EFFICIENT GAIN AND LOSS AMORTIZATION AND OPTIMAL FUNDING IN PENSION PLANS

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ABSTRACT

The authors consider efficient methods of amortizing actuarial gains and losses in defined-benefit pension plans. In the context of a simple model where asset gains and losses emerge as a consequence of random (independent and identically distributed) rates of investment return, it has been shown that direct amortization of such gains and losses leads to more variable funding levels and contribution rates, compared with an indirect and proportional form of amortization that "spreads" the gains and losses. Stochastic simulations are performed and they indicate that spreading remains more efficient than amortization with simple AR(1) and MA(1) rates of return. Similar results are obtained when a more comprehensive actuarial stochastic investment model (which includes economic wage inflation) is simulated. Proportional spreading is rationalized as the contribution control that optimizes mean square deviations in the contributions and fund levels when the funding process is Markovian and the fund is invested in two assets (a random risky and a risk-free asset). Efficient spreading and amortization periods are suggested for the United States, the United Kingdom, and Canada.

1. INTRODUCTION

1.1 Aims

Methods of funding defined-benefit (DB) pension plans and of amortizing actuarial gains and losses are compared. A simplified model of a pension plan is described in Section 1.2 in order to investigate the results of Owadally and Haberman (1999), who suggest that a proportional form of amortization is preferable to the typical practice of direct amortization of gains and losses. These results are reviewed in Section 2. In Section 3, the results of stochastic simulations, with a more realistic treatment of investment returns and inflation, are presented. Finally, in Section 4, the efficiency of one method of amortization over the other is explained by means of the optimization of a quadratic objective criterion.

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1.2 Model and Notation

We consider a strictly DB pension plan in which no discretionary benefit improvement is allowed except for benefit indexation. It is also assumed that provision is made only for a retirement benefit at normal retirement age based on final salary and that actuarial valuations are carried out with the following features:

1. Actuarial valuations take place at regular intervals of one time period.
2. The actuarial valuation basis is invariant in time.
3. The market value of pension plan assets is f_t at time t (no smoothing is used).
4. An "individual" actuarial cost method is used, generating an actuarial liability al_t and a normal cost nc_t at time t.

A simple model for such a pension plan may be projected forward based on the following:

- The pension plan population is stationary (deterministic) from the start.
- Mortality and other decrements are assumed to be contingent as per a life table $\{l'_x\}$.
- A salary scale exactly reflects promotional,

merit-based, or longevity-based increases in salaries (and may be incorporated in the life table, $l_x = s_x l'_x$).

Economic wage inflation (the general increase in wages as measured by a national wages index) may be distinguished from the salary scale. The actuarial valuation basis includes $\{l_x\}$ as well as a valuation discount rate and an assumption as to wage inflation (in order to value final-salary benefits). Actual experience is in accordance with the actuarial valuation assumptions except for inflation and returns on plan assets. The model described above bears similarities to the models described by Trowbridge (1952), Bowers et al. (1979), Dufresne (1988, 1989) and Owadally and Haberman (1999).

The funding level or funded ratio in the plan is defined as the value of plan assets as a percentage of the actuarial liability. The unfunded liability (ul_t) at the start of year $(t, t + 1)$ is the excess of the actuarial liability over the value of plan assets (we assume that cash flows occur at the start of the year): $ul_t = al_t - f_t$.

The outcome of an actuarial valuation at the start of year $(t, t + 1)$ is to recommend a contribution

$$c_t = nc_t + adj_t, \tag{1}$$

where adj_t is a supplementary contribution (or contribution adjustment) paid to amortize past and present experience deviations from actuarial assumptions. These deviations result in actuarial gains or losses.

The actuarial loss l_t is the unanticipated change in the unfunded liability over year $(t - 1, t)$; that is, it is the excess of the unfunded liability at time t over the unfunded liability anticipated at time t based on information and the valuation basis at time $t - 1$. A gain is a negative loss.

1.3 Treatment of Gains and Losses

The calculation of the contribution is crucial to the dynamics of the pension fund when economic experience is volatile. For example, Winklevoss (1982) finds that "the correct treatment of actuarial gains and losses is critical in stochastic simulations because the effect of random fluctuations in salaries and plan assets impact on costs through the funding of such deviations (*Journal of Finance* 37:593)." Typical practice is that the actuarial gain or loss in each year is individually amortized over a fixed term m. The supplementary contribution in year $(t, t + 1)$ is

$$adj_t = \sum_{j=0}^{m-1} l_{t-j}/\ddot{a}_{\overline{m}|} \tag{2}$$

(Dufresne 1989). Alternatively, gains and losses may be spread by paying a proportion k of the unfunded liability:

$$adj_t = kul_t = k(al_t - f_t), \tag{3}$$

(Dufresne 1988), where k is typically defined as

$$k = 1/\ddot{a}_{\overline{m}|}. \tag{4}$$

Equation (3) may be understood as follows. The unfunded liability in the plan is the accumulation with interest of portions of previously incurred losses (as well as of any initial unfunded liability) that have not been fully paid off and have been deferred. When the supplementary contribution in equation (3) is made to the plan, a spread payment equaling a fraction k of the deferred portion of each incurred loss is settled in respect of each loss (as if the loss were taxed at rate k). Therefore, the excess of the unfunded liability over the supplementary contribution in a given year must equal the present value of the unfunded liability, less the newly emergent loss, in the following year: $u(ul_t - kul_t) = ul_{t+1} - l_{t+1}$, where $u = 1 + i$ and i is the rate at which cash flows in the plan are discounted. Ignoring any initial unfunded liability, it follows that $ul_t = \sum_{j=0}^{\infty} (1 - k)^j u^j l_{t-j}$ and, from equation (3),

$$adj_t = \sum_{j=0}^{\infty} k(1 - k)^j u^j l_{t-j}. \tag{5}$$

When $k = m = 1$, gains and losses are not deferred and the unfunded liability consists only of the loss that emerged during the past year, that loss being paid off immediately: $ul_t = adj_t = l_t$. See also Dufresne (1994).

Compare amortization in equation (2) with spreading in equation (5). Under amortization, gains and losses are paid off in level amounts over a finite term m. Under spreading, gains and losses are liquidated in perpetuity by means of exponentially declining payments. A large k (or equivalently a short spreading period m from equation 4) hastens funding as smaller portions of the losses are deferred.

Expressing k as $1/\ddot{a}_{\overline{m}|}$ in equation (4) is a convenient device that enables gains and losses to be removed faster if future cash flows are discounted at a higher rate. A loss is asymptotically liquidated when it is spread because the present value of payments made in respect of a unit loss is $\Sigma_{j=0}^{\infty} k(1-k)^j = 1$ when $m > 1$ (since $0 < i(1+i)^{-1} < k = 1/\ddot{a}_{\overline{m}|} < 1$). Finally, note that initial unfunded liabilities are disregarded in the following since they can be separately amortized and have no permanent effect (Owadally and Haberman 1999).

1.4 Volatility of Funding Process

If actuarial assumptions are unbiased and are realized on average, the loss in any year is *expected* to be zero, and (once any initial unfunded liability is completely amortized) the unfunded liability (an accumulation of unpaid losses) is also expected to be zero, whether gains and losses are amortized or spread. The uncertain nature of deviations from assumed experience (particularly economic experience) means that gains and losses are volatile, however, and the consequent variability of the funding level and of the required contributions must be examined.

The pension plan is regarded in this paper as an entity, set up under trust, whose financial management is separate from that of the company sponsoring pension benefits for its employees. Trustees (who represent plan members) and the plan sponsor are assumed to have convergent, albeit nonidentical, interests in funding the plan on a going-concern basis. One purpose of funding benefits in advance is to maximize the security of benefits (McGill et al. 1996, p. 592). This is an important objective for trustees but also indirectly for the plan sponsor as it plays a part in the recruitment and remuneration of employees.

A reasonable objective of an actuarial funding method is, therefore, to minimize the variability of the funding level or of the unfunded liability in the plan. An ideal measure of security would be based on second and higher moments and would allow for solvency requirements, but the variance is a reasonable and tractable approximation.

Another motivation for prefunding pension benefits is to spread the required future pension contributions and, hence, reduce the strain on the sponsor's cash flows. This is also of interest to the trustees since employees may be contributing to the fund and since successful financial management of the firm benefits not only shareholders but also employees (in terms of continued employment). Trowbridge and Farr (1976, p. 62), thus, refer to the "smoothness of contributions" as a desirable objective. This is often expressed relative to the total payroll for plan members. Minimizing the variability of the contribution or contribution rate (that is, total contribution per payroll dollar) is also a reasonable objective. Funding methods, therefore, seek to reconcile the objectives of plan trustees and sponsor and achieve stable funding levels as well as stable contribution rates.

