Hedging and Reserving for Single-Premium Segregated Fund Contracts

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Abstract

Three methods for determining suitable provision for maturity guarantees for single-premium segregated fund contracts are compared. Actuarial reserving assumes funds are held in risk-free assets, to give a prescribed probability of meeting the guarantee liability. Dynamic hedging uses the Black-Scholes framework to determine the replicating portfolio. Static hedging assumes a counterparty is willing to sell the options required to meet the guarantee. Using a stochastic cash flow projection, we consider how to assess which approach is most profitable. The example given assumes a typical Canadian segregated fund contract.

1. Introduction

Segregated fund (or *unit-linked,* or *equity-linked*) insurance contracts, under which the insurer invests the premiums received in a separate fund with specified assets, comprise an increasing proportion of life insurance contracts written. In many areas it is common to offer guarantees with these contracts with respect to minimum benefits payable on maturity. These guarantees must be provisioned for, either internally or by purchasing suitable options in the market.

In a typical Canadian segregated fund contract, the premiums, net of the annual management charge of 1% to 3% per year, are invested in a fund or several different funds separate from the insurance funds of the office (hence the name segregated fund contracts). When the policy matures (or when the policyholder dies, if earlier) the benefit payable is equal to the market value of the accumulated premiums, subject to a guaranteed minimum amount. The guarantee is generally 75% or 100% of the total gross premiums paid, before management charge deductions. Policyholders may choose in which fund or combination of funds to invest, with the choice generally including bond funds, a general index fund, tracking an appropriate stock index, and a *managed fund* with a combination of bonds and stocks. The funds used by insurers are generally mutual funds independent of the insurer. Currently, there are no mandatory capital requirements for segregated fund maturity guarantees in Canada.

Segregated fund guarantees are financial guarantees and cannot be dealt with adequately using traditional deterministic actuarial approaches, which emphasize mean or median values and rely on diversification. The financial risk under the maturity guarantee is nondiversifiable in that, in the event of serious adverse investment performance, many maturing contracts may require additional funds to cover the guarantee at the same time. The distribution of claims under the guarantee will generally be very highly skewed. Most of the time there will be no cost under the guarantee—as the accumulated premiums will exceed the guarantee—but if the investment performance is very poor over the term of the contract, then the claim can be significant. Given a guarantee of 100% of premiums and a management charge of $100m$% per year, a single premium contract would need to earn a rate of return of more than $100m/(1 - m)$% per year to avoid a payment under the guarantee.

Figure 1 illustrates the annualized return on a ten-year single-premium investment in the TSE 300 index (the main broader-based index of the Toronto Stock Exchange) and the Canadian dollar Standard and Poors 500 index. The upper threshold indicates the point at which a 100% guarantee would have had non-zero cost given a management charge of 2% per annum; the lower line indicates the threshold with no annual charge. The figure shows that, although the

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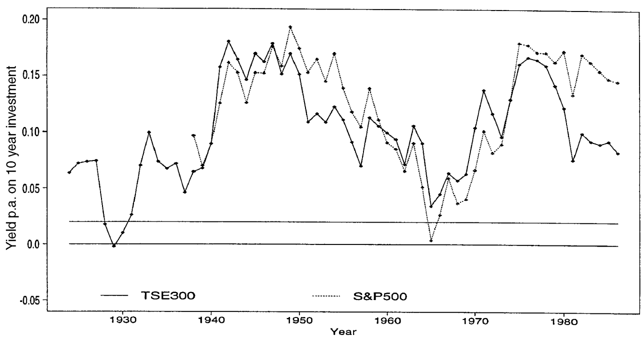
Figure 1
Return Per Annum on 10-Year Single-Premium Investment



guarantee rarely bites, it would not be prudent to ignore the contingent liability altogether.

In Boyle and Hardy (1996) three methods of making provision for this kind of risk are discussed. The first, *actuarial reserving,* assumes a provision is held in risk free bonds until the policy matures; the amount held is calculated to be sufficient to meet the cost of the guarantee with a given probability. This is the method adopted by the Maturity Guarantees Working Party of the Institute and Faculty of Actuaries (Maturity Guarantees Working Party 1980), and is similar in approach to *Value at Risk.* The second method considered is *dynamic hedging.* This utilizes the fact that the guarantee is a European put option—that is, it confers the right of the policyholders to sell their share of the segregated fund for at least the guaranteed amount at the maturity date. Hedging the put option requires a hedge portfolio to be held, which will replicate the guarantee payoff. The hedge portfolio must, in principle, be continuously rebalanced, and this type of hedging is therefore called dynamic. This approach has been adopted by Boyle and Schwartz (1977), Brennan and Schwartz (1976, 1979) and Bacinello and Ortu (1993).

The third method is to buy put options to hedge the guarantee cost, passing the (not inconsiderable) problems of dynamic hedging on to a third party. This is called a *static hedge.* The option price may be high, as these options differ in nature and term from the typical over-the-counter options currently available. The counterparty risk (that the option provider defaults) may also be a factor given the long terms involved.

The main focus of this paper is how to determine which of these methods is preferable. Actuaries have generally used reserving where possible, or the purchase of appropriate options; financial engineers tend to favor hedging. I use the simplest form of the segregated fund contract—the single premium contract—to demonstrate how the three methods may be compared in practice. I ignore here mortality and lapses, and assume that the only segregated fund available is one that tracks the TSE 300 index. Assume that there are no ratchet or reset options for the guarantee. All of these simplifications are made for the purpose of concentrating on the main question: how to determine whether to hedge, reserve, or purchase the options. However, the effects on some of the calculations of exits, reset guarantees, and level annual-premium contracts have been considered in Boyle and Hardy (1996) and Hardy (1999).

2. RESERVING PRINCIPLES

The reserving approach uses a stochastic model of future investment experience to estimate the distribution of the guarantee cost. It may be possible to do this analytically, or stochastic simulation to generate investment scenarios using a given investment model may be necessary.

We use two investment models to project the stock price dynamics in this section. The first is the *i.i.d. lognormal model,* where we assume that the market-to-market rate of return on the assets in the year t to $t + 1$ is $1 + I_t$ where I_t is lognormally distributed, with parameters $\mu = 8.1\%$, $\sigma = 17\%$. These parameters are chosen to be consistent with the second model. This is the *Wilkie model* (Wilkie 1986, 1995) with parameters fitted to Canadian data, from Wilkie (1995)—that is, using data from the years 1925 to 1990.

The Wilkie model is an integrated model of stock returns, interest rates, and inflation rates. It was developed for investigating the effect on U.K. insurers and pension schemes of future uncertainty in investment returns. Designed for long-term applications, it provides an actuarial tool that financial economics, concerned with much shorter-term problems, cannot yet offer. It uses an essentially statistical approach, fitting a model to the appropriate data series. The Wilkie model is used widely in stochastic investigation of insurance processes, by both industry practitioners and academics. This application addresses only the total stock price return, so the fully integrated model is not required.

The reason for projecting the dynamics using two different models in this section is to show the similarity of the results; it is not possible to dismiss the figures as being highly model dependent. In subsequent sections we will use the lognormal model, which has the major advantage of tractability over the Wilkie model.

