we are interested in small values of $1 - W(x)$, so we display upper and lower bounds for this complementary function, dealing with two cases: $\alpha = 3$ and 5. The less α is, the heavier is the tail of F. In both cases, $m_2 < \infty$, and therefore we can use parts (1) from Theorems 1 and 2.

Denote the lower bound of $1 - W$ resulting from (3.3) by $L(x)$. In order to apply Theorem 2, we have to use a test function G. Take $G(x) = (1 + \beta x)^s$ $(0 < s < \alpha)$, with the parameter s to be chosen. We denote by $U_1(x)$ the upper bound of $1 - W$, resulting from (3.18).

Let $U_2(x)$ be the upper bound of $1 - W$ resulting from (4.17). This estimate is defined by parameter b, and $b = \infty$ corresponds to the estimate obtained by

Willmot in the general case. Note that, in the particular case considered here (where F is the DFR d.f.), Willmot (1994) obtained the better inequality (4.14). Hence, his figures are equal to the figures contained in the column $b = \infty$ (see the tables) multiplied by $(1 - q)$. Actually, the difference between these two cases is not significant for our conclusions.

We denote by

$$A(x) = \frac{1}{q(1 + \beta x)^{\alpha}}$$

the correct asymptotic behavior of $1 - W(x)$ when $x \to \infty$. Finally, we denote by $l(x)$ and $u(x)$ lower and upper bounds of $1 - W(x)$ obtained by the recursive algorithm of Dufresne and Gerber (1989).

The results are contained in 18 tables. Tables 1–9 correspond to $\alpha = 3$, whereas Tables 10–18 correspond to $\alpha = 5$. Each group of nine tables is divided into three subgroups referring to different values of probability q: $q = 0.5$, 0.1, and 0.01.

In Tables 1–3, $\alpha = 3$ and $q = 0.5$. Table 1 contains values of $l(x)$, $u(x)$, and $A(x)$ against x. In this case it is possible to obtain tight lower and upper bounds by the recursive algorithm. It is noteworthy that asymptotic approximation $A(x)$ lies out of these bounds in the chosen range of x (and this holds for all other cases).

Table 2 contains values of $L(x)$ and $U_1(x)$ (together with optimal values of parameters a, θ, and s) against x. Beginning from $x = 28$, the bounds $L(x)$ and $l(x)$ are rather close to each other, which testifies that $L(x)$ is not a bad value. The upper bound $U_1(x)$ evidently has poor accuracy. However, the larger x is, the better is the accuracy. Optimal values of the parameter s increase together with x, but, in this range of x, s is far from its limiting value 3. Table 3 contains values of the upper bound $U_2(x)$ corresponding to different values of b (together with optimal values of p giving the equality in (4.16)). The column $b = \infty$ corresponds to the estimate (4.13). The best bound is in the column $b = 0.5$. As we will see, in all similar cases the optimal value of b coincides with q, and there is a motivation of this empirical fact. Nevertheless, even in this best case, bounds from Table 3 are worse than $U_1(x)$ (at least from $x = 24$).

Tables 4–6 correspond to $q = 0.1$. Their structures are the same as those of Tables 1–3, and one can repeat almost all the statements concerning the quality of estimates presented in those tables. The values of $L(x)$ and $U_1(x)$ are closer to each other here as compared to $q = 0.5$. Figures in Table 6 show that the optimal choice of b is 0.1 (= q).

Table 1

x	$l(x)$	$u(x)$	$A(x)$
4	1.34×10^{-1}	1.37×10^{-1}	7.41×10^{-2}
12	1.13×10^{-2}	1.15×10^{-2}	5.83×10^{-3}
20	2.33×10^{-3}	2.35×10^{-3}	1.50×10^{-3}
28	7.98×10^{-4}	8.03×10^{-4}	5.92×10^{-4}
40	2.57×10^{-4}	2.72×10^{-4}	2.16×10^{-4}
64	6.15×10^{-5}	6.35×10^{-5}	5.56×10^{-5}
96	1.81×10^{-5}	1.84×10^{-5}	1.70×10^{-5}

