

TERM STRUCTURE MODELS: A PERSPECTIVE FROM THE LONG RATE*

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ABSTRACT

Term structure models based on dynamic asset-pricing theory are discussed by taking a perspective from the long rate. This paper partially answers two questions about the asymptotic behavior of yields on default-free zero-coupon bonds: in frictionless markets having no arbitrage, what *should* the behavior be; and, in known term structure models, what *can* the behavior be.

In frictionless markets having no arbitrage, yields of all maturities should be positive and uniformly bounded from above. The yield curve should level out as term to maturity increases. Slopes with large absolute values occur only in the early maturities. In a continuous-time framework, the longer the maturity of the yield is, the less volatile it will be. The long rate should be a nondecreasing process. Furthermore, the long rate in continuous-time factor models with nonsingular volatility matrices should be a nondecreasing deterministic function.

In the Black, Derman, and Toy model and factor models with the short rate having the mean reversion property, yields of all maturities are uniformly bounded from above. The long rate in the Duffie and Kan model with the mean reversion property is a constant. The long rate in the Heath, Jarrow, and Morton model can be infinite or a nondecreasing process. Examples with the long rate increasing are given in this paper. A model with the long rate and short rate as two state variables is then obtained.

1. INTRODUCTION

There are two approaches to modeling the term structure of interest rates. The first is fitting curves to data from bond markets using statistical techniques. The objective of this empirical estimation of the term structure of interest rates is to find a smooth function of time to maturity that fits the data sufficiently well. Models in the literature include the polynomial splines of McCulloch (1971), the exponential splines of Vasicek and Fong (1982), the parsimonious functional form of Nelson and Siegel (1987), and others. See Anderson et al. (1996) for a detailed explanation and comparison of these models. This approach takes a static view and considers solely the shape of the term structure of interest rates.

The second approach takes a dynamic view and considers both shapes of the term structure of interest rates and their evolution. This approach is based on models from recent advances in dynamic asset-pricing theory. These models postulate explicit assumptions about the evolution of factors driving interest rates and deduce characterizations of shapes and movements of the term structure of interest rates in frictionless markets having no arbitrage. A partial list of known models is given in Appendix A.

The empirical fitting approach and the dynamic asset-pricing approach are closely related. Some dynamic term structure models (for example, Ho and Lee 1986; Pedersen, Shiu, and Torlacius 1989; Heath, Jarrow, and Morton 1992) take the current term structure, resulting from the empirical fitting approach, as given and then specify the future evolution of the term structure of interest rates. Meanwhile, the functional form for the term structure from the dynamic asset-pricing approach is also used to fit the data. This gives an alternative method for the empirical estimation of the term structure. When analyzing

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fixed-income portfolios, pricing and hedging fixed-income options or other interest-rate-sensitive products, it is not enough to know where interest rates currently are. One also needs to know where interest rates can be in the future. Thus, the dynamic asset-pricing approach is the topic of this paper.

Traditionally, the dynamic asset-pricing approach to term structure modeling takes a perspective from the short rate, which plays a central role in these models. There are two major approaches to dynamic asset pricing: equilibrium pricing and pricing by no-arbitrage. Following the seminal paper of Black and Scholes (1973), pricing by no-arbitrage has been the dominating method in the area of dynamic asset pricing. This pricing method requires that one knows the sequence of short rates for each scenario, so a term structure model must provide this information. Meanwhile, as argued empirically by a number of authors (for example, Litterman and Scheinkman 1991), most of the variation in returns on all fixed-income securities can be explained in terms of three factors, or attributes, of the term structure. One of the three factors describes a common shift of all interest rates in the same direction. This factor alone explains a large fraction of the overall movement of the term structure. Hence, in many instances, valuation with a one-factor model is sufficient. Many term structure models are simply models of the stochastic evolution of the short rate, or models with the short rate as one of the state variables.

There are presently many different term structure models being used in valuation and hedging, but there is little agreement on any one of them being the preferred one. In general, the choice of models depends on the nature of the problem to be solved. See Rogers (1995) for a discussion about the choice of models for the term structure of interest rates.

While the U.S. Treasury market and similar markets in other countries provide a readily observable term structure up to 30 years or so, many finance practitioners require a term structure for 100 years or more. The following are some situations in which the time horizon of *fixed* or *contingent* cash flows extends beyond the limit of the observable term structure: pricing and hedging long-term fixed-income securities, valuation of insurance and annuity contracts, capital budgeting for long-term infrastructure projects, making funding decisions and setting valuation assumptions for private and public retirement systems, and determining settlement values arising from legal decisions (injury, divorce, and so on). To address these needs, we need to have term structure models

that can be used to extrapolate the observable term structure and give appropriate descriptions about the evolution of interest rates of longer maturities.

Hogan (1993) identifies internal inconsistencies linked to the specific parameterization and functional form chosen for the short and long rates in the Brennan and Schwartz (1979) model. (It should be noted that, in Brennan and Schwartz 1979, the long rate is the yield on a console bond that pays coupons continuously and perpetually. In this article, the long rate is the yield on a zero-coupon bond with an infinite maturity.) Dybvig, Ingersoll, and Ross (1996) show that, in frictionless markets having no-arbitrage, the asymptotic long forward and zero-coupon rates never fall. These two results serve as a caution that not every assumption about co-movements of discount rates is consistent with a coherent model of the term structure. They also make clear that it is necessary to examine the behavior of yields of longer maturities in known term structure models before using them for extrapolating the observable term structure and pricing long-term products. Figures 1 and 2 give another reason.

Figures 1 and 2 illustrate two yield curves from the Merton (1970) and the Cox, Ingersoll, and Ross (1985) models. (Graphs are plotted using the parameters estimated in Chan et al. 1992, p. 1218.) The horizontal axis is the time to maturity in years, and the vertical axis is the yield to maturity in percentage. Figure 1 shows these two curves with the time to maturity being less than 30 years. These two curves nearly coincide with each other. However, extending

Figure 1
Yield Curves in the Merton Model (Dashed Curve) and the Cox, Ingersoll, and Ross Model (Solid Curve) with the Time to Maturity up to 30 Years

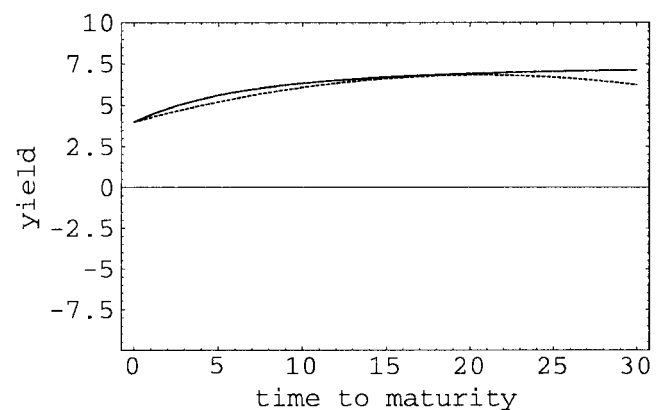
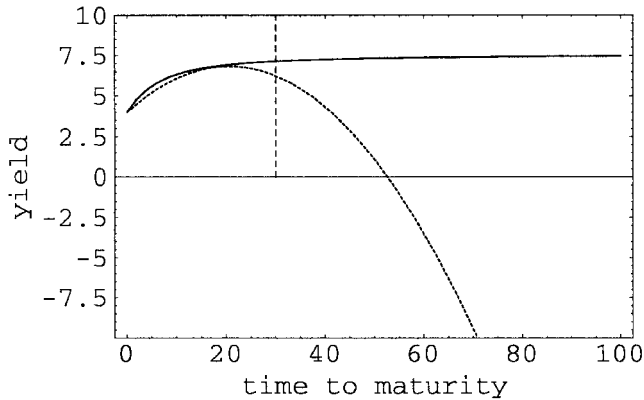


Figure 2
Yield Curves in the Merton Model (Dashed Curve) and the Cox, Ingersoll, and Ross Model (Solid Curve) with the Time to Maturity up to 100 Years



the figure to show the curves with the time to maturity up to 100 years (see Figure 2), we find a dramatic difference between these two curves beyond 30 years. So it is necessary to pay attention to the theoretical as well as the empirical implications of term structure models. This article studies one aspect of the theoretical implications of term structure models, the asymptotic behavior of yields on default-free zero-coupon bonds.