We ignore mortality and the guaranteed death benefit in the following analysis; it is not difficult to incorporate these, but the difference in cost is very small as the death benefit is low cost and (at least in part) diversifiable.

The *reserving standard* is the required probability of having funds sufficient to meet the cost of the guarantee. This means that if we decide on a reserving standard of p_1, then we require reserves to be established such that, by the maturity date n, the reserve is ${}_nV$, the smallest nonnegative amount such that

$$\Pr\,[(F_n + {}_nV) > G] \geq p_1,$$

where F_n is the accumulated fund at time n and G is the guaranteed payment.

This approach is similar to that of the Maturity Guarantees Working Party (1980), but their calculations are based on an aggregate approach, simulating the results for an entire portfolio of policies. The reserve standard which they used was a 99% standard for the entire portfolio, which consisted of policies of different sizes and terms to maturity. We have taken an individual policy approach, which is equivalent to considering reserves separately for each cohort of business. Using this approach with a reserving standard of p implies that the aggregate security of the fund is greater than p, assuming that in the event of a serious market crash, reserve requirements for nonmaturing cohorts may be relaxed slightly to release reserves for the guarantee liability on maturing contracts. The value of p therefore does not give an overall figure for the probability that reserves in aggregate are sufficient to meet the guarantee costs of all policies in the portfolio—a separate exercise would be needed to identify this probability. The individual policy or cohort approach gives more direct information for pricing contracts, because the capital requirements for each contract do not depend on the existing portfolio. It also allows for the gradual release of the reserves over the term of the contract, assuming that returns are favorable, while maintaining the probability that the cohort guarantee costs are met by the cohort reserves. More detail on an approach to releasing reserves is given in Section 4.

The inequality above provides the required reserve at maturity. It is possible to establish the entire discounted reserve at the start of the contract. If we assume a risk-free force of interest of δ for the term of the contract, then the initial reserve would be ${}_nV\,e^{-\delta n}$.

The use of a fixed discount rate assumes essentially that the reserves are invested in a zero coupon bond of term n years. Equities would be an inappropriate medium, because the (random) reserve value would then be negatively correlated with the guarantee cost, meaning that when the reserve is needed is when it is at a minimum. Coupon-paying bonds incorporate a reinvestment risk. It is certainly possible to incorporate a stochastic model of interest rates (it would have to be integrated with the model of stock returns), but with considerable added complexity.

Establishing the full discounted final reserve at the start of the contract is slightly conservative because we can, to some extent, fund the reserve from the annual management charges received. For simplicity we assume that the same fixed rate of interest will be earned on these charges; that is, we ignore the reinvestment risk. A problem, however, is that the management charge is a percentage of the accumulated fund, and is therefore considered as a random variable. We apply a second reserving standard, p_2, such that in each year of the contract we assume the management charge is available to fund the reserve at the lower $100p_2$ percentile—that is, with probability p_2 we will receive at least the management charge assumed. This enables us to set the reserve required at integer duration t, ${}_tV$ for $t = 0, 1, 2, \ldots, n-1$ as follows, where the management charge received at time t is denoted M_t:

$$\Pr\,[({}_tV\,e^{\delta} + M_{t+1}) > {}_{t+1}V] = p_2.$$

This means that given ${}_nV$ we can calculate the initial reserve required, ${}_0V$, allowing for the management charges.

The management charge is used for nonfund costs other than the maturity guarantee, including administration expenses, any commission payable, and the cost of the death-benefit guarantee. The cost of the guaranteed benefit on death is proportional to the mortality rate and is therefore highly age dependent. We have assumed that all the management charge is available for funding reserves, if required; it is very simple to adapt this such that M_t represents the (random) sum available after other costs have been met.

3. RESERVING METHOD

3.1 Simulation

We use the stochastic investment model to generate a number of scenarios, each considered equally likely.

We calculate the cost of the guarantee under each scenario. In most cases the accumulated premiums exceed the guarantee, and the cost is zero. The simulated values of the cost of the guarantee are treated as a sample from the unknown underlying distribution of the random cost of the guarantee. From this sample we can estimate the expected cost of the guarantee, and the terminal reserve required to achieve a given probability that the accumulated reserve is sufficient to pay the cost of the guarantee at maturity. We have calculated reserves allowing for cases in which p_1 and p_2 = 95% and 99%. The mean value and reserve calculations are estimates of true, unknown values from the underlying distribution. We can quantify how good these estimates are by calculating standard errors or confidence intervals. To do so, we do not need to know anything about the true underlying distribution of the accumulated premiums, as we can use the fact that for any distribution, the number of simulations below the true 1% (say) point of the distribution is binomially distributed, with the first parameter equal to the number of simulations and the second equal to 0.01.

The Wilkie model results in the following sections are based on the same set of 45,000 simulations of the investment conditions. Using repeated simulations allows a more direct comparison between the different examples, without having to allow for random differences in simulated conditions.

The confidence intervals quoted below for the reserve estimates are 95% confidence intervals—that is, the probability that the true (unknown) value of the reserve lies in the interval quoted is 95%. The quoted reserve value is the best estimate of the true reserve within that interval. Standard errors are given in parentheses where appropriate.

The confidence intervals allow only for random variation in the stochastic simulations (that is, the possibility that by chance we have an unusually high or low set of simulations). They do not allow for any model mis-specification.

3.2 Calculation

For single-premium contracts it is straightforward to calculate the final reserve using the independent lognormal assumption. This gives an interesting comparison with the Wilkie model and illustrates the extent to which the results depend on the particular investment model. Calculating the initial reserve, assuming a contribution from future management charges, requires either stochastic simulation or use of the lognormal distribution for the sum of the discounted future management charges, as the lognormal distribution can approximate the sum of lognormal distributions.

Let the annual rate of management charge be denoted m, and let $A(n)$ denote the random accumulation factor over n years. We assume that $(1 + I_t)$ is independent and lognormally distributed with parameters μ and σ^2, so that $A(n)$ is also lognormally distributed, with parameters $n\mu$ and $n\sigma^2$, and $A(n)(1 - m)^n$ is lognormally distributed, with parameters $n[\mu + \ln(1 - m)]$ and $n\sigma^2$. For a guarantee of g per \$1 premium, for example, we set the final reserve ${}_nV$ to be the smallest nonzero value such that

$$\Pr\,[P \cdot A(n)(1 - m)^n + {}_nV > gP] \geq p_1,$$

where P is the single premium and p_1 is the reserving standard (0.95 or 0.99). This gives a target maturity reserve, per \$1 single premium, of

$${}_nV = \max\,\{0,\ [g - e^{-z_{p_1}\sqrt{n}\sigma + n\mu}\ (1 - m)^n]\},$$

where $\Phi(z_p) = p$.