In the rest of this paper, the choice between amortizing and spreading volatile gains and losses is discussed in terms of these objectives and under the modeling assumptions of Section 1.2. Gains and losses in the pension plan model arise only as a consequence of unforeseen economic variation from actuarial assumptions, that is, in the returns on plan assets and in inflation on plan liabilities. Contributions may be determined as a level percentage of payroll. The annuities in equations (2) and (4) are then calculated at the valuation discount rate net of assumed inflation on wages.

2. EFFICIENT AMORTIZATION: REVIEW OF MATHEMATICAL RESULTS

The treatment of gains and losses and the volatility of funding are studied by Dufresne (1988, 1989) in a simple mathematical model, whose main features are as in the model described in Section 1.2. In addition, Dufresne (1988, 1989) assumes that (1) pensions in payment are indexed with wage inflation, so that all monetary quantities can be expressed net of wage inflation (as a consequence of the assumptions made, payroll, actuarial liability, etc. are constant when deflated by wage inflation) and (2) the rate of return on the fund (net of wage inflation) is independent and identically distributed from year to year. The second assumption permits mathematical tractability, parsimony, and a search for optimal or robust performance.

The market value of plan assets f_t and the contribution rate c_t are random variables and are governed by the recurrence relation $f_{t+1} = (1 +$

$i_{t+1})(f_t + c_t - B)$, where i_t is the rate of return on the fund and B is the benefit outgo. The moments of f_t and c_t are derived by Dufresne (1988, 1989) when gains and losses are spread (equation 3) as well as amortized (equation 2). The plan is expected to be fully funded eventually: $Eul_t \to 0$ and $Eadj_t \to 0$ as $t \to \infty$, provided actuarial assumptions are unbiased. Dufresne (1988, 1989) shows that the stochastic pension funding process is stationary in the limit as $t \to \infty$, provided that gains and losses are not spread or amortized over very long periods. Since the payroll and actuarial liability are constant (net of wage inflation), lim $\mathrm{Var} f_t$ and lim $\mathrm{Var} c_t$ represent the long-term variances of the contribution rate and funding level, respectively.

The following results pertain to the stationary stochastic pension funding process and are obtained (in the spreading case) by Dufresne (1988) and (in the amortization case) by Owadally and Haberman (1999).

RESULT 1: VARIANCE OF FUNDING LEVEL

The variance of the funding level increases as gains and losses are spread over a longer period. It also increases as gains and losses are amortized over a longer period. Funding levels are less variable under amortization over a fixed term than under spreading over the same term.

Gains and losses are volatile. Spreading or amortizing them over shorter periods and recognizing them faster ensures that full-funded status is reached faster and maintained. This should improve the security of benefits as confirmed by Result 1.

RESULT 2: VARIANCE OF CONTRIBUTION RATE

As the period over which gains and losses are spread increases, the variance of the contribution rate initially decreases, attains a minimum (at m_s^*, say) and then increases. The same occurs as the amortization period increases, with a minimum at $m_a^* > m_s^*$. Contribution rates are less stable under amortization over a fixed term $m < m_a^*$ than under spreading over the same period.

A graph of the variance of contribution rate against spreading or amortization periods exhibits a minimum. It may be thought that spreading or amortizing and, therefore, deferring gains and losses over longer periods leads to a smoother and more stable contribution rate pattern. Result 2 partly contradicts this; for spreading periods longer than m_s^* (or for amortization periods longer than m_a^*), increasing the spreading period (or amortization period) causes increased volatility in the funding level, which feeds back as more volatile gains and losses, and—since supplementary contributions are required to liquidate the gains and losses—also as more volatile contribution rates.

RESULT 3: SPREAD AND AMORTIZATION PERIODS

Based on the criterion of minimizing the variances of contribution rate and funding levels, an efficient range of spread periods is $[1, m_s^*]$ and an efficient range of amortization periods is $[1, m_a^*]$.

For spread periods $m > m_s^*$ (and amortization periods $m > m_a^*$), there will always be a shorter period in the efficient range that yields the same variability in the contribution rate together with less variable funding levels. Dufresne (1988) concludes that, under modern economic conditions, spreading gains and losses over a period $m \in [1, 10]$ is efficient. This result has had some influence on actuarial practice in the United Kingdom (see Wilkie in Dufresne 1994, p. viii). Thornton and Wilson (1992) recommend short spreading periods based partly on Dufresne's (1989) conclusion. Owadally and Haberman (1999) conclude that the common North American practice of amortizing gains and losses over five years is also efficient. Faster defrayal of gains and losses arising from amendments to benefits or to valuation bases, rather than experience deviations, also have been promoted (Kryvicky 1981 and Colbran 1982).

RESULT 4: EFFICIENCY

According to the criterion of minimizing the variances of funding level and contribution rate, it is more efficient to spread gains and losses than to amortize them; for any two spread and amortization periods such that the funding level has the same variance, the variance of the contribution rate is less under spreading than under amortization.

Graphs of the variance of contribution rate against the variance of funding level for spreading and for amortization both exhibit a minimum, but the graph for spreading lies beneath the graph for amortization (except in the trivial case where $m = 1$ and gains and losses are paid off immediately, in which case spreading and amortization coincide).

Compare Result 1 (amortization leads to less variable funding levels than spreading over the same period), Result 2 (spreading leads to less variable contribution rates than amortization over the same period $m < m_a^*$), Result 3 (there exists an efficient spread period range as well as an efficient amortization period range), and Result 4 (overall, spreading is more efficient than amortization).

3. EFFICIENT AMORTIZATION: SIMULATIONS

3.1 Description of Stochastic Simulations

The results of Section 2 are limited in a number of ways. The first limitation concerns the assumption of independent real rates of return on the pension fund from year to year. First, there is considerable debate in the economics literature about serial correlation in equity returns over long horizons (see, e.g., Fama and French 1988). Serial correlation is also demonstrated in the actuarial literature by Panjer and Bellhouse (1980) and Wilkie (1995), among others. Second, debt securities are often held to match certain liability cash flows, and dependence in the returns from such securities will occur (Vanderhoof 1973).

Another limitation of the results in the previous section concerns the fact that inflation and different asset classes were not explicitly modeled. Maynard (1992, p. 245) shows that nominal rates of return have been more volatile than rates of return net of price inflation and that this adds to the volatility of funding for benefits that are not indexed with inflation. Modeling separate asset classes is also more realistic than modeling the return on the whole fund and allows for differences in asset allocation.

It is of interest to consider whether Results 1–4 hold when these limitations are relaxed. Stochastic simulations under two different models were carried out for this purpose. Details of these simulations are given in Appendix A.

3.1.1 Simulation Model 1

The Dufresne (1988) model (see Section 2) was simulated, the only difference being that the logarithmic rate of return (net of wage inflation), denoted by δ_t, is projected as a stationary Gaussian autoregressive process of order 1 or AR(1):

$$\delta_{t+1} - \delta = \varphi(\delta_t - \delta) + e_{t+1}, \qquad (6)$$

where $\{e_t\}$ is a sequence of zero-mean independent and identically normally distributed variables. The process is stationary from the start and $|\varphi| < 1$. The AR model is used by Panjer and Bellhouse (1980) and Dhaene (1989), among others.

3.1.2 Simulation Model 2

An asset-liability projection basis described by Wilkie (1995) was used to carry out projections for three countries: the United States, the United Kingdom, and Canada. The statistical methodology, historical data, and economic theory employed by Wilkie (1995) are exhaustively discussed in the actuarial literature (see Wilkie 1995 for relevant references). This projection model was used because it gives a fairly realistic indication of the relationship between various economic variables that are relevant to pension funding.

There are two noteworthy features of the asset-liability projections. First, stochastic inflation on prices and wages is assumed, and pensions are a fraction of final salary and are not indexed with inflation. This is more realistic than considering *real* rates of return, as was done in Section 2. The distinction between inflation on wages and on prices, which is important for final-salary pensions, is also made in the projections.

Second, two asset classes (equity and long-term government debt) are considered. A proportional rebalancing strategy between the two asset types is assumed, with income from an asset being reinvested in that asset. McGill et al. (1996, p. 665) state that "a 60–40 split between equities and fixed [income securities] is common" in North America, and this was assumed for all the projections. Forty:sixty and 80:20 equity:bond portfolios were also investigated. These portfolios are typical in practice and no suggestion as to their being optimal or otherwise is being made.