This paper attempts to answer partially two questions about the asymptotic behavior of yields on default-free zero-coupon bonds: (1) what can be the behavior in frictionless markets having no-arbitrage and (2) what can be the behavior in known dynamic term structure models? This article first discusses the risk-neutral approach and argues that, in frictionless markets having no-arbitrage, yields of all maturities should be positive and uniformly bounded from above. The yield curve should level out as term to maturity increases; and slopes with large absolute values occur only in the early maturities. It then models arbitrage-free prices of default-free discount bonds as a stochastic differential system and puts factor models and the Heath, Jarrow, and Morton model in this framework. (In general, dynamic term structure models in the literature can be classified as the factor model and the Heath, Jarrow, and Morton model. A review of these two classes of models is given in Back 1996. See also Ang and Sherris 1997 and Vetzal 1994.) A representation for the yield and the long rate is derived from the price and yield relationship. In this setting, I argue that the longer the maturity of the yield is,

the less volatile it will be, and the long rate will be totally determined by the volatility of default-free zero-coupon bond prices. The long rate should be a nondecreasing process. Furthermore, I argue that the long rate in a continuous-time factor model with non-singular volatility matrices should be a nondecreasing deterministic function.

Interest rates have a tendency to be pulled back to some long-run level. This phenomenon is known as mean reversion. I find that explicitly modeling this mean reversion property plays an important role in the boundedness of yields of longer maturities in a term structure model. In factor models with the short rate having the mean reversion property, yields of all maturities are uniformly bounded from above. It is also shown in this paper that in the Black, Derman, and Toy model, yields of all maturities are uniformly bounded from above. Furthermore, the long rate in the Duffie and Kan model with the mean reversion property is a constant.

Following Ho and Lee (1986), many researchers and practitioners model the term structure dynamics by taking the current yield curve as given (for example, Pedersen, Shiu, and Thorlacius 1989; Heath, Jarrow, and Morton 1992), or by making the state variables in factor models time-inhomogeneous and calibrating the model to fit the current yield curve (for example, Hull and White 1990; Black, Derman, and Toy 1990). These methods give a richer behavior of the long rate. However, they may also lead to an infinite long rate, which should be avoided. The long rate from the Heath, Jarrow, and Morton model can be an infinite or a nondecreasing process. This paper gives examples with increasing long rates and a model with the long and short rates as two state variables.

By analyzing the asymptotic behavior of yields on default-free zero-coupon bonds, I also provide a review of dynamic term structure models from the long rate perspective. This can be used to choose among known models for the purpose of pricing long-term fixed-income securities, valuation of insurance and annuity contracts, and so on. This paper also attempts to set a foundation for the future research on modeling yields of longer maturities and pricing and hedging long-term bonds and other interest-rate-sensitive claims.

The outline of the paper is as follows. Section 2 defines concepts that describe the term structure of interest rates. To motivate dynamic term structure models, I first consider a simple situation in which all future movements of interest rates are known with certainty. Section 3 discusses the asymptotic behavior

of yields on default-free zero-coupon bonds in frictionless markets having no-arbitrage. Section 4 investigates the dynamics of the long rates in known factor models. Section 5 gives a numerical example to illustrate the valuation of insurances and annuities in the risk-neutral framework discussed in the paper. I give my conclusion in Section 6.

2. DESCRIPTIONS OF THE TERM STRUCTURE OF INTEREST RATES

Much of the difficulty in discussing the term structure of interest rates is caused by cumbersome notation and inconsistent usage of terminology. To minimize this problem, I now define the notation used in this article and keep the symbols to the minimum required.

A default-free zero-coupon bond maturing at time T is a security that will pay one unit of currency at time T and nothing at any other time. I denote the price at time t of this bond as $P(t, T)$. (I always assume $0 \leq t \leq T$.) At maturity date T , we have $P(T, T) = 1$. The yield (to maturity) is defined as the continuously compounded rate of return for which the bond price rises to one at maturity date T . The yield at time t is denoted as $y(t, T)$. It is determined by the price-yield relationship

$$P(t, T) = \exp\{-\tau y(t, T)\} \tag{2.1}$$

or

$$y(t, T) = -\frac{\log P(t, T)}{\tau}, \tag{2.2}$$

where $\tau = T - t$. As a function of τ , $y(t, T)$ is usually called the (zero-coupon) yield curve at time t . The yield curve describes the term structure of interest rates by specifying the interest rate of any given maturity.

Two concepts of great importance here are the short rate and the (asymptotic) long rate, which are the two ends of a yield curve. The *short rate* or instantaneous spot rate is the yield on the currently maturing bond. Hence, denoting the short rate at time t by $r(t)$, we have

$$r(t) = y(t, t) \equiv \lim_{T \rightarrow t+} y(t, T).$$

The (asymptotic) *long rate* is the yield on the bond with infinite maturity. The long rate prevailing at time t , $l(t)$, is given by

$$l(t) = y(t, \infty) \equiv \lim_{T \rightarrow +\infty} y(t, T).$$

For one unit of currency invested in a money market account at time $t = 0$, its value at time t will be

$$B(t) = \exp \left\{ \int_0^t r(s) ds \right\}.$$

In dynamic asset-pricing theory, the money market account is a benchmark for pricing. The last equation partially explains the importance of the short rate concept in dynamic asset-pricing theory. The importance of the concept of the long rate is its role in describing the behavior of yields of longer maturities, which is important in pricing and hedging long-term fixed-income securities. I discuss this role of the long rate in Section 3.

Another concept used here is the forward rate. The (instantaneous) *forward rate* is the instantaneous rate of return that the bond holder can earn by extending his or her investment for an instant past T . Denote the forward rate at time t as $f(t, T)$. Then it is given by

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T).$$

it follows from integrating the last equation that

$$P(t, T) = \exp \left\{ -\int_t^T f(t, s) ds \right\}. \tag{2.3}$$

From Equations (2.2) and (2.3), we have the relationship between the yield and the forward rate

$$y(t, T) = \frac{1}{\tau} \left(\int_t^T f(t, s) ds \right). \tag{2.4}$$

As functions of τ , the price $P(t, T)$, the yield $y(t, T)$, and the forward rate $f(t, T)$ can be viewed as equivalent descriptions of the term structure of interest rates at time t before maturity data T .

Dynamic term structure models can be better understood by first considering a simplified situation in which all future movements of interest rates are known with certainty. In this case, because of the no-arbitrage condition, all future term structures of interest rates are totally determined by future short rates,

$$P(t, T) = \exp \left\{ -\int_t^T r(s) ds \right\}. \tag{2.5}$$

Differentiating the last equation with respect to t , we have

$$dP(t, T) = r(t)P(t, T) dt, \tag{2.6}$$

which means that the rates of return of all default-free

zero-coupon bonds at time t are equated to $r(t)$. This can be used to motivate term structure modeling in a risk-neutral framework. From Equations (2.3) and (2.5), we derive

$$\int_t^T r(s) ds = \int_t^T f(t, s) ds,$$

Differentiating the last equation with respect to maturity T yields

$$f(t, T) \equiv r(T);$$

hence,

$$f(t, T) \equiv f(0, T),$$

which means that, for each $T > 0$, the forward rate $f(t, T)$ is a constant in a world without uncertainty. This can be used to motivate the Heath, Jarrow, and Morton model.

3. THE ASYMPTOTIC BEHAVIOR OF YIELDS IN AN ARBITRAGE-FREE ECONOMY

In this section I discuss the asymptotic behavior of yields on default-free zero-coupon bonds in frictionless markets having no arbitrage. To do this, I first give a brief description of the risk-neutral approach.