The expected cost of the guarantee per \$1 of single premium, discounting at the risk-free force of interest of δ is,

$$\begin{aligned} &\mathrm{E}[\text{cost}] \\ &= \mathrm{E}[\max(g - A(n)(1 - m)^n, 0)e^{-\delta n}] \\ &= \left\{\Phi\left[\frac{\log(g) - n(\mu + \log(1 - m))}{\sqrt{n}\sigma}\right] g \right. \\ &\quad - (1 - m)^n\, e^{n\mu + (n\sigma^2/2} \\ &\quad \left. \times\ \Phi\left[\frac{\log(g) - n(\mu + \log(1 - m)) - n\sigma^2}{\sqrt{n}\sigma}\right]\right\} e^{-\delta n} \end{aligned}$$

The results are given in Table 1. The 'logN' figures assume i.i.d. lognormal stock appreciation and 'WM' denotes results using the Wilkie model. The risk-free force of interest is assumed to be 6% per annum. A 95% confidence interval is given for the reserves (assuming no management charge contribution). This allows for random variation in the simulations—it does not make allowance for model mis-specification. We can accurately calculate the reserve in the lognormal case, so there is no confidence interval or standard error.

The contribution from the management charge assumes that the full 1% charge is available, if necessary, to fund the reserves. The values show that the lognormal and Wilkie models give similar results (although statistically different at the 5% significance level), both in respect of expected costs and for the percentile reserves. In both cases, the initial reserves required are substantial for short-term contracts, even

Table 1
$100 Single-Premium Contract, 100% Guarantee, 1% Annual Management Charge

95% Reserves					
Term	Asset Model	Expected Cost	No Management Charge Contribution	1% Management Charge Contribution	5% Management Charge Contribution
5 Years	*LogN*	*2.3*	*17.56*	*15.30*	*14.87*
	WM	2.4 (0.03)	17.94 (17.5, 18.4)	15.80	15.35
10 Years	*LogN*	*1.1*	*8.80*	*4.63*	*3.52*
	WM	1.0 (0.02)	7.91 (7.5, 8.4)	3.83	2.71
15 Years	*LogN*	*0.5*	*0.75*	*0.00*	*0.00*
	WM	0.4 (0.01)	0.00 (0.0, 00)	0.00	0.00
99% Reserves					
Term	Asset Model	Expected Cost	No Management Charge Contribution	1% Management Charge Contribution	5% Management Charge Contribution
5 Years	*LogN*	*2.3*	*30.46*	*28.20*	*27.77*
	WM	2.4 (0.03)	30.53 (30.0, 31.2)	28.39	27.94
10 Years	*LogN*	*1.1*	*22.93*	*18.76*	*17.65*
	WM	1.0 (0.02)	20.99 (20.4, 21.6)	16.91	15.79
15 Years	*LogN*	*0.5*	*15.18*	*9.55*	*7.73*
	WM	0.4 (0.01)	12.94 (12.5, 13.5)	7.31	5.52

at the 95% standard. For longer terms, the reserves required at the 95% level are small; a zero reserve indicates that the (estimated) probability of nonzero liability is less than 5%. At the higher level of security, the 15-year reserve, allowing for management charges at the first percentile, is still significant at around 7% to 10% of the single premium.

To adapt these figures to allow for partial allocation of management expenses, one can apply a simple proportionate method to the difference between the fourth and subsequent columns. For example, if a 1% management charge is applied, with only 60% of the management charge available for reserves, the reserve figure for, say, the ten-year contract, with $p_1 = p_2 = 99\%$, would be 18.54 using the Wilkie model and 20.42 using the lognormal model.

4. RELEASING RESERVES

For a conventional insurance policy the reserves required are recalculated annually, allowing for the actual experience. For segregated fund contracts the same follows; in general this will allow the gradual release of reserves back to the company, as in most cases the guarantee liability will be zero. The reserve calculations outlined above will determine values for the provision required given the information available at the start of the contract. We now consider the reserve given information at subsequent policy anniversaries.

We can use the same principles as for the starting reserve: that is, we will require a final reserve at maturity sufficient to pay any guarantee liability with probability p_1, and we will take credit for future management charges at the $(1 - p_2)$ quantile of the distribution.

The information set of experience up to time t, $t = 1, 2, \ldots, n - 1$, is denoted I_t. The reserve at duration t, given the information available at that time, is

$$ {}_tV|I_t = (G - F_t A_{t,n}^{(p_1)}|I_t)\, e^{-(n-t)\delta} - \left(\sum_{k=t+1}^{n-1} (M_k^{(p_2)}|I_t)\, e^{-(k-t)\delta} \right), $$

where the guarantee is G, F_t is the segregated fund value at t (after management charge deduction at t). $A_{t,n}^{(p_1)}|I_t$ is the $(1 - p_1)$ quantile of the accumulation factor from time t to n, net of the management charge (that is, the value at n of a unit investment at t), given the experience of the fund up to time t. $M_k^{(p_2)}|I_t$ is the $(1 - p_2)$ quantile of the management charge at k, given the experience of the fund up to time t. In the case of the independent lognormal model of investment returns, the relevant information up to t is summarized in F_t, the accumulation and annuity factors are independent of the experience, and the reserve is

$$_tV|F_t = (G - F_t A_{t,n}^{(p_1)})\, e^{-(n-t)\delta} - \sum_{k=t+1}^{n-1} (M_k^{(p_2)}|F_t)\, e^{-(k-t)\delta},$$

where F_t affects the management charge by proportion.

In the case of the Wilkie model, the information up to t is used as input to the model for resimulation of future outcomes. It is not generally feasible to use as many second-tier simulations in this case, as the time taken to run the simulations quickly becomes very high.

This approach turns out to be rather too rigid, giving a very high probability that further funding of the contract will be required at some point. For the smooth releasing of reserves it is better to use a corridor approach—that is, release reserves only when the probability of funding the guarantee has risen above an upper threshold, and put more into the reserve only if the probability of meeting the guarantee falls below some lower threshold. This smooths the payments and, with a suitable corridor, substantially reduces the probability of further funding of reserves after the start of the contract (beyond the management charges already accounted for). We allow as before for management charges at the p_2th quantile.

That is, define

$$_tV^{(\alpha)}|I_t = (G - F_t A_{t,n}^{(\alpha)}|I_t)\, e^{-(n-t)\delta} - \left[\sum_{k=t+1}^{n-1} (M_k^{(p_2)}|I_t)\, e^{-(k-t)\delta}\right]$$

so that, given α_1 and α_2, set

$$_tV|I_t = \min\{\max[(_{t-1}V|I_{t-1}\, e^{\delta}),\, _tV^{(\alpha_1)}|I_t],\, _tV^{(\alpha_2)}|I_t\}$$

The mean simulated values for the upper bound, lower bound, and actual reserve held are given in Figure 2. This does not give a clear indication of how the corridor works for individual simulations, so in Figure 3 some examples of the reserve corridor (broken lines) and the actual reserve (solid line) are given. All of these figures use the lognormal model, so that the reserve corridor may be calculated precisely.

For the ten-year single-premium contract we have projected the nonfund cash flows for 5000 simulations, using the lognormal investment model. The starting reserve was calculated with $p_1 = 0.99$, $p_2 = 0.95$. Using $\alpha_1 = 0.925$, $\alpha_2 = 0.998$ gives an estimated 99.4% probability that the maturity reserve is sufficient to pay the guarantee costs. The quantiles of the cash flows from the fund are shown in Figure 4. The probability that further funding is required in the middle durations ($t = 1, 2, \ldots, 9$) is between 2% and 4% in the second to ninth years. It is smaller in the first and in the final years. The overall probability that further funding is required at some point is estimated at approximately 16% (0.005). The probability that funding of more than 10% of the initial premium is required is approximately 2% (0.002).