Table 2

x	$L(x)$	$U_1(x)$	a	θ	s
4	9.85×10^{-4}	9.16×10^{-1}	0.8	0.9	0.85
12	8.60×10^{-4}	1.18×10^{-1}	0.9	0.8	1.15
20	4.15×10^{-4}	2.17×10^{-2}	1.2	0.7	1.35
28	2.09×10^{-4}	5.74×10^{-3}	1.6	0.6	1.50
40	8.86×10^{-5}	1.41×10^{-3}	2.0	0.7	1.70
64	2.54×10^{-5}	2.59×10^{-4}	3.2	0.7	1.90
96	8.16×10^{-6}	6.84×10^{-5}	3.8	0.7	2.00

Table 3

	b			
x	∞	0.5	0.1	0.01
4	3.85×10^{-1}	2.71×10^{-1}	1.34	1.92
12	1.08×10^{-1}	4.13×10^{-2}	6.02×10^{-1}	1.77
20	5.48×10^{-2}	1.51×10^{-2}	2.71×10^{-1}	1.64
28	3.44×10^{-2}	7.60×10^{-3}	1.22×10^{-1}	1.51
40	2.08×10^{-2}	3.60×10^{-3}	4.46×10^{-2}	1.34
64	1.06×10^{-2}	1.32×10^{-3}	1.16×10^{-2}	1.05
96	5.83×10^{-3}	5.49×10^{-4}	3.56×10^{-3}	7.66×10^{-1}
p	0.5	0.740	0.9945	0.999999999

Table 4

x	$l(x)$	$u(x)$	$A(x)$
20	1.33×10^{-1}	1.50×10^{-1}	7.51×10^{-3}
60	3.93×10^{-3}	5.05×10^{-3}	3.36×10^{-4}
100	2.48×10^{-4}	3.08×10^{-4}	7.54×10^{-5}
140	4.97×10^{-5}	5.42×10^{-5}	2.79×10^{-5}
200	1.31×10^{-5}	1.43×10^{-5}	9.71×10^{-6}
320	2.84×10^{-6}	2.97×10^{-6}	2.40×10^{-6}
480	7.70×10^{-7}	8.19×10^{-7}	7.14×10^{-7}

Table 5

x	$L(x)$	$U_1(x)$	a	θ	s
20	5.26×10^{-2}	2.50×10^{-1}	0.8	0.9	0.55
60	9.30×10^{-4}	1.16×10^{-2}	1.1	0.7	0.90
100	4.91×10^{-5}	8.95×10^{-4}	1.6	0.7	1.20
140	1.70×10^{-5}	1.53×10^{-4}	2.1	0.7	1.40
200	6.77×10^{-6}	3.42×10^{-5}	3.5	0.8	1.65
320	1.86×10^{-6}	7.06×10^{-6}	3.9	0.8	1.75
480	5.87×10^{-7}	1.91×10^{-6}	5.1	0.8	1.85

Table 6

x	b			
	∞	0.5	0.1	0.01
20	5.41×10^{-1}	7.46×10^{-2}	1.50×10^{-1}	9.10×10^{-1}
60	3.97×10^{-1}	4.26×10^{-2}	1.13×10^{-2}	6.10×10^{-1}
100	3.42×10^{-1}	3.26×10^{-2}	3.33×10^{-3}	4.09×10^{-1}
140	3.09×10^{-1}	2.72×10^{-2}	1.48×10^{-3}	2.74×10^{-1}
200	2.78×10^{-1}	2.25×10^{-2}	6.20×10^{-4}	1.50×10^{-1}
320	2.41×10^{-1}	1.75×10^{-2}	1.97×10^{-4}	4.56×10^{-2}
480	2.14×10^{-1}	1.41×10^{-2}	7.30×10^{-5}	1.36×10^{-2}
p	0.1	0.18	0.819999999	0.999939999

Tables 7–9 are concerned with $q = 0.01$. In this case, the situation changes radically. The recursive algorithm works badly and requires much time and memory. For large x it turns out to be impossible to obtain numerical results in a reasonable time, and this explains why there are blank entries in Table 7. Whereas $L(x)$ is still close to $l(x)$, bound $U_1(x)$ is better than $u(x)$. In Table 9 the optimal value $b = 0.01$ and $U_2(x) > U_1(x)$, as in all other cases.

In the following group of results, $\alpha = 5$. Therefore, F has a lighter tail than in the case $\alpha = 3$. Tables 10–12 correspond to $q = 0.5$. The picture here is approximately the same as in the case $\alpha = 3$. The optimal bound $U_2(x)$ corresponds to $b = 0.5$, and its values are worse than $U_1(x)$ beginning from $x = 40$.