A major approach in dynamic asset-pricing theory is the risk-neutral approach, which is one form of pricing by no-arbitrage. The risk-neutral approach to dynamic asset pricing is developed by Ross (1976) and Cox and Ross (1976) and extended by Ross (1978), Harrison and Kreps (1979), and Harrison and Pliska (1981). In frictionless markets having no arbitrage, there exists a risk-neutral measure \mathcal{Q} such that, for each non-dividend-paying asset with price at time t denoted as $p(t)$,

$$p(t) = E_t \left[\exp \left(- \int_t^T r(s) ds \right) p(T) \right], \quad (3.1)$$

where $E_t[\cdot]$ denotes the conditional expectation given the information available at time t , taken with respect to the measure \mathcal{Q} . One example of such assets is a default-free zero-coupon bond.

3.1 An Upper Bound for Yields on Default-Free Zero-Coupon Bonds

Because a default-free zero-coupon bond maturing at time T will pay one unit of currency at time T and nothing at any other time, it follows from Equation (3.1) that

$$P(t, T) = E_t \left[\exp \left(- \int_t^T r(s) ds \right) \right]. \quad (3.2)$$

Jensen's Inequality states that, for a random variable X and a convex function u , $E[u(X)] \geq u(E[X])$. See Bowers et al. (1986, Chapter 1) for the details. With $X = -\int_t^T r(s) ds$ and $u(x) = \exp(x)$, it follows from Jensen's Inequality and Equation (3.2) that

$$P(t, T) \geq \exp \left(- \int_t^T E_t[r(s)] ds \right).$$

In frictionless markets having no arbitrage, the short rate should be non-negative. From Equation (3.2) we have

$$P(t, T) \leq 1.$$

Combining the last two inequalities and noting Equation (2.2), we obtain

$$0 \leq y(t, T) \leq \frac{1}{T-t} \left(\int_t^T E_t[r(s)] ds \right), \quad (3.3)$$

which means that the yield to maturity is less than average expected short rate, where the (conditional) expectation is taken with respect to the risk-neutral measure \mathcal{Q} . Taking $T \rightarrow +\infty$ in Equation (3.3) and assuming that the limits exist yield

$$0 \leq l(t) \leq \lim_{T \rightarrow +\infty} \frac{1}{T-t} \left(\int_t^T E_t[r(s)] ds \right).$$

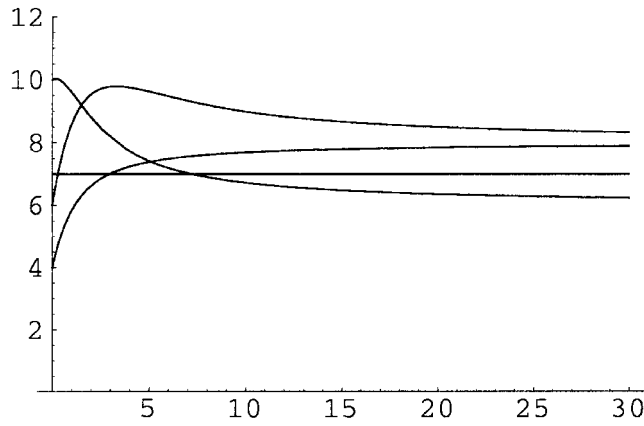
Dybvig, Ingersoll, and Ross (1996) give an example where the long rate does not exist even though yields and forward rates are uniformly bounded.

It is reasonable to assume that the average expected short rate should be uniformly bounded from above. This means that yields of all maturities should be uniformly bounded from above. The long rate *should* be finite if it exists.

3.2 The Yield Curve Should Level Out as Term to Maturity Increases

Generally, smoothed empirical yield curves have approximated one of four forms: the flat curve, the ascending curve, the decreasing curve, and the humped curve. As pointed out by Malkiel (1966, p. 16), the empirical yield curve has the pervasive tendency to level out as term to maturity increases; and slopes with large absolute values occur only in the early maturities. Figure 3 shows these four types of yield curves. The horizontal axis is the time to maturity in years, and the vertical axis is the yield to maturity as a percentage.

Figure 3
The Four Types of Yield Curves



In this section I show that, in frictionless markets having no arbitrage, the yield curve should level out as term to maturity increases. To do this, I differentiate Equation (2.4) with respect to term to maturity $\tau = T - t$:

$$\frac{\partial y(t, T)}{\partial \tau} = -\frac{1}{\tau^2} \left(\int_t^T f(t, s) ds \right) + \frac{f(t, T)}{\tau}.$$

Using Equation (2.4) again, we can rewrite the last equation as

$$\frac{\partial y(t, T)}{\partial \tau} = \frac{f(t, T) - y(t, T)}{\tau}.$$

From Section 3.1 we know that, if the average expected short rate is uniformly bounded from above, yields are also uniformly bounded; then we have, for each t , $\lim_{\tau \rightarrow +\infty} y(t, T)/\tau = 0$. It is reasonable to assume that, for each t , the forward rate should be uniformly bounded from above. In this case, we have, for each t , $\lim_{\tau \rightarrow +\infty} f(t, T)/\tau = 0$. Thus, we have

$$\lim_{\tau \rightarrow +\infty} \frac{\partial y(t, T)}{\partial \tau} = 0,$$

which means that the yield curve levels out as term to maturity increases and slopes with large absolute values occur only in the early maturities.

3.3 The Longer the Maturity Is, the Less Volatile the Yield Is

Although various methods have been used, the most common language for modeling the term structure is that of continuous-time stochastic calculus. In frictionless markets having no arbitrage, for each fixed T ,

the bond price $P(t, T)$ can be represented as a stochastic differential equation of the form

$$\frac{dP(t, T)}{P(t, T)} = r(t)dt - \sigma^P(t, T) \cdot dW(t), \quad (3.4)$$

where $\sigma^P(t, T)$ is a $1 \times n$ vector, and $W(t) = (W_1(t), W_2(t), \dots, W_n(t))^*$ is an n -dimensional standard Brownian motion under a risk-neutral measure \mathcal{Q} . (In this article the asterisk denotes the vector and matrices transpose operation, and the ellipsis the product of vectors and matrices.) Note that Equation (3.4) generalizes Equation (2.6). In Appendix B I give a brief discussion about deriving this equation, and show how to use it to derive the partial differential equation of bond prices under the risk-neutral measure in factor models, and an arbitrage-free characterization of the term structure in terms of forward rates, namely, the Heath, Jarrow, and Morton model.

Next derive a representation for the long rate in the continuous-time arbitrage-free framework. Using Itô's Lemma, we obtain from the last equation

$$d(\log P(t, T)) = r(t)dt - \frac{1}{2} \sigma^P(t, T) \cdot (\sigma^P(t, T))^* dt - \sigma^P(t, T) \cdot dW(t).$$

Changing t to s in the last equation and integrating from $s = 0$ to $s = t$ yields

$$\begin{aligned} \log P(t, T) &= \log P(0, T) + \int_0^t r(s) ds \\ &\quad - \frac{1}{2} \int_0^t \sigma^P(s, T) \cdot (\sigma^P(s, T))^* ds \\ &\quad - \int_0^t \sigma^P(s, T) \cdot dW(s). \end{aligned}$$

Recalling that $y(t, T) = \log P(t, T)/\tau$ with $\tau = T - t$, we have

$$\begin{aligned} y(t, T) &= \frac{T}{\tau} y(0, T) - \frac{1}{\tau} \int_0^t r(s) ds \\ &\quad + \frac{1}{2} \int_0^t \frac{\sigma^P(s, T) \cdot (\sigma^P(s, T))^*}{\tau} ds \\ &\quad + \int_0^t \frac{\sigma^P(s, T)}{\tau} \cdot dW(s). \end{aligned}$$

Letting $T \rightarrow +\infty$, and so $\tau \rightarrow +\infty$, we have the representation for the long rate

$$l(t) = l(0) + \frac{1}{2} \int_0^t \delta(s) ds + \int_0^t \Delta(s) \cdot dW(s),$$

where $\delta(s) = \lim_{T \rightarrow +\infty} [\sigma^P(s, T) \cdot (\sigma^P(s, T))^*] / \tau$ is a non-negative scalar, and $\Delta(s) = \lim_{T \rightarrow +\infty} \sigma^P(s, T) / \tau$ is an $n \times 1$ vector. It is easy to see that if $\delta(s)$ is a well-defined function, then $\Delta(s)$ must be zero. So we have

$$l(t) = l(0) + \frac{1}{2} \int_0^t \delta(s) ds, \tag{3.5}$$

which is a nondecreasing process. Thus, the long rate is determined by the volatility of default-free zero-coupon bond prices. Also note that $\sigma^P(t, T) / (T - t)$ is the (instantaneous) volatility of the yield on a default-free zero-coupon bond with maturity T . It follows from the finiteness of $\delta(s)$ that $\lim_{T \rightarrow +\infty} \sigma^P(t, T) / (T - t) = 0$. Hence, in general, the longer the maturity of the yield is, the less volatile it is.