3.2 Simulation Results

More details about the simulations are given in Appendix A. The results are discussed here. For model 1 (AR(1) and equation 6), the scaled

Table 1

AR(1) Logarithmic Rates of Return, $\varphi = +0.3$

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	34.6	30.5	61.24	75.83
$m_s^* \approx 5$	51.0	41.2	54.77	67.08
$m_a^* \approx 7$	69.3	52.0	56.57	61.24
10	109.5	64.8	67.08	63.25
15	273.9	96.4	122.47	70.71
20	[a]	148.3	[a]	89.44
25	[a]	214.5	[a]	111.80

[a]Indicates nonstationarity.

standard deviations of the funding level and contribution rate are given in Table 1 ($\varphi = +0.3$), Table 2 ($\varphi = +0.5$), and Table 3 ($\varphi = -0.1$). For model 2 (asset-liability projections), the standard deviations of the funding level and contribution rate at the time horizon of the simulation study are shown in Tables 4 and 5 for the U.S. and Canadian projections, respectively, (assuming a 60:40 equity:bond portfolio). The same results for the U.K. projection, with an 80:20 equity:bond portfolio, are shown in Table 6.

3.2.1 Variance of the Funding Level (Result 1)

The variance of the funding level in Tables 1–6 increases as spreading and amortization periods increase. This is true both for the AR(1) model (including values of φ not tabulated here for the sake of brevity) and for all three countries in the asset-liability simulations. Result 1 appears to be borne out.

3.2.2 Variance of the Contribution Rate (Result 2)

Numerical results, not tabulated here for the sake of brevity, indicate that, when the rate of return is highly positively (or negatively) autocorrelated at lag 1, contribution rate variability increases (or decreases) monotonically as both spreading and amortization periods increase. See, for example, the graphs for $\varphi = +0.8$ and -0.3 in Figure 1. This feature is also reported by Haberman (1994) in the case of spreading.

For AR(1) rates of return that are moderately autocorrelated at lag 1 (Tables 1–3), the variance of contribution rates decreases and then increases as spreading and amortization periods increase, as stated in Result 2. This is also true for all three countries in the asset-liability projections (Tables 4–6). For the U.S. projections in Table 4, the variance of the contribution rate is minimized at a spreading period of $m_s^* \approx 7$ years

Table 2

AR(1) Logarithmic Rates of Return, $\varphi = +0.5$

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
2	31.3	27.4	80.62	90.83
$m_s^* \approx 3$	43.6	34.6	77.46	88.03
4	57.4	44.7	79.06	86.60
$m_a^* \approx 5$	74.2	52.9	82.16	85.15
6	97.5	61.6	90.83	86.60
7	122.5	70.7	104.88	88.03
8	167.3	81.9	122.47	94.87

Table 3
AR(1) Logarithmic Rates of Return, $\varphi = -0.1$

	Standard Deviation			
	Funding Level		Contribution Rate	
m	Spreading	Amortization	Spreading	Amortization
1	19.1%	19.1%	95.26%	95.26%
3	24.5	23.5	43.01	54.77
5	30.7	27.6	33.91	43.87
$m_s^* \approx 10$	44.7	37.4	27.84	34.28
15	54.8	44.7	28.28	31.62
$m_a^* \approx 20$	80.6	53.9	30.82	31.22
25	104.9	63.2	34.64	32.02
30	130.4	72.1	40.62	33.17

and an amortization period of $m_a^* \approx 13$ years. For $m < 13$, contribution rates are higher under amortization than under spreading. The same feature is reproduced in the other tables. Result 2 thus appears to hold under practical conditions.

3.2.3 Spread and Amortization Periods (Result 3)

The simulations, therefore, indicate that an efficient range of spreading periods and amortization periods exists under practical conditions (for moderately autocorrelated rates of return). Table 4 suggests that gains and losses in the United States should be spread over periods no longer than seven years and amortized over no more than 13 years. (The corresponding numbers are 15 and 35 for the United Kingdom and 13 and 20 for Canada.) The gain/loss amortization period range of up to five years prescribed by ERISA for single-employer pension plans is well within our suggested range, but the range of gain/loss amortization periods for multi-employer plans under ERISA is between 1 and 15 years (McGill et al. 1996, p. 597).

3.2.4 Efficiency (Result 4)

The variance of the contribution rate is plotted against the variance of the funding level in Figure 1 for the AR(1) model and in Figure 2 for the asset-liability projections. The "spreading" curve lies beneath the corresponding "amortization" curve in all the cases considered. Ignoring country-specific regulatory requirements, we conclude that, for the most stable funding levels and contribution rates, gains and losses should be paid off by being spread rather than amortized. Result 4 appears to hold.

3.2.5 Further Comments

Simulations also were carried out with the logarithmic rate of return process (net of wage inflation) being projected as a stationary Gaussian

Table 4
U.S. Projections with a 60:40 Equity:Bond Portfolio

	Standard Deviation			
	Funding Level		Contribution Rate	
m	Spreading	Amortization	Spreading	Amortization
1	15.7%	15.7%	42.46%	42.46%
3	23.2	22.1	21.79	28.87
5	29.8	27.1	17.45	23.54
$m_s^* \approx 7$	37.0	30.8	16.12	21.04
10	52.0	35.9	16.84	17.84
$m_a^* \approx 13$	75.9	42.0	20.04	17.40
15	99.9	46.4	23.74	17.41
17	135.7	51.4	29.57	17.56
20	235.0	60.8	45.97	19.00

Table 5
Canada Projections with a 60:40 Equity:Bond Portfolio

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	14.0%	14.0%	37.55%	37.55%
3	18.0	17.6	16.81	22.50
5	21.7	20.5	12.69	17.47
10	30.5	25.6	9.87	12.57
$m_s^* \approx 13$	36.3	28.5	9.62	11.46
15	40.7	30.4	9.71	11.02
17	45.4	32.3	9.94	10.66
$m_a^* \approx 20$	53.3	35.0	10.46	10.29
23	62.0	38.1	11.16	10.64
25	68.3	40.3	11.71	10.69
30	85.7	46.9	13.30	11.21

moving average process of order 1 or MA(1): $\delta_t - \delta = e_t - \phi e_{t-1}$, where $\{e_t\}$ is as in the AR(1) model and $|\phi| < 1$ (for invertibility). The results in the MA(1) case were very similar to those in the AR(1) case. (The detailed results of these simulations are not shown here for the sake of brevity.) For $\phi = -0.5, -0.3, 0, +0.1, +0.3$, Results 1–4 all appear to hold. Finally, Wilkie's (1995) economic time series are essentially linear and autoregressive. It is not entirely surprising that similar results are obtained from the AR(1) and the asset-liability simulations. The feature of an efficient range of spreading periods is indeed reproduced by Haberman and Smith (1997), who perform simulations of Wilkie's (1995) time series based on U.K. economic data.

4. Optimal Funding When Returns Are Independent

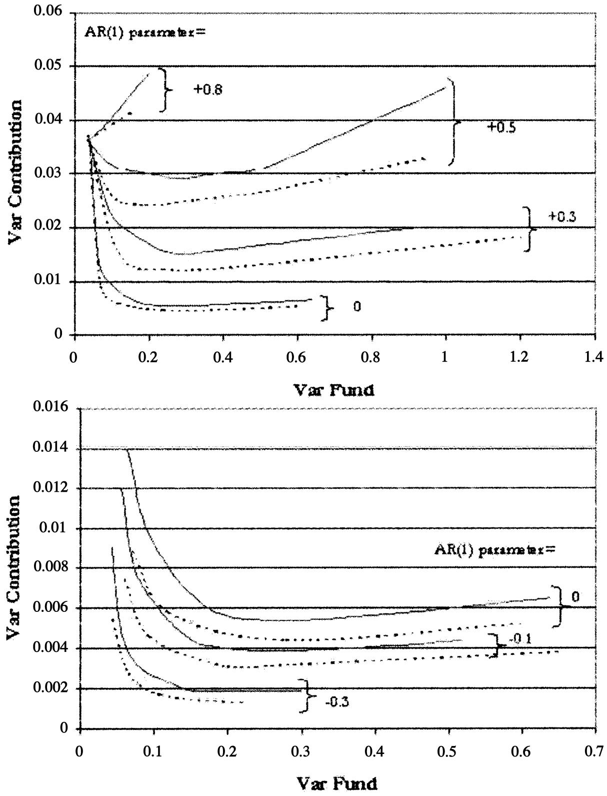
4.1 An Optimization Problem

The efficiency of spreading, as compared to direct gain/loss amortization, may be explained by the fact that, in the former, the supplementary contribution is proportional to the contemporaneous unfunded liability in the plan (equation 3). Proportional adjustment represents an optimal form of contribution control when a quadratic measure of the variability of contribution and market value of plan assets is to be minimized. This result was obtained by O'Brien (1987) and Haberman and Sung (1994) assuming random rates of invest-