Tables 13–15 correspond to $q = 0.1$. Here the accuracy of $L(x)$ and $U_1(x)$ is higher than for the case $\alpha = 3$. Optimal values of s are far from its limiting value $s = 5$. However, the larger x is, the closer is s to 5. Asymptotic formula (1.5) works badly for relatively small x.

In the last collection of tables, $q = 0.01$. One can see from Table 16 that the recurrence algorithm works badly in this case. The bounds $L(x)$ and $U_1(x)$ shown in Table 17 are better than $l(x)$ and $u(x)$, and the larger x is, the more noticeable is the difference between these two types of bounds. This is of great value, as the cases $q \to 0$ and $x \to \infty$ are difficult for direct calculations although they are important in applications.

If we compare values from columns $b = 0.1$ and 0.01 in Table 3 with corresponding values in Table 12, we discover their similarity (column $b = 0.01$ is the most impressive). The same effect can be observed on comparing Table 6 with Table 15 and Table 9 with Table 18. Such a similarity is quite natural. To explain this, let us take column $b = 0.01$ in Tables 3 and 12. This corresponds to $x^* = 298$ when $\alpha = 3$ and $x^* = 496$ when $\alpha = 3$ (see Examples 2 for notation). But $x < 96$ in both cases. This means that the function

Table 7

x	$l(x)$	$u(x)$	$A(x)$
200	9.44×10^{-2}	1.86×10^{-1}	9.71×10^{-5}
600	8.88×10^{-4}	6.62×10^{-3}	3.67×10^{-6}
1000	5.04×10^{-6}	4.20×10^{-4}	7.95×10^{-7}
1400	4.40×10^{-7}	3.16×10^{-5}	2.90×10^{-7}
2000	1.22×10^{-7}	4.70×10^{-6}	9.97×10^{-8}
3200			2.44×10^{-8}
4800			7.22×10^{-9}

Table 8

x	$L(x)$	$U_1(x)$	a	θ	s
200	1.25×10^{-1}	1.00			
600	2.24×10^{-3}	2.93×10^{-3}	0.5	0.7	0.70
1000	4.07×10^{-5}	6.31×10^{-5}	0.7	0.7	1.00
1400	9.23×10^{-7}	2.04×10^{-6}	1.5	0.9	1.45
2000	7.97×10^{-8}	1.71×10^{-7}	3.1	0.9	1.70
3200	2.10×10^{-8}	3.66×10^{-8}	4.8	0.9	1.85
4800	6.54×10^{-9}	1.08×10^{-8}	6.8	0.9	1.90

Table 9

x	b			
	∞	0.5	0.1	0.01
200	8.79×10^{-1}	1.11×10^{-1}	3.47×10^{-2}	1.36×10^{-1}
600	8.51×10^{-1}	1.04×10^{-1}	2.50×10^{-2}	6.75×10^{-3}
1000	8.38×10^{-1}	1.01×10^{-1}	2.14×10^{-2}	1.53×10^{-3}
1400	8.30×10^{-1}	9.86×10^{-2}	1.94×10^{-2}	5.78×10^{-4}
2000	8.21×10^{-1}	9.64×10^{-2}	1.74×10^{-2}	2.05×10^{-4}
3200	8.10×10^{-1}	9.37×10^{-2}	1.51×10^{-2}	5.22×10^{-5}
4800	8.00×10^{-1}	9.15×10^{-2}	1.33×10^{-2}	1.60×10^{-5}
p	0.01	0.02	0.1	0.97

Table 10

x	$l(x)$	$u(x)$	$A(x)$
4	1.37×10^{-1}	1.40×10^{-1}	6.24×10^{-2}
12	6.57×10^{-3}	6.74×10^{-3}	1.95×10^{-3}
20	6.21×10^{-4}	6.34×10^{-4}	2.57×10^{-4}
28	1.09×10^{-4}	1.10×10^{-4}	6.10×10^{-5}
40	1.69×10^{-5}	1.88×10^{-5}	1.24×10^{-5}
64	1.66×10^{-6}	1.76×10^{-6}	1.41×10^{-6}
96	2.27×10^{-7}	2.35×10^{-7}	2.05×10^{-7}