Figure 4 illustrates how the (instantaneous) volatility of the yield to maturity decreases as term to maturity increases in two popular models. The horizontal axis is the time to maturity in years, and the vertical axis is the (instantaneous) volatility of the yield to maturity. The two curves represent the volatility of the yield to maturity in the Vasicek (1977) and the Cox, Ingersoll, and Ross (1985) models, respectively.

3.4 The Long Rate Should Be Deterministic in Factor Models: The Nonsingular Volatility Matrix Case

Most, if not all of, known term structure models are factor models. A partial list of known models is given as Appendix A. In factor models the term structure of

interest rates is determined by a finite number of state variables $X = (X_1, X_2, \dots, X_n)^*$, which are assumed to be governed by a stochastic differential system of the form

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t) \cdot dW(t),$$

where $\mu(X, t)$ is an $n \times 1$ vector, $\sigma(X(t), t)$ is an $n \times n$ matrix, and $W(t)$ is an n -dimensional standard Brownian motion under a risk-neutral measure Q .

In factor models the bond price $P(t, T)$ is assumed to be a function of state variables X . From the yield and price relationship, yields are also functions of state variables X , and so is the (asymptotic) long rate. Using Itô's Lemma, we have

$$dl = \left\{ \frac{\partial l}{\partial t} + \frac{\partial l}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 l}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] \right\} dt + \frac{\partial l}{\partial x} \cdot \sigma \cdot dW(t). \tag{3.6}$$

For notional simplicity, we have omitted variable dependence. Here,

$$\frac{\partial l}{\partial x} = \left(\frac{\partial l}{\partial x_1}, \frac{\partial l}{\partial x_2}, \dots, \frac{\partial l}{\partial x_1}, \dots, \frac{\partial l}{\partial x_n} \right)$$

is a $1 \times n$ vector of the first-order partial derivatives,

$$\frac{\partial^2 l}{\partial x^2} = \left(\frac{\partial^2 l}{\partial x_i \partial x_j} \right)_{n \times n}$$

is an $n \times n$ matrix comprising the second-order partial derivatives,

and $\text{tr}[\cdot]$ is the trace of a matrix. Comparing Equation (3.6) with Equation (3.5), we obtain

$$\frac{\partial l}{\partial x} \cdot \sigma \equiv 0.$$

If volatility matrices of state variables are nonsingular, then

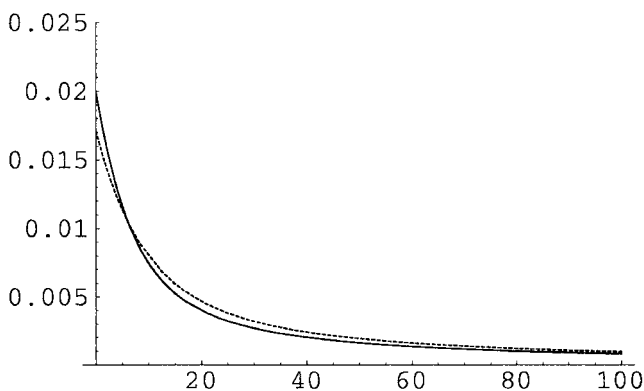
$$\frac{\partial l}{\partial x} \equiv 0;$$

that is, the long rate $l(t)$ in factor models is a non-decreasing function of time t and does not depend on the state variables.

In the next section examples from the Heath, Jarrow, and Morton framework are given to show that, if the volatility matrix of state variables is singular, then the long rate $l(t)$ can be an increasing process.

Figure 4

The (Instantaneous) Volatility of the Yield to Maturity in the Vasicek Model (Solid Curve) and the Cox, Ingersoll, and Ross Model (Dashed Curve)



4. THE ASYMPTOTIC BEHAVIOR OF YIELDS IN KNOWN TERM STRUCTURE MODELS

Many known factor models explicitly mean reversion of interest rates. Meanwhile, many practitioners use the Heath, Jarrow, and Morton model or time-inhomogeneous models of the Hull and White type. I analyze how these two phenomena affect the asymptotic behavior of yields on default-free zero-coupon bonds in known term structure models.

4.1 Mean Reversion of the Short Rate and a Bound for Yields

Many one-factor models appearing in the literature explicitly model mean reversion of interest rates and are of the form

$$dr(t) = [\alpha_1(t) - \alpha_2(t)r(t)]dt + \sigma(r(t), t)dW(t),$$

where α_1 and α_2 are deterministic functions of t . Some examples are found in Merton (1970), Vasicek (1977), Dothan (1978), Courtadon (1982), Cox, Ingersoll, and Ross (1985), Ho and Lee (1986), Hull and White (1990), and Chan et al. (1992).

Applying the method of integrating factors to the last equation yields

$$r(T) = r(t)e^{-\int_t^T \alpha_2(s)ds} + \int_t^T \alpha_1(U)e^{-\int_t^U \alpha_2(s)ds} dU + \int_t^T \sigma(U)e^{-\int_t^U \alpha_2(s)ds} dW(U).$$

This gives an expression for the expectation of $r(T)$ conditional on the information available at time t :

$$E_t[r(T)] = r(t) \exp \left\{ -\int_t^T \alpha_2(s) ds \right\} + \int_t^T \alpha_1(U) \exp \left\{ -\int_t^U \alpha_2(s) ds \right\} dU.$$

If there exist two positive constants C_1 and C_2 such that $\alpha_1(t) \leq C_1$ and $\alpha_2(t) \geq C_2$ for all t , then we have from the last equation

$$E_t[r(T)] \leq r(t) \exp\{-C_2(T - t)\} + C_1 \int_t^T \exp\{-C_2(T - U)\} dU.$$

Noting that $\exp\{-C_2(T - t)\} \leq 1$ and

$$\int_t^T \exp\{-C_2(T - U)\} dU = \frac{1 - \exp\{-C_2(T - t)\}}{C_2} \leq \frac{1}{C_2},$$

we obtain

$$E_t[r(T)] \leq r(t) + \frac{C_1}{C_2},$$

that is, $E_t[r(T)]$ is uniformly bounded from above. Then, from Section 3.1, yields of all maturities are uniformly bounded from above.

This condition for $\alpha_1(t)$ and $\alpha_2(t)$ can be easily satisfied by time-homogeneous models, in which the parameters $\alpha_1(t)$ and $\alpha_2(t)$ are constants. A counterexample is the continuous-time equivalent of the Ho and Lee model. In this model, the short rate is given by

$$dr(t) = \left[\frac{d}{dt} f(0, t) + t\sigma^2 \right] dt + \sigma dW(t).$$

In this case,

$$E[r(T)] = f(0, T) + \frac{T^2\sigma^2}{2}.$$

So $E[r(T)]$ approaches positive infinity as does T . As shown in Dybvig, Ingersoll, and Ross (1996), the long rate is positive infinity, that is, $l(t) = +\infty$.