Table 6
U.K. Projections with an 80:20 Equity:Bond Portfolio

m	Standard Deviation			
	Funding Level		Contribution Rate	
	Spreading	Amortization	Spreading	Amortization
1	16.7%	16.7%	29.67%	29.67%
3	20.3	20.1	12.57	16.82
5	23.9	22.9	9.31	12.83
10	32.4	28.0	7.00	8.83
$m_s^* \approx 15$	41.1	33.0	6.52	7.74
20	51.3	37.7	6.68	7.16
25	64.3	42.3	7.28	7.01
30	81.2	47.0	8.29	6.97
$m_a^* \approx 35$	102.2	51.8	9.65	7.11
40	127.4	57.0	11.33	7.41
45	156.5	62.3	13.29	7.46
50	188.9	67.6	15.47	8.09
60	258.2	83.8	20.09	10.25
70	323.9	111.1	24.40	11.84

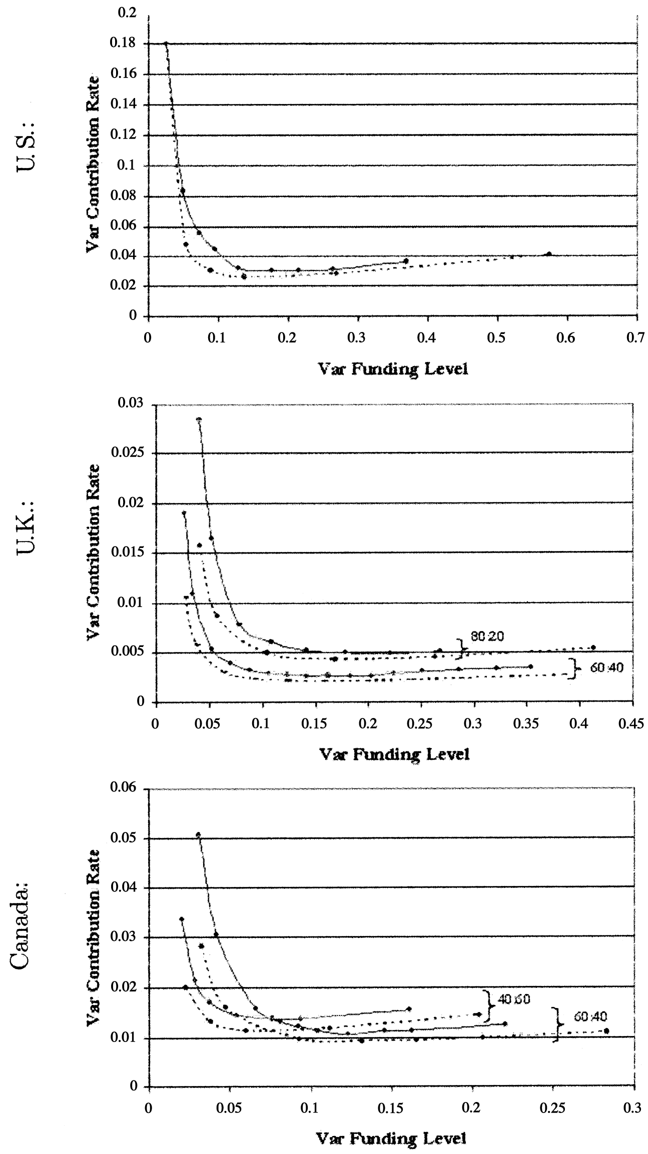
Figure 1
Projections with AR(1) Rates of Return for Various Values of φ. Spreading (Dashed Lines) is More Efficient Than Amortization (Solid Lines)



ment return on the plan assets. Boulier et al. (1995) obtained a similar result when asset allocation between two assets was also a decision variable and when the pension plan was being valued continuously and indefinitely.

A linear contribution adjustment is also obtained when regular valuations and cash flows occur at discrete intervals and a finite time horizon is assumed. Consider the pension plan model of Section 1.2. Assume that the fund is invested in two assets: a riskless asset earning risk-free rate r and a risky asset earning $r + \alpha_{t+1}$ in year $(t, t + 1)$, where α_{t+1} is a random risk premium. Let y_t be the proportion of the fund invested in the risky asset in year $(t, t + 1)$, and $1 - y_t$ be the proportion invested in the riskless asset. The arithmetic rate of return on the fund in year $(t, t + 1)$ is $r + y_t\alpha_{t+1}$. It is further assumed that $\{\alpha_t\}$ is a sequence of independent and identically distributed random variables over time, with mean $\alpha > 0$ and variance σ^2. It is also simpler to disregard inflation at this stage. Since the plan population is assumed to be stationary, the payroll, actuarial liability, and benefit payout are constant. The variability of contributions corresponds to the contribution rate variability, while the variability

Figure 2
Projections for the United States (60:40 Equity:Bond Portfolio), the United Kingdom (60:40 and 80:20) and Canada (60:40 and 40:60). Spreading (Dashed Lines) is More Efficient Than Amortization (Solid Lines)



of market values of plan assets corresponds to funding level variability.

The pension fund can be considered as a random system,

$$f_{t+1} = (1 + r + y_t\alpha_{t+1})(f_t + c_t - B), \quad (7)$$

where the market value of plan assets f_t is a state variable and c_t and y_t are contribution and asset allocation control variables, respectively. B represents the retirement benefits paid out yearly. By virtue of the independence over time of $\{\alpha_t\}$, f_t exhibits the Markov property: $E[f_{t+1}|W_t] = E[f_{t+1}|f_t, y_t, c_t]$, where W_t represents information available up to time t.

The objectives of the funding process are to stabilize contributions, defray any unfunded liabilities, and pay off actuarial losses and gains as they emerge. The performance of the pension fund may be judged in terms of the deviations in the values of plan assets and contributions from their desired levels (say FT_t and CT_t, respectively) relating to the actuarial liability and normal cost. The "cost" incurred for any such deviation at time $0 \le t \le N - 1$ may be defined as

$$C(f_t, c_t, t) = \theta_1(f_t - FT_t)^2 + \theta_2(c_t - CT_t)^2 \quad (8)$$

Different weights ($\theta_1 > 0$ and $\theta_2 > 0$) are placed on the twin long-term objectives of fund security and contribution stability. The cost in equation (8) reflects a quadratic utility function. Minimizing the cost also minimizes the risks of contribution instability and of fund inadequacy.

The performance of the fund may be given different importance over time. At the end of the given control period N, a closing cost is incurred if an unfunded liability still exists: $\mathbb{C}_N = \theta_0(f_N - FT_N)^2$. The *discounted* cost of deviation occurring t years ahead is $\beta^t C(f_t, c_t, t)$, where β is a time preference rate and $0 < \beta < 1$. For $0 \le t \le N - 1$, the discounted cost-to-go or discounted cost incurred from time t to N is

$$\mathbb{C}_t = \sum_{s=t}^{N-1} \beta^{s-t} C(f_s, c_s, s) + \beta^{N-t}\mathbb{C}_N. \quad (9)$$

An objective criterion for the performance of the pension funding system over period N may therefore be defined to be

$$E[\mathbb{C}_0|W_0] = E\left[\beta^N\mathbb{C}_N + \sum_{s=0}^{N-1} \beta^s C(f_s, c_s, s)\middle| W_0\right]. \quad (10)$$

Define the value function $J(f_t, t)$ as the minimum, over the remaining asset allocation and contribution decisions, of the expected discounted cost-to-go from time t given information at time t: $J(f_t, t) = \min_\pi E[\mathbb{C}_t|W_t]$, where $\pi = \{c_t, y_t, c_{t+1}, y_{t+1}, \ldots, c_{N-1}, y_{N-1}\}$. Objective criterion (10) may be minimized using the Bellman optimality principle (see, e.g., Bertsekas 1976); the minimizing values of c_t and y_t (say, c_t^* and y_t^*, respectively) in the optimality equation

$$J(f_t, t) = \min_{c_t y_t}\{C(f_t, c_t, t) + \beta E[J(f_{t+1}, t+1)|f_t, c_t, y_t]\}, \quad (11)$$

with boundary condition $J(f_N, N) = \mathbb{C}_N = \theta_0(f_N - FT_N)^2$, are the optimal contribution and asset allocation controls.