In Figure 4 the initial negative cash flow shows the cost of establishing the initial reserve, less the initial management charge of \$1. The substantially relaxed lower threshold, compared with the 99% initial reserve standard keeps the risk of subsequent negative cash flows reasonably low. In many cases, the returns are sufficiently high that the reserve required falls to zero, and the small positive cash flows represent the management charge income. In the final year, no management charge is received and the lower 10th, 25th, and 50th percentile cash flows are all zero. In the cases where the final reserve is not zero, some or all could be returned to the company in the final year, which explains the upper percentiles.

Reducing the width of the threshold will increase the probability of future cash calls for the contract. For example, setting $\alpha_1 = 0.95$, $\alpha_2 = 0.99$ increases the simulated probability to 34%.

5. THE OPTION APPROACH

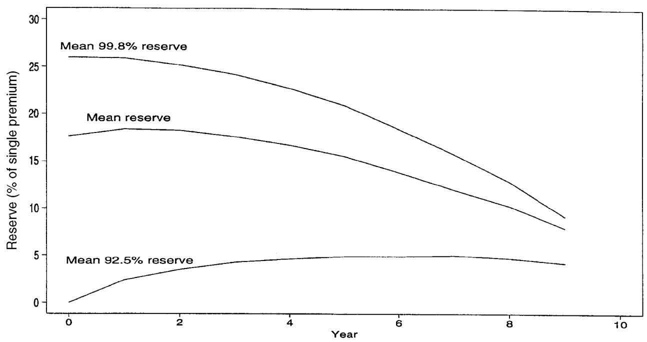
5.1 Dynamic Hedging

Bearing in mind that the maturity guarantee can be viewed as a put option, we can estimate the cost of the hedging portfolio that replicates the payoff.

A European put option gives the holder the right to sell an underlying asset (or portfolio) for a specific price at a specific date. In the case of the single-premium segregated fund guarantee, the policyholder pays a premium of, say, \$100. A management charge is deducted and the balance is paid into a segregated fund—for example, tracking the TSE 300 index. The premium accumulates at the maturity date, T, to the random amount $100(1 - m)^T A(T)$, where $A(T)$ is the value of T of \$1 invested in the index at time 0, with dividends reinvested.

The strike price for a put option is the amount which the option seller will pay for the assets at the maturity date if the option is exercised. Under the 100% segregated fund guarantee, if the assets are worth less than \$100, the insurer must pay the policyholder \$100. The situation is equivalent to that of the European put option, with strike price $K = 100$ and initial stock held of $S(t_0) = 100(1 - m)^n$ (in terms of the cumulative effect, it is immaterial whether the

Figure 2
Mean Upper and Lower Bounds for Reserves, 10-Year Single-Premium



management charge is deducted annually or in a single initial charge).

The Black Scholes price for the put option maturing at T, valued at t, is

$$\mathrm{BSP}\ (t) = K\, e^{-r(T-t)}\, N[-d_2(t)] - S(t)\, N[-d_1(t)],$$

where r is the risk-free rate (force) of interest, N denotes the cumulative normal density function, $S(t)$ is the asset price at t, and

$$d_1(t) = \frac{\ln\left(\frac{S(t)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2(t) = d_1(t) - \sigma\sqrt{T-t}.$$

The parameter σ is the volatility of the asset return, which corresponds to the lognormal parameter used in Section 3 above, where $\sigma = 0.17$ is the estimate from the historic data for the TSE 300 index. (The Black-Scholes price formula is identical to the expected cost formula for the single-premium contract in Section 3.2, but here the lognormal parameter μ is replaced with $r - \sigma^2/2$, as we have changed to the risk-neutral measure).

By setting t to the start date, t_0, the Black-Scholes formula gives us the initial price of the hedge portfolio; it also tells us that the initial hedge portfolio comprises $N[-d_2(t_0)]$ units of a risk-free bond with the maturity date corresponding to the option maturity date, together with a short position of $N[-d_1(t_0)]$ units in the asset.

In theory, the hedge portfolio is continuously readjusted, such that at any time $t \geq t_0$ the portfolio comprises $N[-d_2(t)]$ units of risk-free bond, together with a short position of $N[-d_1(t)]$ units in the asset. However, this ideal arrangement does not allow for transaction costs, or for the fact that, in practice, the rebalancing of the hedge portfolio can only occur at discrete intervals.

We have estimated by simulation the distribution of the transactions costs and hedging error associated with monthly rebalancing of the hedge portfolio, assuming transactions costs of 0.5% of the value of the stocks bought or sold. The hedging error is the difference between the prices of the hedge portfolio before and after rebalancing. That is, for the transactions at time t_v, we require a hedge portfolio of

$$K\, e^{-r(T-t_v)}\, N[-d_2(t_v)] - S(t_v) N[-d_1(t_v)],$$

but we have, before rebalancing, the hedge portfolio set up as at the previous rebalancing at time t_{v-1}, that is,

$$K\, e^{-r(T-t_v)}\, N[-d_2(t_{v-1})] - S(t_v) N[-d_1(t_{v-1})].$$

The difference between these is the hedging error, and it may be positive or negative. The transactions cost is assumed to be

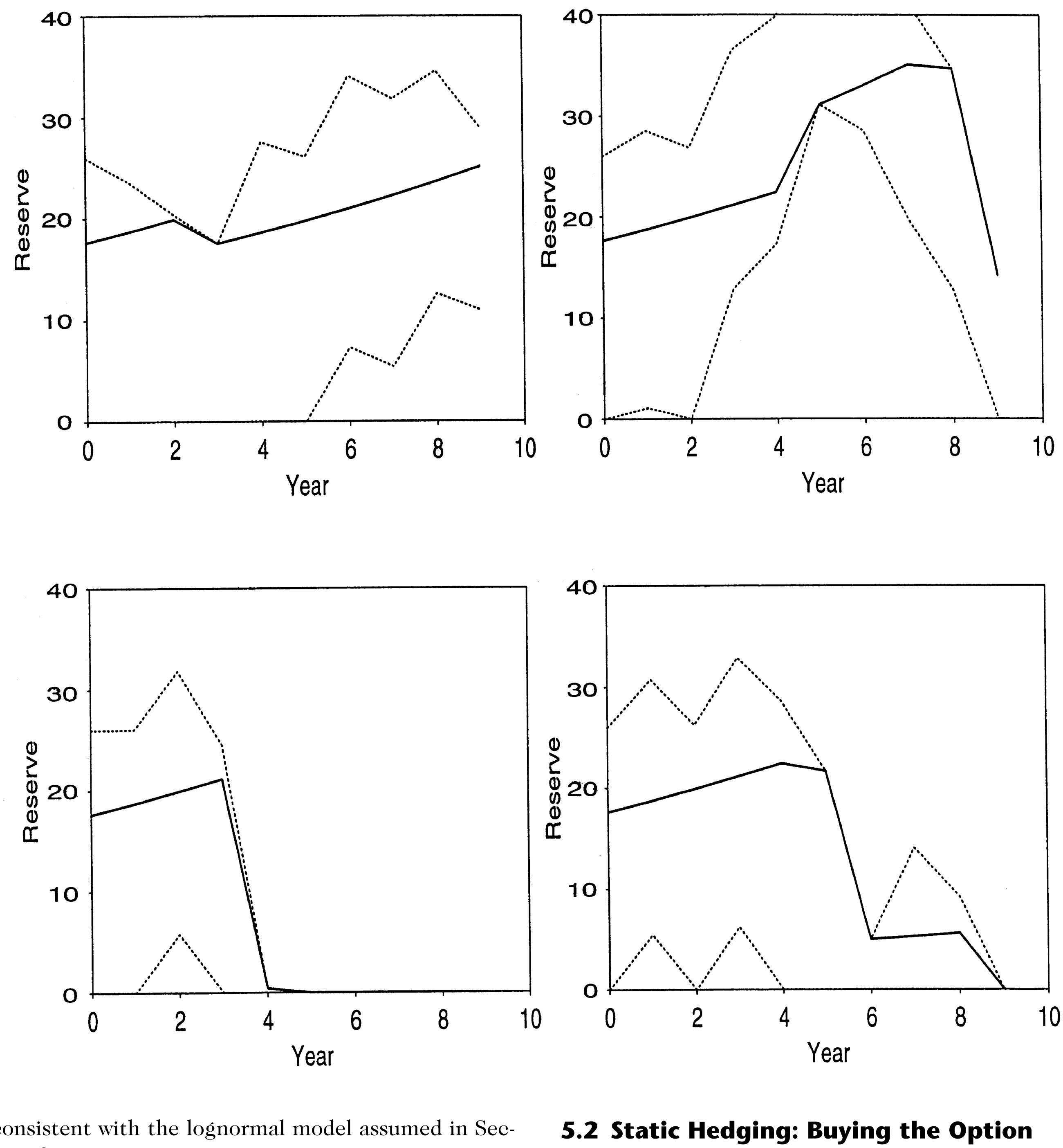
$$0.005\{S(t_v)\ |N[-d_1(t_v)] - N[-d_1(t_{v-1})]|\}$$