Table 11

x	$L(x)$	$U_1(x)$	a	θ	s
4	1.54×10^{-3}	8.10×10^{-1}	0.9	0.9	1.55
12	3.56×10^{-4}	7.70×10^{-2}	1.2	0.8	2.05
20	7.78×10^{-5}	9.72×10^{-3}	1.4	0.7	2.30
28	2.26×10^{-5}	1.68×10^{-3}	1.7	0.7	2.55
40	5.22×10^{-6}	2.09×10^{-4}	2.2	0.7	2.85
64	6.50×10^{-7}	1.43×10^{-5}	2.9	0.7	3.15
96	9.89×10^{-8}	1.64×10^{-6}	3.7	0.7	3.35

Table 12

	b			
x	∞	**0.5**	**0.1**	**0.01**
4	3.53×10^{-1}	2.71×10^{-1}	1.34	1.92
12	6.25×10^{-1}	1.63×10^{-2}	6.02×10^{-1}	1.77
20	2.26×10^{-2}	3.44×10^{-3}	2.71×10^{-1}	1.64
28	1.10×10^{-2}	1.14×10^{-3}	1.21×10^{-1}	1.51
40	4.98×10^{-3}	3.34×10^{-4}	3.66×10^{-2}	1.34
64	1.67×10^{-3}	6.27×10^{-5}	4.32×10^{-3}	1.05
96	6.40×10^{-4}	1.42×10^{-5}	6.29×10^{-4}	7.66×10^{-1}
p	0.5	0.769	0.99961999	0.999999994

Table 13

x	*l*(*x*)	*u*(*x*)	*A*(*x*)
20	1.31×10^{-1}	1.48×10^{-1}	1.29×10^{-3}
60	2.54×10^{-3}	3.58×10^{-3}	9.54×10^{-6}
100	5.27×10^{-5}	9.01×10^{-5}	8.42×10^{-7}
140	1.42×10^{-6}	2.61×10^{-6}	1.65×10^{-7}
200	5.01×10^{-8}	8.05×10^{-8}	2.90×10^{-8}
320	3.83×10^{-9}	4.17×10^{-9}	2.87×10^{-9}
480	4.73×10^{-10}	5.06×10^{-10}	3.86×10^{-10}

Table 14

x	*L*(*x*)	$U_1(x)$	*a*	θ	*s*
20	7.38×10^{-2}	2.18×10^{-1}	0.9	0.9	0.90
60	1.12×10^{-3}	7.20×10^{-3}	0.9	0.6	1.25
100	1.71×10^{-5}	2.70×10^{-4}	1.1	0.5	1.55
140	3.50×10^{-7}	1.31×10^{-5}	1.4	0.6	2.05
200	2.09×10^{-8}	4.28×10^{-7}	2.2	0.7	2.65
320	2.24×10^{-9}	1.61×10^{-8}	3.5	0.8	3.15
480	3.18×10^{-10}	1.64×10^{-9}	4.5	0.8	3.35

Table 15

	b			
x	∞	**0.5**	**0.1**	**0.01**
20	4.54×10^{-1}	2.37×10^{-2}	1.50×10^{-1}	9.10×10^{-1}
60	2.78×10^{-1}	9.14×10^{-3}	3.54×10^{-3}	6.10×10^{-1}
100	2.18×10^{-1}	5.71×10^{-3}	3.71×10^{-4}	4.09×10^{-1}
140	1.85×10^{-1}	4.16×10^{-3}	8.16×10^{-5}	2.74×10^{-1}
200	1.56×10^{-1}	2.97×10^{-3}	1.62×10^{-5}	1.50×10^{-1}
320	1.23×10^{-1}	1.90×10^{-3}	1.88×10^{-6}	4.52×10^{-2}
480	1.01×10^{-1}	1.28×10^{-3}	2.91×10^{-7}	9.14×10^{-3}
p	0.1	0.194	0.93	0.99999994

Table 16

x	*l*(*x*)	*u*(*x*)	*A*(*x*)
200	9.27×10^{-2}	1.86×10^{-1}	2.90×10^{-7}
600	8.08×10^{-4}	6.42×10^{-3}	1.27×10^{-9}
1000	2.79×10^{-6}	3.94×10^{-4}	1.00×10^{-10}
1400	6.70×10^{-9}	2.82×10^{-5}	1.88×10^{-11}
2000		3.63×10^{-6}	3.17×10^{-12}
3200			3.03×10^{-13}
4800			4.00×10^{-14}