4.2 Yields in the Black, Derman, and Toy Model

In the continuous-time equivalent of the Black, Derman, and Toy model, the short rate is given by

$$d[\log r(t)] = [\theta(t) - \phi(t) \log r(t)]dt + \sigma(t)dW(t),$$

where θ , ϕ , and σ are deterministic functions of t . For simplicity, we denote $R(t) = \log r(t)$. Applying the method of integrating factors yields

$$R(T) = R(t)e^{-\int_t^T \phi(s)ds} + \int_t^T \theta(U)e^{-\int_t^U \phi(s)ds} dU + \int_t^T \sigma(U)e^{-\int_t^U \phi(s)ds} dW(U).$$

This gives expressions for the conditional expectation and variance of $R(T)$

$$E_t[R(T)] = R(t)e^{-\int_t^T \phi(s)ds} + \int_t^T \theta(U)e^{-\int_t^U \phi(s)ds} dU,$$

$$\text{Var}_t[R(T)] = \int_t^T \sigma^2(U)e^{-2\int_t^U \phi(s)ds} dU.$$

Because $R(T)$ is normally distributed, $r(T)$ is lognormally distributed. We have

$$E_t[r(T)] = \exp \left\{ E_t[R(T)] + \frac{1}{2} \text{Var}_t[R(T)] \right\}.$$

If $E_t[r(T)]$ is uniformly bounded above, then, from Section 3.1, yields of all maturities are uniformly bounded from above. One condition that guarantees this is the existence of positive constants C_1, C_2 , and C_3 such that $\phi(t) \geq C_1$, $\theta(t) \leq C_2$, and $\sigma(t) \leq C_3$ for all times t .

4.3 The Long Rate in the Duffie and Kan Affine Models

Duffie and Kan (1996) consider a class of term structure models characterized by an affine relation between the drift and diffusion coefficients of the stochastic process describing the evolution of the state variables. It is assumed that the state variables X are square-root processes of form

$$dX(t) = (K \cdot X(t) + K_0)dt + \sqrt{V(X(t))} \cdot \Sigma \cdot dW(t), \quad (4.1)$$

where K is an $n \times n$ matrix, K_0 is an $n \times 1$ vector, and $\sqrt{V(X(t))}$ is the diagonal matrix

$$\text{diag} \left[\sqrt{\alpha_1 + \beta_1 \cdot X(t)}, \sqrt{\alpha_2 + \beta_2 \cdot X(t)}, \dots, \sqrt{\alpha_n + \beta_n \cdot X(t)} \right],$$

where, for each i , α_i is a scalar, β_i is a $1 \times n$ vector, and Σ is an $n \times n$ matrix that is positive semi-definite and symmetric. This class of models includes many parametric factor models in the literature or in industry practice as the special cases. Some examples are found in Merton (1970), Vasicek (1977), Cox, Ingersoll, and Ross (1985), Longstaff and Schwartz (1992), Chen and Scott (1992), and Chen (1996).

In this model, the bond prices $P(t, T)$ can be written in the form

$$P(t, T) = \exp\{A(\tau) + B(\tau) \cdot X(t)\},$$

where B is a vector-valued function satisfying the Riccati equation

$$\frac{dB(\tau)}{d\tau} = \frac{1}{2} \text{tr}[B^* \cdot B \cdot \beta] + B \cdot K + \gamma_i, \quad (4.2a)$$

and A is a scalar function satisfying the equation

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2} \text{tr}[B^* \cdot B \cdot \alpha] + B \cdot K_0 + \gamma_0, \quad (4.2b)$$

with $A(0) = 0$, $B(0) = 0$, and $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$.

If all eigenvalues of the matrix K have a negative real part, that is, the state variables have the mean reversion property, then the solution $B(\tau)$ of Equation (4.2a) is uniformly bounded above. (See, for example, Braun 1993, pp. 386 and 387.) So we have

$$\delta(s) = \lim_{\tau \rightarrow +\infty} \left[\frac{B \cdot \Sigma \cdot V \cdot \Sigma \cdot B^*}{\tau} \right] = 0.$$

Hence, from Section 3.3, the long rate in the Duffie-Kan model is a constant.

Now several simple examples can explain the technical condition. The simple models are the Merton (1970), Vasicek (1977), and Cox, Ingersoll, and Ross (1985) models. In these three models the short rate itself is the state variable. In the Merton model the short rate is given by

$$dr(t) = \theta dt + \sigma dW(t).$$

The yields in this model are

$$y(t, T) = r(t) + \frac{1}{2} \theta \tau - \frac{1}{6} \sigma^2 \tau^2.$$

If $\sigma \neq 0$, then the long rate in the Merton model is $l(t) = -\infty$. This example shows that nonsingularity of the volatility matrix of state variables is not enough to guarantee the finiteness of the long rate. In the Vasicek model the short rate is given by

$$dr(t) = k[\theta - r(t)]dt + \delta dW(t).$$

The yields in this model are

$$y(t, T) = l + [r(t) - l] \frac{1 - e^{-kT}}{kT} + \frac{\sigma^2(1 - e^{-kT})^2}{4k^3T},$$

with $l = \theta - \sigma^2/2k^2$. If $k > 0$, then the long rate in the Vasicek model is

$$l(t) \equiv l = \theta - \frac{\sigma^2}{2k^2}.$$

However, if $k \leq 0$, then the long rate in the Vasicek model is $l(t) = -\infty$. In the Cox, Ingersoll, and Ross model, the short rate is given by

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t).$$

If $\sigma \neq 0$, then the long rate in the Cox, Ingersoll, and Ross model is

$$l(t) = \frac{2k\theta}{k + \sqrt{\sigma^2 + k^2}}$$

4.4 The Long Rate in the Heath, Jarrow, and Morton Model: Examples

Heath, Jarrow, and Morton (1992) modeled the dynamics of the entire forward rate curve. Mathematically, for each fixed T , the forward rate at time t is assumed to satisfy the stochastic differential equation of the form

$$df(t, T) = \sigma_T(t) \cdot \left(\int_t^T \sigma_S(t) dS \right)^* dt + \sigma_T(t) \cdot dW(t), \quad (4.3)$$

where $\sigma_T(f(t, T), t)$ is a $1 \times n$ vector, and $W(t)$ is an n -dimensional standard Brownian motion under the risk-neutral measure \mathbf{Q} .

As shown in Appendix B, the Heath, Jarrow, and Morton model can be put in the risk-neutral framework discussed in Section 3. So from Section 3.3, if

$$\delta(t) = \lim_{T \rightarrow +\infty} \frac{\left(\int_t^T \sigma_S(t) dS \right) \cdot \left(\int_t^T \sigma_S(t) dS \right)^*}{T}$$

is a well-defined function, then the long rate exists and is a nondecreasing process with instantaneous volatility being zero. It is easy to find an example in which the long rate is infinite. Heath, Jarrow, and Morton (1992) give a simple model:

$$df(t, T) = \left[\sigma_1^2 \tau + \frac{2\sigma_2^2}{\lambda} e^{-(\lambda/2)\tau} (1 - e^{-(\lambda/2)\tau}) \right] dt + \sigma_1 dW_1(t) + \sigma_2 e^{-(\lambda/2)\tau} dW_2(t),$$

where $\tau = T - t$. In this model $l(t) = +\infty$. In the following discussion the goal is to find models in which the long rate $l(t)$ can be an increasing function.

In general, the evolution of the term structure in the Heath, Jarrow, and Morton model could depend on the entire path taken by the term structure since it was initialized. Under special volatility restrictions, the path dependence can be completely removed, and closed-form solutions are obtained for bond prices. Here I discuss only a one-dimensional case, which is considered by Cheyette (1992), Jamshidian (1991), Ritchken and Sankarsubramanian (1995), and Shiu and Yao (1998). In the one-dimensional case, to remove the path dependence, we can choose

$$\sigma_T(t) = \sigma(t) \exp \left[- \int_t^T k(x) dx \right],$$

where $k(x)$ is a deterministic function of time x . In this case the price of a default-free discount bond is given by

$$P(t, T) = \exp \left\{ - \int_t^T f(0, s) ds - B(t, T)[r(t) - f(0, t)] - \frac{1}{2} [B(t, T)]^2 \phi(t) \right\},$$

and the dynamic movement of the short rate is determined by

$$dr(t) = \left\{ k(t)[f(0, t) - r(t)] + \phi(t) + \frac{d}{dt} f(0, t) \right\} dt + \sigma(t) dW_1(t),$$

where $f(0, T)$ is the current forward rate,

$$B(t, T) = \int_t^T \exp \left[- \int_t^U k(x) dx \right] dU,$$

$$\phi(t) = \int_0^t \left\{ \sigma(s) \exp \left[- \int_s^t k(x) dx \right] \right\}^2 ds,$$

and $W_1(t)$ is a standard Brownian motion under the risk-neutral measure \mathbf{Q} . This model can be viewed as a factor model by setting $X_1(t) = r(t)$, $X_2(t) = \phi(t)$, and letting $\sigma(t) = \sigma(r(t), \phi(t), t)$ be a function of $r(t)$, $\phi(t)$, and t . In this case

$$dX_1(t) = \left\{ k(t)[f(0, t) - X_1(t)] + X_2(t) + \frac{d}{dt} f(0, t) \right\} dt + \sigma(t) dW_1(t),$$

$$dX_2(t) = \{\sigma^2(t) - 2k(t)X_2(t)\}dt.$$

Obviously the volatility matrix of these two state variables is singular.