For a ten-year single-premium contract, assuming volatility $\sigma = 0.17$ (consistent with the lognormal model used in Section 3), and with a 1% per year management charge, the put option price is \$3.525, and the (simulated) expected present value of transactions costs and hedging error is \$0.592 (0.008). The 95th percentile of the transactions and hedging costs is \$1.372 and the 99th percentile is \$3.257. This assumes that negative hedging errors are allowed to offset positive errors, which means maintaining a transactions costs and hedging error (T&H) reserve, assumed invested in bonds, used to fund the increase in market value when required by rebalancing. Where the rebalancing yields a surplus, the surplus is deposited in the T&H reserve.

Figure 5 shows the median, 95th, and 99th percentiles of the simulated discounted future hedging costs at each month's end for a 10-year single-premium contract. The figure shows that the 99th percentile of future hedging costs holds relatively stable at between 3.4% and 3.8% of the single premium until the final year when it drops to 2.8%.

These results do not allow for model risk, that is, the risk that the underlying model is inaccurate. The length of the contract increases model risk. We also have made no allowance for parameter risk, the risk that the parameters used are misestimated. The model and parameters assumed in this section are

Figure 3
Examples of Individual Reserve Corridors



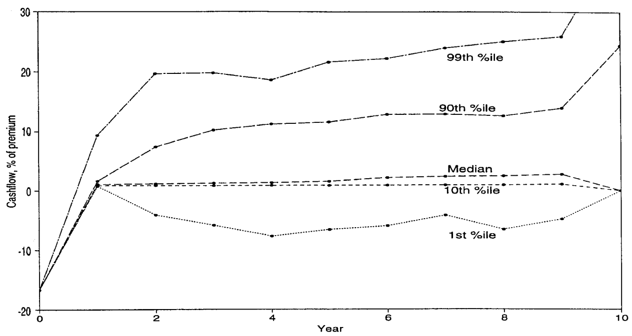
consistent with the lognormal model assumed in Section 3.

For more frequent hedging, the hedging error would decrease and the transactions costs would increase. A more efficient approach appears to be move-based hedging, where the fund is adjusted when the stock price moves by a given trigger amount. This is discussed further in Boyle and Hardy (1996).

5.2 Static Hedging: Buying the Option

A ten-year option is very long, well beyond the duration of most over-the-counter markets. In Boyle and Hardy (1996) it was suggested that volatility margin of 500 basis points might be used for seven-year options. If we assume the same margin for a 10-year option, the price of a put option (allowing for the 1% per annum management charge) would be \$6.56 for

Figure 4
Percentiles of Cash Flows, 10-Year Single-Premium Contract



a $100 single premium. If a more extensive market developed in these longer options, the margin could be reduced to, say, 300 basis points, which, allowing for 1% management charge, would give an option cost of $5.30 for a $100 single premium.

If the maturity guarantee is hedged by external purchase of the option, then model risk becomes the problem of the option provider. However, the counterparty risk—the risk that the option provider defaults—would be an important consideration.

6. Profit Testing

6.1 Expected Costs

We have described three approaches to the provision for the maturity guarantee. We can compare the expected costs and the initial outgo for these three approaches, using consistent assumptions. We use the ten-year single-premium contract, and make no allowance here for model or counterparty risk. The results

Figure 5
Percentile Hedging Reserves, 10-Year Single-Premium Contract

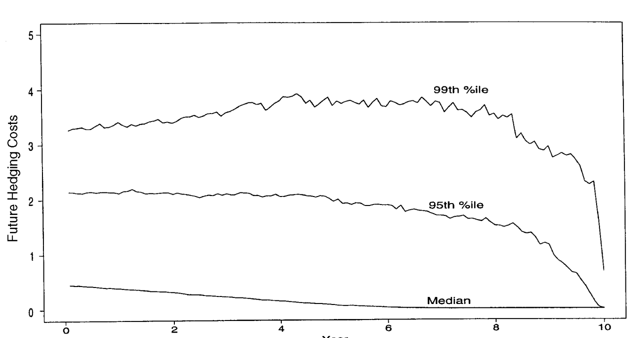


Table 2
10-Year Single-Premium Contract, Premium $100, Management Charge 1% per annum

Method of Provision	Expected Cost	Initial Outgo (99% Standard)
Reserve in Risk-Free Bonds	1.10	17.65
Dynamic Hedging (Monthly)	4.12	7.05
Static Hedging (σ = 0.22)	6.56	6.56

are given in Table 2. It is not at all clear from this table which method is preferable, even in terms of expected cost, because the expected cost shown does not allow for the cost of capital involved in setting up reserves. In general, the reserves will be released back to the company, but the cost of establishing reserves depends on how quickly they are released.

6.2 Nonfund Cash Flows

One useful exercise is to estimate the expected value of the profit emerging each year from a given contract, considering only the nonsegregated fund income and outgo. Nonfund income from the policyholder comes from the expense or management charges and surrender charges, if any. Nonfund outgo to the policyholder arises from guarantee costs on death or maturity. In addition there are costs arising from the reserve requirements.

In the following we consider the results of profit testing for the ten-year single-premium contract. The premium is assumed to be $100. Management charges of 1% per year are charged in advance, so the initial management charge is always $1. We consider the results of profit tests with the three methods described—reserving, hedging (with additional reserves for hedging error and transactions costs, assuming monthly hedging), and external purchase of put options. The guarantee is $100, that is, 100% of the single premium.