The pension planning objectives above were set over a finite period N. The plan is assumed to remain solvent and not discontinued during these N years, so that the funding process does not terminate unexpectedly. When the pension plan is regarded as a going concern, an infinite planning horizon may be usefully envisaged as a reasonable approximation to long-term funding.

No closing cost is incurred in the infinite horizon case. Suppose that the funding process is time-homogeneous, in that the fund and contribution targets are constant. The instantaneous cost at time t is as in equation (8) with $FT_t = FT$, $CT_t = CT\ \forall t$. The discounted cost-to-go and objective criterion are as in equations (9) and (10), respectively, except that $\mathbb{C}_N = 0$ and an infinite summation of discounted costs is taken. One naturally expects that the control rules that are optimal at time t based on the infinite discounted cost-to-go are also the optimal control rules at times $t + 1, t + 2, \ldots$ since the same infinite discounted cost-to-go exists at all times.

The dynamic programming algorithm in equation (11) can in fact be shown to converge as $N \to \infty$ (see Bertsekas 1976, p. 251) as it involves a contraction mapping and the instantaneous costs in equation (8) are non-negative and discounted. The optimal contribution and asset allocation over an infinite horizon (say, c_t^∞ and y_t^∞, respectively) are the minimizing values of c_t and y_t in equation (11) when the value function and the control variables are fixed or time-invariant functions of the state variable f_t.

4.2 Solution of the Optimization Problem

It is shown in Appendix B that the solution to the Bellman equation (11) in the finite-horizon case is

$$J(f_t, t) = P_t f_t^2 - 2Q_t f_t + R_t, \quad (12)$$

where

$$P_t = \theta_1 + \theta_2\beta\sigma^2(1 + r)^2\tilde{P}_{t+1}P_{t+1}, \quad (13)$$

$$\tilde{P}_{t+1} = [\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1 + r)^2P_{t+1}]^{-1}, \quad (14)$$

$$Q_t = \theta_1 FT_t + \theta_2\beta\sigma^2(1 + r)\tilde{P}_{t+1} \times [Q_{t+1} - P_{t+1}(1+r)(CT_t - B)], \quad (15)$$

with boundary conditions $P_N = \theta_0$ and $Q_N = \theta_0 FT_N$ (with R_t representing some additional terms independent of f_t).

It is also shown in Appendix B that the equivalent solution in the infinite-horizon case is

$$J(f_t) = Pf_t^2 - 2Qf_t + R, \quad (16)$$

where $P > \theta_1$ is the positive root of the quadratic equation

$$P^2[\beta\sigma^2(1 + r)^2] + P[\theta_2(\alpha^2 + \sigma^2) - (\theta_1 + \theta_2) \times \beta\sigma^2(1 + r)^2] - \theta_1\theta_2(\alpha^2 + \sigma^2) = 0 \quad (17)$$

and $P = \lim_{t\to\infty} P_t$, and where

$$Q = [\theta_1 FT + (P - \theta_1)(B - CT)] \times P(1 + r)/(Pr + \theta_1) \quad (18)$$

and $Q = \lim_{t\to\infty} Q_t$, and R contains terms independent of f_t.

Define $\Theta_t = \theta_2(\alpha^2 + \sigma^2)\tilde{P}_t = \theta_2(\alpha^2 + \sigma^2)/[\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1 + r)^2P_t]$ in the finite-horizon setting and correspondingly $\Theta = \theta_2(\alpha^2 + \sigma^2)/[\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1 + r)^2P]$ in the infinite-horizon setting. The following proposition is proven in Appendix B.

Proposition 1

For $0 \leq t \leq N - 1$ over a finite horizon, the optimal contribution is

$$c_t^* = \Theta_{t+1}CT_t + (1 - \Theta_{t+1}) \times [B - f_t + Q_{t+1}P_{t+1}^{-1}(1 + r)^{-1}], \quad (19)$$

and the optimal amount invested in the risky asset is

$$y_t^*[f_t + c_t^* - B] = [Q_{t+1}P_{t+1}^{-1}(1 + r)^{-1} - (f_t + c_t^* - B)]\alpha(1 + r)(\alpha^2 + \sigma^2)^{-1}. \quad (20)$$

The corresponding optimal decisions over an infinite horizon are:

$$c_t^\infty = \Theta CT + (1 - \Theta)[B - f_t + QP^{-1}(1 + r)^{-1}], \quad (21)$$

$$y_t^\infty[f_t + c_t^\infty - B] = [QP^{-1}(1 + r)^{-1} - (f_t + c_t^\infty - B)]\alpha(1 + r)(\alpha^2 + \sigma^2)^{-1}. \quad (22)$$

4.3 Optimal Funding Decisions

The behavior of the optimal funding decisions as the fund level varies is described in the following proposition, which is proven in Appendix B.

Proposition 2

The optimal contribution and asset allocation decisions are decreasing functions of the fund level: in the finite-horizon case, $\partial c_t^*/\partial f_t < 0$ and $\partial y_t^*/\partial f_t < 0$; in the infinite-horizon case, $\partial c_t^\infty/\partial f_t < 0$ and $\partial y_t^\infty/\partial f_t < 0$.

Proposition 2 states that the optimal proportion invested in the risky asset decreases as f_t increases, whatever the planning horizon. A similar result is obtained by Boulier et al. (1995) and Cairns (1997) for an infinite horizon and for continuous pension plan valuations. The better the investment performance of the risky asset, the better funded the plan is, the more the pension fund should be invested in the riskless asset. This is a contrarian strategy that entails buying as the market falls and selling as the market rises. It is reasonable in the sense that, first, liabilities need to be hedged so as to minimize the volatility of both surpluses and contributions and, second, any available surpluses should be "locked in" by being invested in less risky assets (Exley et al. 1997).

Conversely, the optimal strategy requires that an underfunded plan takes a riskier investment position than an overfunded plan, all other things being equal. For instance, the assets of a poorly funded immature pension plan (with a young membership) could arguably be invested more aggressively in the early years than the assets of a comparable plan with a healthy surplus. The optimal strategy here may be contrasted with portfolio insurance strategies (Black and Jones 1988) that require riskier investment as the value of plan assets less some minimum value (possibly determined by a solvency requirement) increases. The contrarian strategy is evidently a consequence of the quadratic utility function implied in criterion (8), which is simplistic as it is symmetric and continuous, and does not admit solvency and full-funding constraints. The risk that the plan sponsor winds up the plan or defaults on pension obligations was also disregarded.

The optimal contribution is linear in f_t, whatever the planning horizon; from equation (19), c_t^* may be written as $\Lambda_t - (1 - \Theta_{t+1})f_t$, where $1 - \Theta_{t+1} > 0$ while, from equation (21), c_t^∞ may be written as $\Lambda - (1 - \Theta)f_t$, where $1 - \Theta > 0$. The optimal contribution at the start of year $(t, t + 1)$ is, therefore, similar to the contribution calculated when gains and losses are spread (equations 1 and 3) in that they both depend in a decreasing linear way on the *current* market value of assets. This result is based on the assumption of serially independent rates of return and a Markovian funding process. The market value of plan assets represents the *state* of the funding process and, conditional on knowing the current state of funding and current funding decisions, the future evolution of the fund is statistically independent of its past.

The optimal contribution is, therefore, a function of the current state only. This contrasts markedly with the amortization of gains and losses where the contribution (equations 1 and 2) is a function of unanticipated changes in the state of the funding process over the past m years. This analysis helps to justify the efficiency of spreading over amortization as stated in Result 4 (which was also based on quadratic or second-moment criteria), despite the simplifying assumptions of a quadratic utility function and of zero inflation.

5. CONCLUSION

The funding of final-salary DB pension plans was considered and methods of amortizing actuarial gains and losses arising from unforeseen economic experience were investigated within simple models. Dufresne (1988) and Owadally and Haberman (1999) show that efficient periods over which to spread or amortize gains and losses exist. Spreading the gains and losses by paying a proportion of the current unfunded liability appears to yield less volatile funding levels and more stable contribution rates than amortizing past and present gains/losses.

These results depend on the assumption that rates of return on the fund are independent from year to year. Stochastic projections simulating Gaussian AR(1) and MA(1) logarithmic rates of return indicated that these results are robust when rates of return are moderately autocorrelated. Further simulations of pension plan assets and liabilities based on published actuarial stochastic time series models of equities, long-term government debt and wage inflation in three separate jurisdictions and based on typical asset portfolios supported the results. Gains and losses in U.S. pension plans should be amortized over no more than 13 years, but more stable funding levels and contribution rates emerge if gains and losses are spread over suggested periods of no more than seven years.