To obtain an example in which the long rate $l(t)$ can be an increasing function, we can choose

$$k(x) = \frac{1}{2(x + c)},$$

where c is a positive constant. In this case the volatility of bond price has the form

$$\sigma^P(t, T) = 2\sigma(t)\sqrt{t + c}(\sqrt{T + c} - \sqrt{t + c}),$$

and from Section 3.3 the long rate is given by

$$l(t) = l(0) + \frac{1}{2} \int_0^t \delta(s) ds = l(0) + 2 \int_0^t [\sigma(s)]^2 (s + c) ds.$$

Note that, in this case, $\phi(t) = 1/(t + c) \int_0^t [\sigma(s)]^2 (s + c) ds$, and $l(t) - l(0) = 2(t + c)\phi(t)$. So we can obtain a two-factor model in which the two state variables are the short rate and the long rate. Figure 5 shows examples of yield curves resulting from this model. The horizontal axis indicates time to maturity from time t in years, and the vertical axis is the yield to maturity as a percentage. The three curves represent three types of yield curves from the model.

By choosing $\sigma(t) = \sigma[r(t)]^\gamma$, we can obtain models in which the long rate is an unbounded increasing process of time t . To obtain an example in which the long rate is a bounded increasing nondeterministic function of time t , we can choose

$$\sigma(t) = \frac{\sigma_1(t)}{\sqrt{t + c}},$$

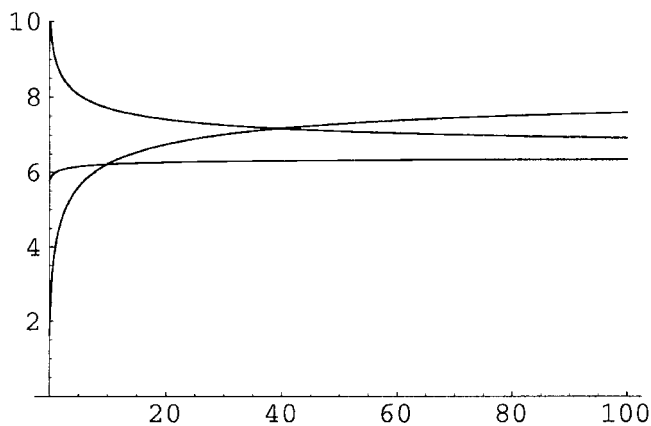
where $\sigma_1(t)$ is a function with $\int_0^{+\infty} [\sigma_1(s)]^2 ds < \infty$. One example for $\sigma_1(t)$ can be

$$\sigma_1(t) = \exp\{-t - [r(t)]^2\}.$$

5. A NUMERICAL EXAMPLE

This section illustrates the valuation of insurances and annuities in the no-arbitrage framework discussed in this article with a numerical example. To be simple and specific, the whole-life annuity-due of 1 payable at

Figure 5
Examples of Yield Curves in a Model with the Short Rate and the Long Rate as Two State Variables



the beginning of each year while a given life survives is used as the example. The actuarial present value of this policy is given by

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kP_x,$$

where $v = 1/(1 + i)$ is a constant. Table 1 gives the values of this policy under different interest rate assumptions using the illustrative life table in Appendix 2A of Bowers et al. (1986).

To value this policy in the no-arbitrage framework, use Equation (3.1). From this equation and assuming independence between mortality and interest rates under the risk-neutral measure, value the whole-life annuity-due using the formula

$$V_x = \sum_{k=0}^{\infty} E \left[\exp \left(- \int_0^k r(s) ds \right) \right] {}_k p_x.$$

Note that, from Equation (3.2), $E[\exp(-\int_0^k r(s) ds)] = P(0, k)$ is the price of the default-free zero-coupon bond with maturity k . So we have

$$V_x = \sum_{k=0}^{\infty} P(0, k) {}_k p_x.$$

To calculate this value, we need to know the prices of default-free discount bonds with maturities up to 110 years, which are calculated from one of the term structure models.

In the Merton (1970) and Ho and Lee (1986) models the long rates are infinite. So these two models cannot be used to price long-term bonds or other fixed-income securities. In the Vasicek (1977) and Langetieg (1980) models, interest rates are normally distributed. There is a positive probability of negative interest rates, which implies arbitrage opportunities. These two models are not used here. See Rogers (1996) for detailed discussions. In many applications a closed-form solution for the default-free discount bond price is a desired property of term structure models. This makes us choose the Cox, Ingersoll, and Ross (1985) model or its generalizations, in which the long rate is a constant. To incorporate empirical evidence discussed in El Karoui, Frachot, and German (1998), we may choose the model with the short rate and the long rate as two state variables discussed in Section 4, which is a special case of the Heath, Jarrow, and Morton model and has a closed-form formula for the bond price.

Take the Cox, Ingersoll, and Ross (1985) model as an illustration. In the Cox, Ingersoll, and Ross model, the short rate is given by

Table 1
Actuarial Present Values of the Whole-Life Annuity-Due under
Different Constant Interest-Rate Assumptions

Age	<i>i</i>					
	0.05	0.06	0.07	0.08	0.09	0.10
0	19.6427	16.8009	14.6702	13.0296	11.7300	10.6790
1	19.9829	17.0982	14.9340	13.2629	11.9396	10.8689
2	19.9589	17.0870	14.9294	13.2617	11.9402	10.8704
3	19.9312	17.0731	14.9228	13.2589	11.9394	10.8707
4	19.9001	17.0567	14.9141	13.2545	11.9373	10.8700
5	19.8658	17.0387	14.9036	13.2486	11.9341	10.8683
6	19.8285	17.0167	14.8919	13.2415	11.9299	10.8657
7	19.7883	16.9935	14.8777	13.2330	11.9246	10.8624
8	19.8453	16.9682	14.8623	13.2235	11.9184	10.8583
9	19.6996	16.9409	14.8456	13.2128	11.9114	10.8535
10	19.6513	16.9118	14.8274	13.2010	11.9036	10.8481

$$dr(t) = k[\theta - r(t)]dt + \sigma\sqrt{r(t)}dW(t),$$

and the prices of default-free discount bonds are given by

$$P(t, T) = A(t, T)\exp[-B(t, T)r],$$

where

$$A(t, T) = \left[\frac{2\gamma \exp[(k + \gamma)(T - t)/2]}{(k + \gamma) \{ \exp[\gamma(T - t)] - 1 \} + 2\gamma} \right]^{2k\theta/\sigma^2},$$

$$B(t, T) = \frac{2\{\exp[\gamma(T - t)] - 1\}}{(k + \gamma)\{\exp[\gamma(T - t)] - 1\} + 2\gamma},$$

$$\gamma = \sqrt{k^2 + 2\sigma^2}.$$

As an illustration, choose the parameters of the Cox, Ingersoll, and Ross model as follows: $k = 0.2339$, $\theta = 0.081$, $\sigma = 0.085$, which are estimated in Chan et al. (1992, p. 1218). The long rate in the Cox, Ingersoll, and Ross model is a constant and is given by $l = 2k\theta / (k + \sqrt{\sigma^2 + k^2}) = 0.076$. Table 2 gives yields on default-free discount bonds in the Cox, Ingersoll, and Ross model under different short rate assumptions. Table 3 gives values of the whole-life annuity-due under different short rate assumptions using the illustrative life table in Appendix 2A of Bowers et al. (1986).