For the reserving approach, the cash flow at time t for a ten-year contract, ignoring exits, is

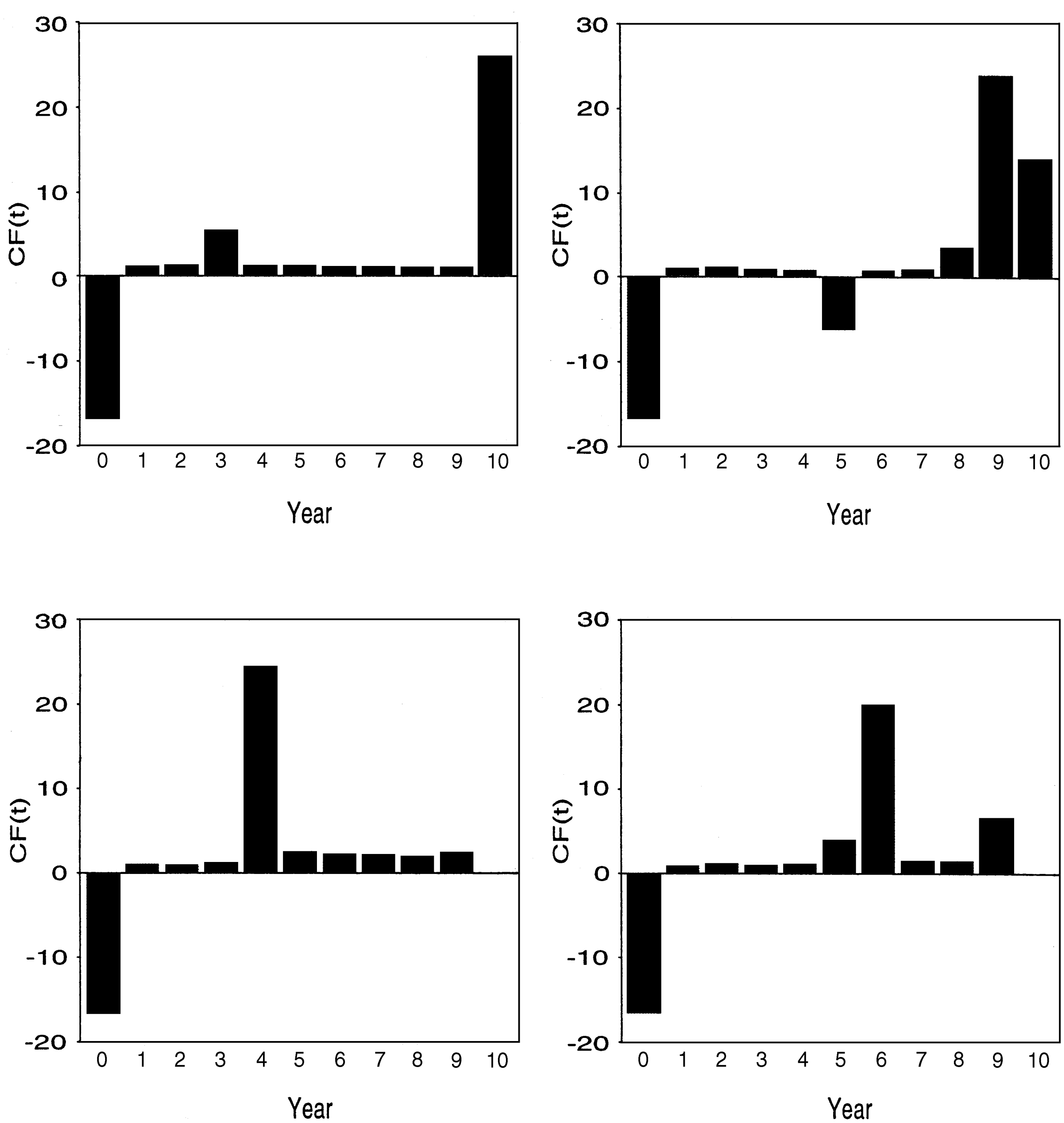
$$CF_t = {}_{t-1}V\, e^r + M(t) - {}_tV, \qquad t = 1, 2, 3, \ldots, 9$$

where $M(t)$ is the management charge received at t, ${}_tV$ is the reserve held at t, and r is the force of interest earned on the reserve. We also have $CF_0 = 1 - {}_0V$, as the initial management charge is $1, and $CF_{10} =$

${}_9Ve^r - [100 - F(10)]^+$, where F_t is the segregated fund value at t.

Figure 6 gives some examples of the simulated cash flows under the reserving method. These examples correspond to the examples of reserves shown in Figure 3, so that, for example, in the top left diagram, the extra release of capital in the fourth contract year is a result of hitting the upper reserve barrier. In the same diagram, the large cash flow in the final year shows that the reserves built up were not all needed and some or all were released at the end of the contract. In the top right diagram, there is a negative cash flow corresponding to the point where the reserve hits the bottom barrier. The funds are released subsequently. In the bottom left diagram the required reserve falls to zero in the fifth year and stays there. This gives a release of reserves in that year, with subsequent cash flows coming from the management charges alone.

Figure 6
Examples of Simulated Cash Flows, 10-Year Single-Premium Contract



See Hardy (1999) for a discussion of how to adapt the profit test to allow for lapses and deaths.

For the hedging approach, let ${}_tV^{(t\&h)}$ be the T&H reserve required at time t. We adopt a 99% standard and assume that reserves of ${}_0V^{(t\&h)} = 3.5$ are required at the start of the contract. Subsequent reserves are determined dynamically by second tier simulation. This means that at each projection data, a further 100 simulations are carried out of the discounted hedging and transactions costs from the projection date to the maturity date. The second-largest simulated value is taken as an approximation to the 99th. The compound simulation method is very computer intensive, which explains the small number of second-tier simulations. However, the results are very similar to those found using a fixed (nondynamic) value for the vector of T&H reserves, differing in that reserves are released a little more quickly under the dynamic method. An assumption of fixed values for ${}_tV^{(t\&h)}$, found using the results plotted in Figure 5, would also be a valid approach.

We have assumed that the hedging reserve is held in risk-free assets, earning a force of interest of r per annum. Let C_t represent the total cost in the year $t - 1 \rightarrow t$ of the hedging error and transactions costs arising from the monthly rebalancing, accumulated to the year end. The cash flows generated assuming the hedging approach, and ignoring exits, are

$$CF_t = M(t) + {}_{t-1}V^{(t\&h)}\, e^r - C_t - {}_tV^{(t\&h)}, \qquad t = 1, 2, 3, \ldots, 9$$

with $CF_0 = 1 - BSP - {}_0V^{(t\&h)}$, where BSP represents the Black-Scholes put option price for the contract, and $CF_{10} = {}_9V^{(t\&h)}\, e^r - C_{10}$.

The cash flows generated are considerably less variable than under the reserving method. The hedging error and transactions costs are generally small compared with the management charge income and the release of the T&H reserve.

Finally, for the static hedge, the cash flows are

$$CF_t = M(t) \qquad t = 1, 2, 3, \ldots, 9$$

with $CF_0 = 1 - BSP(\sigma = 0.22)$ and $CF_{10} = 0.0$ as, in the final year, there is no management charge and the option covers the guarantee liability.