Finally, the efficiency of spreading over amortization was explained by the fact that the optimal contribution, based on a quadratic utility function and irrespective of the optimization period, for a pension fund invested in one risk-free and one random risky asset resembles the contribution calculated when gains and losses are spread. Both are a function of the current level of funding rather than the past and present gains or losses.

Further work should address the fairly constraining modeling assumptions that were made. In particular, gains and losses arising from uncertain mortality, withdrawal, and early retirement experience were ignored. Statutory and regulatory requirements also were ignored. Other areas requiring investigation include: considering a more realistic utility function incorporating solvency and full-funding constraints; modeling other asset classes, particularly shorter-term and inflation-indexed bonds, and using asset allocation strategies other than proportional rebalancing (e.g., constant proportion portfolio insurance); introducing a dynamic valuation basis, possibly based on corporate bond yields; and investigating efficient amortization mechanisms for expensing purposes.

APPENDIX A

DETAILS OF STOCHASTIC SIMULATIONS

Two thousand scenarios are simulated with a time horizon of 300 years each. The randomization routine generates the same set of 300 × 2000 random numbers so that sampling error does not occur when results are compared. The standard deviations of the funding level and contribution rate at the time horizon are calculated.

Simulation Model 1

For the AR(1) rate of return model, a simple final-salary pension plan as in Section 1.2 is assumed. Pensions in payment are assumed to be indexed with economic wage inflation. All quantities may therefore be considered net of wage inflation and, since the pension plan population is stationary, the liability structure of the pension plan is stable in time. The payroll in real dollars (net of wage inflation) is constant. The valuation discount rate (net of wage inflation) is assumed to be 5%. The actuarial liability (*AL*), normal cost (*NC*) and yearly benefit outgo (*B*) (all deflated by wage inflation) are also constant and hold in equilibrium ($B = NC + AL \times 4.76\%$).

The standard deviation of the funding level is calculated ($\sqrt{\text{Var } F/AL}$), whereas the standard deviation of the contribution rate (contribution per payroll dollar) is proportional to $\sqrt{\text{Var } C/NC}$. The arithmetic rate of return (net of wage inflation) in year $(t-1, t)$ is $\exp(\delta_t) - 1$, its mean is 5% (that is, it is equal to the valuation discount rate net of salary inflation), and its standard deviation is 20%. Similar modeling assumptions were made with the MA(1) model to which allusion is made at the end of Section 3.2.

Simulation Model 2

In the asset-liability projections, time series models fitted by Wilkie (1995) for equity returns, long bond yields, and price inflation in Canada, the United Kingdom, and the United States are used. Time series models, fitted by Wilkie (1995) and Sharp (1993) for economic wage inflation in the United Kingdom and Canada, respectively, are also used. Assets are valued at market and the projected unit credit method with a time-invariant valuation basis is used to value the liabilities.

Contributions are calculated so as to be a level percentage of payroll. The pension fund is invested in two asset classes, equities and long bonds. For the United States, payroll increases at a constant 4.5% every year. For the United Kingdom, payroll increases in line with Wilkie's (1995) economic wage inflation time series. For Canada, payroll increases in line with Sharp's (1993) model for Canadian wage inflation.

Economic valuation assumptions are as follows. For the United States and Canada, wage inflation is assumed at 4.5% and a real (net of wage inflation) discount rate of 4.5% is assumed. For the United Kingdom, wage inflation is assumed at 6.5% and a real (net of wage inflation) discount rate of 4.5% is assumed. No early retirement and no salary scale was assumed. For the United States and Canada, mortality follows the 1983 Group Annuitant Mortality table for males. For the United Kingdom, mortality follows English Life Table No. 14 for males. Demographic valuation and projection assumptions are identical. Demographic experience does not deviate from the actuarial valuation basis, and gains and losses emerge only as a result of unforeseen economic experience.

APPENDIX B

SOLUTION OF BELLMAN EQUATION AND PROOF OF PROPOSITION 1

PROOF OF PROPOSITION 1: FINITE HORIZON

Consider first a finite horizon. Let

$$\Phi_t = f_t + c_t - B, \tag{23}$$

$$\Psi_t = 1 + r + y_t\alpha, \tag{24}$$

and note that

$$y_t^2 = \alpha^{-2}\Psi_t^2 - 2\alpha^{-2}(1+r)\Psi_t + \alpha^{-2}(1+r)^2, \tag{25}$$

$$(c_t - CT_t)^2 = \Phi_t^2 - 2(f_t - B + CT_t)\Phi_t + (f_t - B + CT_t)^2. \tag{26}$$

Since α_t is independent and identically distributed over time, it follows from equation (7) that

$$\text{E}[f_{t+1}|f_t] = \Psi_t\Phi_t, \tag{27}$$

$$\text{Var}[f_{t+1}|f_t] = \sigma^2\Phi_t^2 y_t^2, \tag{28}$$

using equation (25),

$$E[f_{t+1}^2|f_t] = (1 + \sigma^2\alpha^{-2})\Phi_t^2\Psi_t^2 - 2\sigma^2\alpha^{-2}(1 + r)\Phi_t^2\Psi_t + \sigma^2\alpha^{-2}(1 + r)^2\Phi_t^2. \quad (29)$$

The Bellman optimality equation (equation 11) is

$$J(f_t, t) = \min_{c_t y_t} J, \quad (30)$$

where

$$J = \theta_1(f_t - FT_t)^2 + \theta_2(c_t - CT_t)^2 + \beta E[J(f_{t+1}, t + 1)|f_t], \quad (31)$$

with boundary condition, at time $t = N$,

$$J(f_N, N) = \theta_0(f_N - FT_N)^2. \quad (32)$$

A trial solution for equation (30) is

$$J(f_t, t) = P_t f_t^2 - 2Q_t f_t + R_t. \quad (33)$$

The boundary condition in equation (32) certainly satisfies the trial solution, with $P_N = \theta_0$ and $Q_N = \theta_0 FT_N$.

Proceeding by induction, suppose that the right-hand side of equation (33) is a solution of the Bellman equation (30) at $t + 1$. Then,

$$E[J(f_{t+1}, t + 1)|f_t] = P_{t+1}E[f_{t+1}^2|f_t] - 2Q_{t+1}E[f_{t+1}|f_t] + R_{t+1} = (1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t^2\Psi_t^2 - 2[Q_{t+1}\Phi_t + \sigma^2\alpha^{-2}(1 + r)P_{t+1}\Phi_t^2]\Psi_t + \sigma^2\alpha^{-2} \times (1 + r)^2P_{t+1}\Phi_t^2 + R_{t+1}, \quad (34)$$

where use is made of equations (27) and (29).

J may be written as a quadratic expression in Ψ_t, Φ_t and f_t, by substituting equations (26) and (34) into equation (31):

$$J = [\beta(1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t^2]\Psi_t^2 - 2\beta \times [Q_{t+1}\Phi_t + \sigma^2\alpha^{-2}(1 + r)P_{t+1}\Phi_t^2]\Psi_t + [\theta_2 + \beta\sigma^2\alpha^{-2}(1 + r)^2P_{t+1}]\Phi_t^2 - 2\theta_2 \times (f_t - B + CT_t)\Phi_t + \beta R_{t+1} + \theta_2(f_t - B + CT_t)^2 + \theta_1(f_t - FT_t)^2. \quad (35)$$

Upon completing the squares in Ψ_t and Φ_t,

$$J = \Psi_t^A(\Psi_t - \Psi_t^B)^2 + \Phi_t^A(\Phi_t - \Phi_t^B)^2 + P_t f_t^2 - 2Q_t f_t + R_t, \quad (36)$$

where P_t and Q_t are as in Section 4, R_t represents additional terms independent of f_t, and

$$\Psi_t^A = \beta(1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t^2, \quad (37)$$

$$\Psi_t^B = \frac{\sigma^2\alpha^{-2}(1 + r)P_{t+1}\Phi_t + Q_{t+1}}{(1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t}, \quad (38)$$

$$\Phi_t^A = \tilde{P}_{t+1}^{-1}(\alpha^2 + \sigma^2)^{-1}, \quad (39)$$

$$\Phi_t^B = \theta_2(\alpha^2 + \sigma^2)\tilde{P}_{t+1}(f_t - B + CT_t) + \beta\sigma^2(1 + r)\tilde{P}_{t+1}Q_{t+1}, \quad (40)$$

$$\tilde{P}_{t+1} = [\theta_2(\alpha^2 + \sigma^2) + \beta\sigma^2(1 + r)^2P_{t+1}]^{-1}. \quad (41)$$

J has a unique minimum in Ψ_t and Φ_t provided $\Psi_t^A > 0$ and $\Phi_t^A > 0$. It is sufficient that $P_t > 0$ for $t \in [1, N]$ for both these conditions to be satisfied (since $P_t > 0 \Rightarrow \tilde{P}_t > 0$). (It is assumed that the plan is partially funded and invests in the two assets at all times and $\Phi_t = f_t + c_t - B > 0$.) The minimum occurs when $\Psi_t = \Psi_t^B$ and $\Phi_t = \Phi_t^B$ simultaneously. There is a direct linear relationship between c_t and Φ_t (equation 23) and between y_t and Ψ_t ($\alpha > 0$ in equation 24). Therefore, $\min_{c_t, y_t} J = P_t f_t^2 - 2Q_t f_t + R_t$ which is in the form postulated in the trial solution (33). Since the solution holds for $t = N$, it holds for $t \in [1, N]$. Since $P_N = \theta_0 > 0$, then $P_t > 0$ for $t \in [1, N]$ and the sufficient condition for the existence of a single minimum is satisfied.