6. CONCLUSION

Presently many different term structure models are being used in valuation and hedging interest-rate-sensitive claims, but there is little agreement on any one of them being the preferred model. This is the result of compromises between the properties a term

structure model should have and the features a model can have.

In this article I discuss the asymptotic behavior of yields on default-free discount bonds in the arbitrage-free framework. In this setting the yields of all maturities should be positive and uniformly bounded from above. The yield curve should level out as term to maturity increases, and slopes with large absolute values occur only in the early maturities. The longer the maturity of the yield is, the less volatile it is. The long rate should be a nondecreasing process. Furthermore, the long rate in continuous-time factor models with nonsingular volatility matrices for the factors should be a nondecreasing deterministic function of time.

Many known factor models explicitly model the mean reversion property of interest rates; this results in the boundedness of the long rate from these term structure models. In many known factor models, the volatility matrices are nonsingular. These two properties make the long rate in many known models a constant or a nondecreasing deterministic function.

Many practitioners use the Heath, Jarrow, and Morton model or time-inhomogeneous factor models of the Hull and White type. These models give a richer behavior of the long rate. The long rate from these models can be infinite or a nondecreasing process. When using these models in pricing and hedging long-maturity products, one should be aware of the possibility of an unbounded long rate.

Future research should be done with respect to the jump and default effects on the dynamics of the long rates. Empirical testing and comparison of known term structure models can be also done by taking a perspective from the long rate.

Table 2
Yields on Default-Free Discount Bonds in the Cox, Ingersoll, and Ross Model under Different Short-Rate Assumptions

Time to Maturity	r					
	0.05	0.06	0.07	0.08	0.09	0.10
1	0.0533	0.0622	0.0711	0.0800	0.0889	0.0978
2	0.0560	0.0640	0.0720	0.0798	0.0878	0.0958
3	0.0583	0.0654	0.0726	0.0797	0.0868	0.0940
5	0.0618	0.0677	0.0734	0.0792	0.0850	0.0908
7	0.0644	0.0692	0.0740	0.0788	0.0836	0.0884
10	0.0671	0.0708	0.0745	0.0783	0.0820	0.0857
20	0.0712	0.0732	0.0752	0.0772	0.0793	0.0813
30	0.0728	0.0742	0.0755	0.0769	0.0782	0.0795
50	0.0741	0.0750	0.0757	0.0765	0.0773	0.0781
70	0.0746	0.0752	0.0758	0.0763	0.0770	0.0778
100	0.0751	0.0754	0.0759	0.0762	0.0767	0.0771
110	0.0752	0.0755	0.0759	0.0762	0.0766	0.0769

Table 3
Arbitrage-Free Values of the Whole-Life Annuity-Due under Different Short-Rate Assumptions

Age	r					
	0.05	0.06	0.07	0.08	0.09	0.10
0	14.1587	13.7421	13.3397	12.9513	12.5762	12.2140
1	14.4135	13.9888	13.5787	13.1828	12.8005	12.4312
2	14.4119	13.9873	13.5773	13.1814	12.7992	12.4301
3	14.4084	13.9840	13.5741	13.1784	12.7963	12.4273
4	14.4031	13.9790	13.5693	13.1738	12.7919	12.4231
5	14.3963	13.9724	13.5630	13.1678	12.7861	12.4176
6	14.3880	13.9644	13.5554	13.1604	12.7791	12.4108
7	14.3782	13.9550	13.5464	13.1518	12.7708	12.4028
8	14.3672	13.9444	13.5362	13.1420	12.7614	12.3938
9	14.3549	13.9326	13.5249	13.1311	12.7509	12.3838
10	14.3414	13.9197	13.5124	13.1192	12.7394	12.3727

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Appendix A

A Partial List of Known Term Structure Models

Author(s)	Model Specifications	Long Rate
Merton (1970)	$dr(t) = \theta dt + \sigma dW_1(t)$	$-\infty$ (if σ is positive)
Vasicek (1977)	$dr(t) = k[\theta - r(t)]dt + \delta dW_1(t)$	$\theta - (\sigma^2/2k^2)$ (if k is positive)
Dothan (1978)	$dr(t) = \sigma r(t) dW_1(t)$	A finite nondecreasing deterministic function of time
Brennan-Schwartz (1979)	$dr(t) = \beta_1(r, l, t) dt + \eta_1(r, l, t) dW_1(t)$ $dl(t) = \beta_2(r, l, t) dt + \eta_2(r, l, t) dW_2(t)$	The long rate is nondecreasing, so it is required that $\eta_2(r, l, t) \equiv 0$ and $\beta_2(r, l, t) \geq 0$
Langsetieg (1980)	$dX(t) = (K \cdot X(t) + K_0) dt + \Sigma \cdot dW(t)$ $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$	A finite constant
Courtadon (1982)	$dr(t) = k[\theta - r(t)] dt + \sigma r(t) dW_1(t)$	A finite nondecreasing deterministic function of time
Cox-Ingersoll-Ross (1985)	$dr(t) = k[\theta - r(t)] dt + \sigma\sqrt{r(t)} dW_1(t)$	$2k\theta/(k + \sqrt{\sigma^2 + k^2})$
Ho-Lee (1986)	$dr(t) = \left[\frac{d}{dt} f(0, t) + t\sigma^2 \right] dt + \sigma dW(t)$	$+\infty$
Hull-White (1990)	$dr(t) = [\theta(t) + \alpha(t)(b - r(t))] dt + \sigma(t) dW_1(t)$, or $dr(t) = [\theta(t) + \alpha(t)(b - r(t))] dt + \sigma(t) \sqrt{r(t)} dW_1(t)$	Can be infinite or a finite nondecreasing function of time
Black-Derman-Toy (1990)	$d(\log r(t)) = [\theta(t) - \phi(t) \log r(t)] dt + \sigma(t) dW_1(t)$	A finite nondecreasing deterministic function of time
Chan-Karolyi-Longstaff-Sanders (1992)	$dr(t) = k[\theta - r(t)] dt + \sigma[r(t)]^\nu dW_1(t)$	A finite nondecreasing deterministic function of time t (if k is positive)
Longstaff-Schwartz (1992) Chen-Scott (1992)	$dX(t) = [a - bX(t)]dt + c\sqrt{X(t)}dW_1(t)$ $dY(t) = [d - eY(t)]dt + f\sqrt{Y(t)}dW_2(t)$ $r(t) = \gamma_1 X(t) + \gamma_2 Y(t)$	A finite constant (if b and e are positive)
Heath-Jarrow-Morton (1992)	$df(t, T) = (\sigma_r(t))^* \cdot \left(\int_t^T \sigma_s(t) dS \right) dt + \sigma_r(t) \cdot dW(t)$ f is the forward rate	Can be infinite or a nondecreasing process
Jamshidian (1991) Cheyette (1992) Richken-Sankar (1995)	$dr(t) = \left\{ k(t)[f(0, t) - r(t)] + \phi(t) + \frac{df(0, t)}{dt} \right\} dt + \sigma(t)dW_1(t)$ $d\phi(t) = \{\sigma^2(t) - 2k(t)\phi(t)\}dt$	Can be infinite or a nondecreasing process
Chen (1996)	$dr(t) = k[\theta(t) - r(t)]dt + \sqrt{V(t)}dW_2(t)$ $d\theta(t) = v[\bar{\theta} - \theta(t)]dt + \zeta\sqrt{\theta(t)}dW_2(t)$ $dV(t) = \mu[\bar{\sigma} - V(t)]dt + \eta\sqrt{V(t)}dW_3(t)$	A finite constant (if k , v , and μ are positive)
Duffie-Kan (1996)	$dX(t) = (K \cdot X(t) + K_0)dt + \sqrt{V(X(t))} \cdot \Sigma \cdot dW(t)$ $r(t) = \gamma_0 + \sum_{i=1}^n \gamma_i X_i(t)$	A finite constant (if all the eigenvalues of matrix K have a negative real part)

APPENDIX B

FACTOR MODELS AND HEATH, JARROW, AND MORTON MODEL IN THE RISK-NEUTRAL FRAMEWORK

In Section 2 $B(t)$ is denoted as $\exp\{\int_0^t r(s) ds\}$. From Equation (3.1) we have

$$\frac{p(t)}{B(t)} = E_t \left[\frac{p(T)}{B(T)} \right], \quad t \leq T; \quad (\text{A.1})$$

that is, the process $\{p(t)/B(t)\}$ is a martingale with respect to the measure \mathbf{Q} . Equation (A.1) is called here the martingale pricing equation.