Table 17

x	*L*(*x*)	$U_1(x)$	*a*	θ	*s*
200	1.28×10^{-1}	1.00			
600	2.30×10^{-3}	2.76×10^{-3}	0.2	0.4	0.55
1000	4.13×10^{-5}	5.36×10^{-5}	0.2	0.3	0.65
1400	7.42×10^{-7}	1.05×10^{-6}	0.4	0.4	1.10
2000	1.79×10^{-9}	2.96×10^{-9}	0.6	0.6	2.00
3200	2.72×10^{-13}	8.08×10^{-13}	3.3	0.9	3.35
4800	3.62×10^{-14}	7.51×10^{-14}	5.7	0.9	3.60

Table 18

	b			
x	∞	**0.5**	**0.1**	**0.01**
200	8.30×10^{-1}	3.72×10^{-2}	5.39×10^{-3}	1.37×10^{-1}
600	7.86×10^{-1}	3.34×10^{-2}	3.31×10^{-3}	2.76×10^{-3}
1000	7.66×10^{-1}	3.17×10^{-2}	2.63×10^{-3}	2.19×10^{-4}
1400	7.54×10^{-1}	3.07×10^{-2}	2.26×10^{-3}	4.12×10^{-5}
2000	7.40×10^{-1}	2.96×10^{-2}	1.93×10^{-3}	6.99×10^{-6}
3200	7.23×10^{-1}	2.82×10^{-2}	1.56×10^{-3}	6.74×10^{-7}
4800	7.09×10^{-1}	2.71×10^{-2}	1.30×10^{-3}	8.95×10^{-8}
p	0.01	0.02	0.09	0.997

$T^*(x)$ is the same in the two cases, and therefore the bounds are also the same.

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DISCUSSION

JOHN A. BEEKMAN*

The author is to be congratulated for writing an excellent paper, with a thorough analysis of an important risk theory topic.

Five key ideas in his paper are a geometric sum, lower and upper bounds on $\psi(u)$, algorithms for obtaining such bounds, various claim distributions, and tables of bounds of $\psi(u)$ for certain claim distributions. Each of those ideas prompts me to offer further references.

The author's series (1.1) was used by this discussant in Beekman (1968, 1985); see also Shiu (1988), with its further references.

Goovaerts and DeVylder (1984) and Gerber and Dufresne (1989) obtain upper and lower bounds on $\psi(u)$ and develop algorithms to compute such bounds. Beekman and Fuelling (1995) used those bounds and algorithms to obtain initial surplus values in eight examples. The claim distributions were: (1) exponential, (2) Swedish nonindustry fire insurance, (3) modified lognormal distribution, (4) a second modified lognormal distribution, (5) Weibull distribution, (6) a γ distribution, (7) a second γ distribution, and (8) a Pareto distribution. We determined values u_1, u_2, and u_3 such that $\psi(u_1) = 0.1$, $\psi(u_2) = 0.05$, and $\psi(u_3) = 0.01$ for each example. Our Pareto distribution was

$$P(x) = 1 - (1 + 2x)^{-1.5},$$

whereas the author considers the two Pareto distributions

$$F(x) = 1 - (1 + 0.5x)^{-3},$$
$$F(x) = 1 - (1 + 0.25x)^{-5}.$$

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We used $\theta = 0.3$, which would equate to $q = 0.23$ in the paper's equation (1.1). The q values for the author's tables are 0.5, 0.1, and 0.01. Section 6 indicates that the new bounds yield closer numerical values than those obtained from the recursive algorithm of Gerber and Dufresne, for many values of u.

The discussant would be interested in seeing the author apply his bounds in the eight examples considered in Beekman and Fuelling (1995), as well as in the inverse Gaussian distribution examples of Beekman (1985).

There are several other references that may prove useful to the author, namely, Panjer (1986) and Ramsay (1992a, 1992b). The last reference uses the first four sample moments of the claim distribution and illustrates the method with 14 tables. Its algorithm compares favorably with previously developed approximations to $\psi(u)$.

In summary, this excellent paper is a substantial contribution to actuarial uses of the infinite time ruin function from collective risk theory.

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AUTHOR'S REPLY

VLADIMIR KALASHNIKOV

It is a great honor for me to have such a discussant as Dr. Beekman. I thank him for his valuable remarks and very useful references.

I appreciate greatly the proposal to apply my bounds in the cases indicated by the discussant. This will definitely be done. And I should mention that, after the submission of this paper, some new bounds were obtained; they will also be tested on those cases.

Additional discussions on this paper can be submitted until October 1, 1999. The author reserves the right to reply to any discussion. Please see the Submission Guidelines for Authors for instructions on the submission of discussions.