The cash flows generated are smaller (on average) and less variable than the hedging cash flows. They are also strictly positive (after the initial strain), which is not true for either of the other methods.

6.3 Net Present Values

We will compare the costs ignoring exits. The net present value (NPV) of the cash flows is a basis for deciding on the value of the contract to the insurer under the different strategies. Let

$$NPV = \sum_{t=1}^{10} CF_t v^t$$

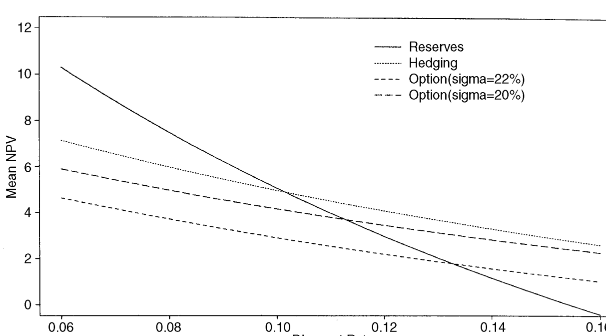
where v is an appropriate discount factor.

In Figure 7 we show the means of the simulated NPVs for the three methods of guarantee provision, at risk discount rates of 6% to 16%. The appropriate discount rate to use depends on the risk involved, with a suggested range given, for example, by Lewin et al. (1995) of 2% to 6% real, corresponding approximately to 6% to 12% nominal. These rates, however, are lower than those currently regularly used.

The risk involved, as measured for example by downside variability, is greatest using the reserving method, followed by dynamic hedging, followed by external purchase of options (assuming a suitably low-risk choice of counterparty). Hence, we should use a higher discount rate for reserving, followed by dynamic hedging, followed by static hedging. This does not allow for the effect of model risk.

We see from Figure 7 that over this range of discount rates, both reserving and hedging appear more attractive than purchasing options at the assumed volatility of $\sigma = 22\%$. Of course, this depends crucially on the option price offered. At $\sigma = 22\%$, the price is 6.56; if the mark-up in volatility is 300 basis points rather than the 500 assumed here, then the price is reduced to 5.30, which (because the management charges are unchanged) increases the mean NPV for all risk discount rates by 1.26. This gives the fourth line graphed in Figure 7, which at low discount rates

Figure 7
Mean Net Present Values—Reserving, Hedging, and Buying Options

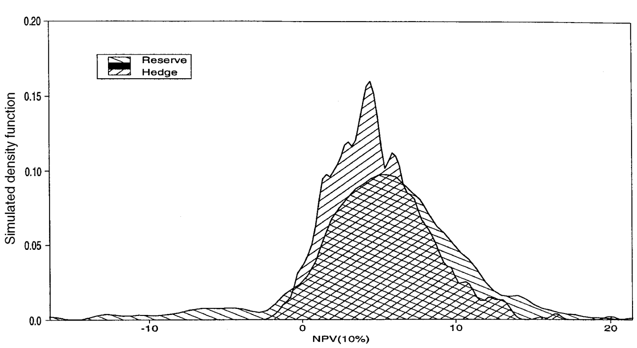


(appropriate for the lower risk involved) gives results competitive with reserving and hedging, assuming higher discount rates for these strategies.

To assist with the comparison between hedging and reserving, in Figure 8 the simulated density functions of the NPVs for the contracts are given, using a discount rate of 10% in both cases. If we assume that a discount rate of 10% per annum is appropriate for the reserving strategy, then it follows that, for this contract, there is very little difference between the mean NPV arrived at through dynamic hedging compared with that provided by reserving. Above 10%, the added cost of capital associated with the reserving method outweighs the advantage in expected cost of reserving over hedging. Figure 8 shows that, at a 10% discount rate, the mean and upside outturns for the NPVs for the reserving and hedging strategies are similar; however, the downside risk is very much worse for the reserving strategy than for the dynamic hedging strategy.

These figures are intended only to demonstrate how the comparison may be made. For example, different assumptions with respect to transactions costs or lapse rates, and different contract details, can lead to different conclusions. Also, the results are sensitive to the investment parameters assumed; a lower volatility has a more substantial effect on the reserving method than on the hedging method, making the question of which method is preferable more balanced than it is using these parameters.

Figure 8
Simulated Density Functions for NPV(10%) for Reserving and Hedging Approaches



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Discussions on this paper can be submitted until October 1, 2000. The author reserves the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.

rity Benefits" by Robert J Myers. *North American Actuarial Journal* 3(3):153–5.

"Hedging and Reserving for Single-Premium Segregated Fund Contracts," Mary R. Hardy, April 2000

MARTIN LE ROUX*

The question that Dr. Hardy addresses—whether it is more profitable to reserve passively or hedge actively against segregated fund maturity guarantees—is a topical one among Canadian insurance companies. These contracts have become very popular in Canada over the past few years, with annual sales in the billions of dollars. At present it appears that only a handful of the 40-odd companies underwriting this business are hedging the risk, although a number of others have partial reinsurance in place. However, this state of affairs may be about to change. As Dr. Hardy mentions in the introduction, there are currently no mandatory capital requirements for these contracts, but the Canadian insurance regulator has recently announced its intention to change this (OSFI 1999). The new requirements have not yet been finalized, but they may prompt many companies to reassess their current risk management strategies.

As Dr. Hardy states, the examples in her paper are deliberately based on a simplified contract and make a number of other simplifying assumptions. In practice, an insurer undertaking such an analysis will likely encounter many complexities, including the effect of ancillary contract provisions, the nature of underlying assets, models of market returns, and hedging strategies.

EFFECT OF ANCILLARY CONTRACT PROVISIONS

In some cases ancillary provisions are relatively straightforward to model. For example, death benefit guarantees, surrenders, and partial withdrawals can be modeled in traditional actuarial fashion by overlaying a deterministic decrement table on a stochastic asset model. However, other features can be much more difficult to analyze. For example, insurers commonly provide guarantees on a "family of funds" basis and allow switches between different fund types. Some insurers also allow contractholders to reset their guarantees, in effect providing a penalty-free lapse and re-entry option. At present there seems to be no satisfactory way to model such provisions, bearing in mind that policyholder behavior is rarely perfectly efficient when it comes to making use of them. Part of the difficulty is a lack of experience studies. Even determining the point at which these options are theoretically optimal to exercise can be a highly complex exercise. For example, see Boyle, Kolkiewicz, and Tan (1999) and Windcliff, Vetzal, and Forsyth (1999) for discussions of reset options.

NATURE OF UNDERLYING ASSETS

A further complication is that contracts are not usually based directly on indices such as the Toronto Stock Exchange (TSE) 300 or the Standard & Poor's (S&P) 500. Instead, most contracts are based on mutual funds, whose performance may deviate significantly from the relevant benchmark index performance. For example, total returns for the TSE 300 for the year ending Dec. 31, 1999, were 31.7%, whereas the average return for Canadian equity mutual funds over the same period was only 21.7% (*Globe and Mail* 2000). There have been attempts to model the difference between fund performance and index performance, but these models have met with mixed success (see Carriere and Hill 1999). Moreover, the underlying fund may not have been in existence long enough for meaningful statistical analysis to be feasible.