Assume that, for each fixed T , the bond price $P(t, T)$ can be represented as a stochastic differential equation of the form

$$\frac{dP(t, T)}{P(t, T)} = \mu^P(t, T) dt - \sigma^P(t, T) \cdot dW(t),$$

where $\mu^P(t, T)$ is a scalar, $\sigma^P(t, T)$ is a $1 \times n$ vector, and $W(t)$ is an n -dimensional standard Brownian motion under the risk-neutral measure \mathbf{Q} . In factor models and the Heath, Jarrow, and Morton (1992) model, the bond prices have this form. Using Itô's Lemma, we obtain from Equation (A.1)

$$\begin{aligned} \frac{d[P(t, T)/B(t)]}{P(t, T)/B(t)} &= [\mu^P(t, T) \\ &\quad - r(t)]dt - \sigma^P(t, T) \cdot dW(t). \end{aligned}$$

Because the process $\{P(t, T)/B(t)\}$ is a martingale and the drift of a martingale is zero, we have $\mu^P(t, T) = r(t)$, which means that the rates of return of all default-free zero-coupon bonds at time t are equal to $r(t)$. Therefore we obtain an arbitrage-free characterization of the term structure in terms of bond prices

$$\frac{dP(t, T)}{P(t, T)} = r(t) dt - \sigma^P(t, T) \cdot dW(t). \quad (\text{A.2})$$

Factor Models

In most factor models, the state variables $X = (X_1, X_2, \dots, X_i, \dots, X_n)^*$ are governed by a stochastic differential equation of the form

$$dX(t) = \mu(X(t), t) dt + \sigma(X(t), t) \cdot dW(t),$$

where $\mu(X(t), t)$ is an $n \times 1$ vector and $\sigma(X(t), t)$ is an $n \times n$ matrix.

In factor models the bond price $P(t, T)$ is assumed to be a function of state variables X . Using Itô's Lemma, we have

$$\begin{aligned} dP &= \left\{ \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] \right\} dt \\ &\quad + \frac{\partial P}{\partial x} \cdot \sigma \cdot dW(t), \end{aligned}$$

where, for simplicity in notation, we have omitted variable dependence. From Equation (A.2), we obtain

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] = r(t)P.$$

Rearranging the last equation yields

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} \cdot \mu + \frac{1}{2} \text{tr} \left[\frac{\partial^2 P}{\partial x^2} \cdot \sigma \cdot \sigma^* \right] - r(t)P = 0,$$

which is the partial differential equation for bond prices under the risk-neutral measure.

The Heath, Jarrow, and Morton Model

The Heath, Jarrow, and Morton model starts from modeling the dynamics of the entire forward rate curve. Mathematically, for each fixed T , the forward rate at time t , $t \leq T$, is assumed to satisfy the stochastic differential equation of the form

$$\begin{aligned} df^t(t, T) &= \mu_T(f^t(t, T), t)dt \\ &\quad + \sigma_T(f^t(t, T), t) \cdot dW(t), \quad (\text{A.3}) \end{aligned}$$

where $\mu_T(f^t(t, T), t)$ is scalar and $\sigma_T(f^t(t, T), t)$ is a $1 \times n$ vector. For notational simplicity, $\mu_T(t)$ and $\sigma_T(t)$ are used instead of $\mu_T(f^t(t, T), t)$ and $\sigma_T(f^t(t, T), t)$ in the following paragraphs. Recall Equation (2.3), which is $P(t, T) = \exp\{-\int_t^T f(t, s) ds\}$. Using Itô's Lemma, we obtain

$$\begin{aligned} \frac{dP(t, T)}{P(t, T)} &= \left\{ r(t) - \int_t^T \mu_S(t) dS \right. \\ &\quad \left. + \frac{1}{2} \left(\int_t^T \sigma_S(t) dS \right) \cdot \left(\int_t^T \sigma_S(t) dS \right)^* \right\} dt \\ &\quad - \left(\int_t^T \sigma_S(t) dS \right) \cdot dW(t). \end{aligned}$$

Comparing with Equation (A.2), which states that the expected rate of return of each bond at time t is $r(t)$ under the risk-neutral measure, we have

$$r(t) - \int_t^T \mu_S(t) dS + \frac{1}{2} \left(\int_t^T \sigma_S(t) dS \right) \cdot \left(\int_t^T \sigma_S(t) dS \right)^* = r(t).$$

Differentiating the last equation with respect to T and rearranging give

$$\mu_T(t) = \sigma_T(t) \cdot \left(\int_t^T \sigma_S(t) dS \right)^*. \quad (\text{A.4})$$

Substituting Equation (A.4) into Equation (A.3), we obtain an arbitrage-free characterization of the term structure in terms of forward rates

$$df(t, T) = \sigma_T(t) \cdot \left(\int_t^T \sigma_S(t) dS \right)^* dt + \sigma_T(t) \cdot dW(t).$$

This specification for forward rates is the essence of the so-called Heath, Jarrow, and Morton model.

DISCUSSION

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There are now many term structure models available for determining yields for present-valuing cash flows based on no-arbitrage or equilibrium assumptions. No-arbitrage and equilibrium models differ mainly in the way the price of interest rate risk is determined rather than in the process assumed for the short rate. Appendix A of the paper gives a list of many of these models. Given such a wide range of choices, it is apparent that some basis is required in order to select a suitable model for actuarial applications to the long-term cash flows that occur in insurance and pensions. Because all of the models assume a stochastic process for the short rates, it is possible to use time series analysis to determine how well these models fit the data. Empirical studies often demonstrate poor fit to the data of the standard term structure models.

It is also possible to use cross-sectional data at a single time point to examine the fit of the models to

market data. This approach is used in the paper by Jacques Carriere "Long-Term Yield Rates for Actuarial Valuations." By increasing the number of parameters used, these models can be made to fit current yield curves arbitrarily well. In the time series context this is equivalent to introducing a time varying parameter. The price of interest rate risk is effectively assumed to be time varying.

There are obvious problems using these approaches to extrapolate the yield curve to long maturities required for the cash flows often found in actuarial problems. In this case, there is usually no market data to test the reasonableness of the model for the long maturities, and there will be many models that fit the available data to choose from. A theoretical basis is desirable when deciding which model to use for long-term cash flows.

This paper provides an analysis of the behavior of the long interest rate in various models and identifies features of the long rate that a desirable model should have. This is an excellent paper for actuaries interested in selecting an appropriate model for valuing long-term cash flows. It provides a theoretical analysis of a number of often-used models in terms of the behavior of the long rate. In some cases this identifies the need for caution in selecting off-the-shelf models that may not be designed for use in long-term valuation.

Some models not covered in the paper that may also be worth examining include those where the discrete compounding interest rates, rather than the continuous compounding interest rates, are modeled as log-normal. These are the bases of models used to produce market formulas used for interest rate options.

Finally, the concept of the long rate as the yield on a zero-coupon bond with infinite maturity is interesting. If an investor purchases such a financial instrument they will receive no income nor a return of capital during any finite time. The price of such an instrument will be zero. Yet the behavior of the yield on this instrument can be useful in assessing term structure models for actuarial applications.

Additional discussions on this paper can be submitted until January 1, 2000. The author reserves the right to reply to any discussion. See the Submission Guidelines for Authors on the inside back cover for detailed instructions on the submission of discussions.

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