Let $\Theta_t = \theta_2(\alpha^2 + \sigma^2)\tilde{P}_t$. Then, $1 - \Theta_{t+1} = \beta\sigma^2(1 + r)^2\tilde{P}_{t+1}P_{t+1}$, from equation (41). Then,

$$\Phi_t^* = \Phi_t^B = \Theta_{t+1}(f_t + CT_t - B) + (1 - \Theta_{t+1})Q_{t+1}P_{t+1}^{-1}(1 + r)^{-1}. \quad (42)$$

Using equation (23), $c_t^* = \Phi_t^* + B - f_t$ is readily obtained. Replacing $\Phi_t = \Phi_t^*$ in the right-hand side of equation $\Psi_t = \Psi_t^B$ gives

$$\Psi_t^* = \frac{Q_{t+1} + \sigma^2\alpha^{-2}(1 + r)P_{t+1}\Phi_t^*}{(1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t^*}. \quad (43)$$

Now, $[1 + r + \alpha y_t^*][f_t + c_t^* - B] = \Psi_t^*\Phi_t^* = [Q_{t+1} + \sigma^2\alpha^{-2}(1 + r)P_{t+1}(f_t + c_t^* - B)](1 + \sigma^2\alpha^{-2})^{-1}P_{t+1}^{-1}$, from which equation (20) follows.

It is also possible to express y_t^* independently of c_t^* by using equations (24) and (43), giving

$$\alpha y_t^* = \Psi_t^* - (1 + r) = \frac{Q_{t+1} - (1 + r)P_{t+1}\Phi_t^*}{(1 + \sigma^2\alpha^{-2})P_{t+1}\Phi_t^*}, \quad (44)$$

which, upon substitution of Φ_t^* from equation (42), readily yields

$$y_t^* = \frac{\alpha\Theta_{t+1}(1+r) \times [Q_{t+1} - P_{t+1}(1+r)(f_t + CT_t - B)]}{(\alpha^2+\sigma^2)[(1-\Theta_{t+1})Q_{t+1} + \Theta_{t+1}P_{t+1} \times (1+r)(f_t + CT_t - B)}. \quad (45)$$

PROOF OF PROPOSITION 1: INFINITE HORIZON

The proof for the infinite-horizon case follows closely the preceding proof for the finite-horizon case and is only sketched.

CT_t and FT_t are constant and may be replaced by CT and FT. The value function does not depend directly on time, and $J(f_t, t)$ and $J(f_{t+1}, t+1)$ may be written as $J(f_t)$ and $J(f_{t+1})$. The optimality equation has no boundary condition and backwards induction is not necessary. Since the optimality equation does converge, its solution in the infinite-horizon case is the steady-state or equilibrium solution of the finite-horizon case. A suitable trial solution is $J(f_t) = Pf_t^2 - 2Qf_t + R$, where $P_t \to P$, $Q_t \to Q$ and $R_t \to R$ as $t \to \infty$. All P_t, Q_t and R_t in the preceding proof may be replaced by constant P, Q and R. $\mathrm{E}[J(f_{t+1})|f_t, c_t, y_t]$ and J are as in equations (34) and (35), with the appropriate constant terms described above. J may be simplified to a quadratic in Ψ_t and Φ_t. It turns out that J may be minimized, leaving a quadratic in f_t, thereby confirming the trial solution:

$$J = \Psi^A(\Psi_t - \Psi^B)^2 + \Phi^A(\Phi_t - \Phi^B)^2 + f_t^2[\theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P] - 2f_t[\theta_1 FT + \theta_2\beta\sigma^2 \times (1+r)P(Q - (1+r)P(CT - B))] + R, \quad (46)$$

where

$$\tilde{P} = [\theta_2(\alpha^2+\sigma^2) + \beta\sigma^2(1+r)^2P]^{-1}, \quad (47)$$

$$\Psi^A = \beta(1+\sigma^2\alpha^{-2})P\Phi_t^2, \quad (48)$$

$$\Psi^B = \frac{\sigma^2\alpha^{-2}(1+r)P\Phi_t + Q}{(1+\sigma^2\alpha^{-2})P\Phi_t}, \quad (49)$$

$$\Phi^A = \tilde{P}^{-1}(\alpha^2+\sigma^2)^{-1}, \quad (50)$$

$$\Phi^B = \theta_2(\alpha^2+\sigma^2)\tilde{P}(f_t - B + CT) + \beta\sigma^2(1+r)\tilde{P}Q], \quad (51)$$

and R includes additional terms independent of f_t and not required here.

By comparing the coefficients of f_t^2 and f_t in equation (46) with those in the trial solution, it is clear that P satisfies $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$, which may be rewritten as equation (17), while Q satisfies $Q = \theta_1 FT + \theta_2\beta\sigma^2(1+r)\tilde{P}[Q - (1+r)P(CT - B)]$. This may be simplified into the form given in equation (18) by noting that $\theta_2\beta\sigma^2(1+r)\tilde{P} = (P - \theta_1)/(P + Pr)$, given that $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$.

J in equation (46) has a unique minimum in Ψ_t and Φ_t, provided that $\Psi^A > 0$ and $\Phi^A > 0$. It is sufficient that $P > 0$ for both these conditions to hold. (The plan is assumed to be partially funded at all times and $\Phi_t > 0$.) Since the coefficient of P^2 and the constant term in the quadratic equation (17) are positive and negative, respectively, P must have one negative real root and one positive real root, the latter being the admissible solution. Then, $\tilde{P} > 0$ in equation (47), and $P > \theta_1$ since $P = \theta_1 + \theta_2\beta\sigma^2(1+r)^2\tilde{P}P$. J is minimized when $\Psi_t = \Psi^A$ and $\Phi_t = \Phi^A$ simultaneously and, exploiting again the direct linear relationship between Φ_t and c_t and between Ψ_t and y_t, the minimizing values of c_t and y_t, denoted respectively by c_t^∞ in equation (21) and by y_t^∞ in equation (22), may be found as in the finite-horizon case. Again, y_t^∞ may also be written as

$$y_t^\infty = \frac{\alpha\Theta(1+r)[Q - P(1+r)(f_t + CT - B)]}{(\alpha^2+\sigma^2) \times [(1-\Theta)Q + \Theta P(1+r)(f_t + CT - B)]}. \quad (52)$$

PROOF OF PROPOSITION 2

Since $P_N = \theta_0 > 0$, backward recursion in the Riccati difference equation for P_t formed by equations (13) and (14) shows that $P_t > 0$, $\tilde{P}_t > 0$ for $t \in [1, N]$. Clearly, $0 < \Theta_t < 1$ for $t \in [1, N]$. It is reasonable to assume that the fund and contribution targets in any year are such that $FT_t > 0$ and $CT_t < B$ (as otherwise there is no sense to funding in advance for retirement benefits). Then, $Q_t > 0$ for $t \in [1, N]$ from equation (15) and the boundary condition $Q_N = \theta_0 FT_N > 0$. In the infinite-horizon case, $P > \theta_1 > 0$ and clearly $0 < \Theta < 1$. With $FT > 0$ and $B > CT$, it follows that $Q > 0$ (equation 18).

It is immediately observed that $\partial c_t^*/\partial f_t < 0$ from equation (19). From equation (20), it is easy to show that $\partial y_t^*/\partial[f_t + c_t^* - B]$ is directly proportional to $-Q_{t+1}P_{t+1}^{-1}(1+r)^{-1}$, which is negative. Since $\partial[f_t + c_t^* - B]/\partial f_t = \Theta_{t+1} > 0$, it follows that $\partial y_t^*/\partial f_t < 0$. Likewise, $\partial c_t^\infty/\partial f_t < 0$ and $\partial y_t^\infty/\partial f_t < 0$.