MODELS OF MARKET RETURNS

While the independent lognormal (ILN) model used in this paper has many convenient properties, it is well known that it fails to capture the "fat-tailed" nature of market return distributions observed in practice. When reserving against segregated fund guarantees one is, of course, concerned mainly about the tails of the distribution.

In 1999, Hardy considered a number of alter-

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natives to the ILN model, concluding that a simple regime-switching lognormal (RSLN) model (a type of stochastic volatility model) provides a better fit to the data than either the ILN model or more complex ARCH models, while being parsimonious in the number of parameters required. The downside risk of a reserving strategy appears to be much greater under an RSLN model than under the ILN model. Hardy (1999) also points out that the Wilkie model is not a good choice for testing hedging strategies because it generates only annual returns, whereas hedging strategies generally require rebalancing to take place much more frequently.

HEDGING STRATEGIES

The hedging strategy tested in this paper—a delta-neutral strategy with rebalancing at fixed monthly intervals—is likely to perform poorly in volatile markets. In practice, an insurer may aim for gamma-neutrality as well as delta-neutrality, which can be achieved by including options in the hedge portfolio. While such a strategy may be more robust than a simple delta-neutral hedge, it presents a number of modeling challenges:

- Because there are more instruments involved, it becomes more complex to model the future composition of the hedge portfolio.
- The hedging strategy may be based on index-linked futures and options, in which case the hedge portfolio will be imperfectly correlated with the underlying segregated fund assets. The resulting basis risk may be a major source of hedging error and should be modeled explicitly.
- If the hedging strategy requires the insurer to maintain a rolling position in short-dated options then it becomes necessary to model future implied volatility as well as future realized volatility.
- One of the major risks of a dynamic hedging strategy—and one of the most difficult to model—is liquidity risk. This can be a particular problem if the strategy calls for rapid rebalancing in response to major market movements.

In one respect, however, the situation faced by Canadian insurers may be slightly less complex than the one described in Hardy's current paper. The reserving principles described in section 2 envisage not only that accumulated funds at maturity should be adequate to meet the guarantees (with probability p_1), but also that there should be no future injections of capital over the life of the contract (with probability p_2). Moreover, p_1 is not necessarily constant but may fluctuate within a defined "corridor." Although Canadian reserving standards are currently in a state of flux, most proposals to date have taken the simpler approach of focusing solely on the adequacy of the accumulated funds at maturity (for example, see CIA 1998). This approach also provides a uniform reserving framework regardless of whether the insurer is hedging or passively accumulating assets. From a reserving perspective, the difference between these strategies is that, in the former case, one has to model the accumulated value of an actively managed hedge portfolio instead of a passive bond portfolio; in other respects, the reserving process is essentially the same.

A final issue concerns the use of quantiles to determine reserves, that is, a value-at-risk approach. It is sometimes taken for granted that reserves have to be set using quantiles, but these have a number of disadvantages. In particular, they are technically "incoherent" (Artzner 1999), disregard much information about the shape of the loss distribution, and suffer from the practical disadvantage that they can be highly volatile in response to minor changes in fund market values (le Roux 1999). Wirch and Hardy (2000) provide a summary of various alternatives. One of these is the conditional tail expectation (CTE), sometimes known as the expected policyholder deficit or TailVaR. The CTE has a number of advantages: It has a simple, intuitive definition (the CTE at 100%–x% is essentially the mean of the worst x% of outcomes), addresses the problems associated with quantiles, and is less prone to estimation error than a quantile measure. As a rule of thumb, the CTE at 100%–x% provides a level of protection comparable to the quantile at 100%–½x%.

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"Measuring Foreign Exchange Risk in Insurance Transactions," John I. Mange, April 2000

HANGSUCK LEE*

Mr. Mange has written an interesting and timely paper. My comments concern Section 4 on "Insurance Models of Foreign Exchange." The author developed several formulas related to continuous term life insurance and whole life annuity with benefits payable in a foreign currency. The models of life insurance and annuities are constructed by mixing the traditional actuarial models that assume a constant force of interest with an exchange rate model. As pointed out by the author, these models need to be generalized by using stochastic force of interest models. In this discussion, I shall give alternative proofs and corrections of Equations (6), (7), (9) and (11). I shall also point out a paradox arising from the model.

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Let $T = T(x)$ be the random variable of the future lifetime of (x). Let t be the current time. Consider an n-year term life insurance issued to (x) with the death benefit being one unit in foreign currency. The value in domestic currency of the death benefit payable at the time of death, $t + T$, is $e^{S(t+T)}$. An n-year term life insurance provides for a payment only if the insured dies within n years. Let $I(\cdot)$ denote the indicator function. Then $I(T < n) = 1$ if death occurs within n years, and $I(T < n) = 0$ otherwise. Hence, $e^{S(t+T)-S(t)-\delta T}I(T < n)$ is the random variable of the time-t value in foreign currency of the death benefit. In Equation (6) of the paper, the indicator random variable, $I(T < n)$, is missing. Also, $S(t)$ is known, not random at time t. Below is a derivation of Equation (6). From Equation (4) of the paper, it follows that

$$S(T+t) - S(t) = \eta T + \sigma[Z(T+t) - Z(t)], \quad \text{(D1)}$$

which has the same distribution as $\eta T + \sigma Z(T)$. Here, $\{Z(s), s \geq 0\}$ is a standard Brownian motion. Under the assumption that T and $\{S(\tau), \tau \geq 0\}$ are independent, the expected present value in foreign currency of the term insurance can be derived as follows.

$$\begin{aligned}\overline{A'}{}^{1}_{x:\overline{n}|} &= \mathrm{E}[e^{S(t+T)-S(t)-\delta T}I(T<n)]\\ &= \mathrm{E}[\mathrm{E}[e^{S(t+T)-S(t)-\delta T}I(T<n)|T]]\\ &= \mathrm{E}[e^{-\delta T}I(T<n)\mathrm{E}[e^{S(t+T)-S(t)}|T]]\\ &= \mathrm{E}[e^{-\delta T}I(T<n)\ \exp(\eta T + \tfrac{1}{2}\sigma^2 T)]\\ &= \mathrm{E}[\exp\{-(\delta - \eta - \tfrac{1}{2}\sigma^2)T\}I(T<n)]\\ &= \bar{A}^{1}_{x:\overline{n}|} \text{ @ force of interest } \delta - \eta - \tfrac{1}{2}\sigma^2.\end{aligned} \quad \text{(D2)}$$

If we replace n in Equation (D2) by ∞, we obtain the expectation of the present value in foreign currency of the death benefit payable in foreign currency at the moment of death,

$$\begin{aligned}\overline{A'}_x &= \mathrm{E}[e^{S(t+T)-S(t)-\delta T}]\\ &= \bar{A}_x \text{ @ force of interest } \delta - \eta - \tfrac{1}{2}\sigma^2.\end{aligned} \quad \text{(D3)}$$

The j^{th} moment of the random variable $e^{S(t+T)-S(t)-\delta T}I(T < n)$ can be calculated as follows.

$${}^{j}\overline{A'}{}^{1}_{x:\overline{n}|} = \mathrm{E}[e^{j[S(t+T)-S(t)-\delta T]}I(T<n)]$$