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Discussions on this paper can be submitted until July 1, 2004. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.

Valuation paradigms are something new in accounting practice. Before fair value, accountants depended on a set of rules for recording transactions and rolling the results into summaries. The guiding principle, if any, has been to "follow the cash". The valuation has been whatever comes out the tailpipe. In fact, where rational valuation is the goal, some items of cash flow are ambiguous and subject to partition. The most obvious of these, as I emphasized heavily in the paper, is the credit penalty incurred by a borrower (or the party on the short end of any obligation). We may define it as the reduction in the cash proceeds compared to the terms enjoyed by the most favored borrower. Traditional accounting amortizes this penalty over the term of the obligation. A correct treatment should, in my view, treat this penalty as an immediate expense, i.e. prepaid interest. If the transaction is treated in this manner, equivalent obligations lead to equal liabilities regardless of credit standing; and earnings are not exposed to taxation before their time. (A short paper on this patter point has been submitted for publication consideration).

I can only speculate why the financial economics community has, in large part, failed to recognize the essential differences between asset and liability valuation. Financial economists know that liabilities should be recorded on a default-free basis. But it seems that they have avoided (or evaded) a direct collision with the accounting community by arguing that the corporate default option – an asset of the owners rightly enough – is somehow also an asset of the corporation, giving the same equity measurement as traditional accounting. Therefore an "economic", "fair" valuation of the liabilities requires deducting the value of the default option, giving liability values equal to the market value of the countervailing assets. (The author and friends are preparing a paper for presentation at an upcoming conference on the question of where to put the default option.)

In arriving at this conclusion, both camps have ignored some vital contextual issues. Markets derive asset valuations by implicitly averaging over many scenarios. Among the contingencies considered is the survival of the firm that is obligated to fulfill the asset. Markets know that a firm may fail, and asset valuations always consider the possibility of default. Liabilities, on the other hand, are valued by management sitting inside the firm and managing it either as a going concern or not. Managements cannot average, they must choose. The middle ground is occupied by the judge of a bankruptcy court, who must be informed if management thinks that the firm is in danger of failure. A going concern is not expected to default on its obligations, which as liabilities, must be valued on a default-free basis. Values adjusted for default are to be booked only after negotiation and under judicial supervision. Traditional accounting practice is in clear conflict with due process of law and has been tolerated only because it is older than any bankruptcy code. The present consensus in financial economics has apparently been arrived at by, inexplicably, ignoring economics and according precedence to accounting issues over legal issues.

The most seductive of the counter-arguments opposing default-free valuation – that is, the "buy-back" argument – must be considered in the light of the foregoing. Many debt contracts, e.g. callable bonds, contain embedded buyback options and should be valued accordingly, but always under the assumption that the obligation will be performed as written. Buyback windfalls can occur, but they cannot be anticipated.

Those who know me know that this is not my job. I write without authority, relying only on the appeal to reason and common sense. These writings have no value apart from the assent kindled in other minds. For this reason, Mr. Chambers' review is particularly welcome and gives me the hope, absent before, that this whole sorry mess may yet be put right. I wish to thank Mr. Chambers for his careful and incisive reading and his helpful comments.

"Efficient Gain and Loss Amortization and Optimal Funding in Pension Plans," M. Iqbal Owadally and Steven Haberman, January 2004

JEREMY GOLD*

As evidenced by the Vancouver Symposium (SOA 2003), a globe-circling intellectual battle line has been drawn between two schools of pension

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actuarial thought. Each school is working to apply the concepts and tools of economics to the science and practice of pension actuaries. Professors Owadally and Haberman have brought us some of the best-written work from one side of the line of battle.

Because I hail from the other side, I am cautious about ascribing properties to the Owadally-Haberman school. Nonetheless, I think it is fair to characterize it as scheme-centric, statistical, and friendly to the inclusion of equities in defined benefit (DB) plans. My side may be counter-characterized as principal-centric (recognizing shareholders, plan beneficiaries, and taxpayers as principals), capital structure intensive, and not friendly to the inclusion of equities.[1]

Jon Exley (2003) has most sharply defined the divide between the two schools with respect to asset allocation:

> "At first sight, deciding the asset allocation for a defined benefit pension scheme . . . might seem like an obvious application of portfolio selection theory. . . A natural way to proceed might be to . . . pick a strategy that maximises return subject to some acceptable degree of risk. . ."
>
> "[We argue] that the basic flaw in traditional asset and liability modelling is that asset allocation of institutional funds should not be treated as a classical portfolio selection issue. The traditional models are looking in the wrong direction. The machinery commonly applied by many practitioners is designed to solve the wrong problem. . . [We argue] that institutional asset allocation problems should be addressed . . . by the corporate financial theory of the firm. This is an entirely different branch of economic theory that deals with issues such as optimal capital structures for firms rather than individuals' portfolio construction" (p. 29–31).

In terms of the economic literature, Exley (2003) is pointing us toward Modigliani and Miller (1958), Treynor (as Bagehot 1972), Jensen and Meckling (1976), Miller (1977), Black (1980) and Tepper (1981), and away from Markowitz (1952) and Sharpe (1964) and their myriad successors. Certainly, he means no disrespect to the portfolio selection branch and its brilliant developers. Rather, he challenges DB plans, their sponsors, actuaries, and consultants to identify and apply the pertinent tools to the task at hand.

The present paper is clearly consistent with the portfolio selection literature. Owadally and Haberman define *efficient* methods of gain and loss amortization as those that offer the least volatile funding for a given level of contribution rate volatility or, equivalently, the least volatile contribution rates for a given funding volatility. Comments by Section follow.

SECTION 1

Section 1 lays out a clear beginning to a paper that remains clear and well-organized throughout. Owadally and Haberman begin by assuming that deterministic (demographic) actuarial assumptions are perfectly met and that inflation and returns on assets are stochastic. This readily brings us to focus on gains and losses arising from the economics of DB plans.

Asset returns and inflation (and from here on I will ignore inflation as, for the most part, do Owadally and Haberman) are hedgable in the capital markets. This is a critical divide for the two contending pension actuarial schools. The corporate-finance side argues that shareholders and taxpayers can take on or dispose of asset risks and rewards on virtually the same terms as those available to DB plans. Thus, I assert that the proper problem definition asks not how much equity risk principals[2] shall take in their aggregate portfolios but rather where such risks shall reside (especially as to whether inside or outside of tax-sheltered plans; see Bader 2003a).

Asset hedgability should allow us to ask, and should incline Owadally and Haberman to address, the question of why DB plans generate any economic gains or losses. Because DB plans need not take unhedged economic risk, why should they? Owadally and Haberman create utility functions at the plan level without ever telling us whose utility should apply. In my view, this is where the scheme-centric view breaks down.

In Section 1.4, Owadally and Haberman correctly note that expected losses and unfunded

[1] Except as justified by external (e.g., PBGC) guarantees and weak funding (Harrison and Sharpe 1983).

[2] The immediate model ignores any insolvency risk to employees and, thus, so do I.

liabilities equal zero under their model. Gains and losses, therefore, must arise from deliberate asset-liability mismatches or, more crudely, from gambling at the scheme level.

Owadally and Haberman refer to a pension plan under a trust that is independent of the firm. While trust management may be independent (perhaps more so in the United Kingdom, less so in the United States), the financial outcomes of the trust fall upon plan principals and, under the no-default assumption applied throughout much of the paper, principally upon shareholders and taxpayers.

Owadally and Haberman identify benefit security as "one purpose of funding benefits in advance. . . ." The corporate-finance view of the matter is that benefit security is the paramount purpose of funding. In a model incorporating potential insolvency of the plan and its sponsor,[3] we can argue that the efficient contract (here we define *efficient* in terms of maximizing the total value for the owners and the beneficiaries—subject to negotiated allocation thereof) includes full funding. Bader (2003b) demonstrates that underfunding is equivalent to borrowing from one's employees (the sponsor puts its "bond" into the plan instead of cash) and that the employees' inability to diversify firm-specific risk makes them inefficient (expensive) sources of financing.

Viewed from this corporate-finance perspective, the efficient funding and investing rules are quite straightforward: Fund fully and invest in bonds. Tepper (1981) makes this same recommendation based on Modigliani and Miller (1958) and Miller (1977) and the tax regimes of Anglo-American nations. That benefit-security and tax-arbitrage considerations lead to the same conclusion is coincidental; nonetheless, both point to full funding and no equities.

Thus, Owadally and Haberman must, as a prerequisite to their scheme-based utility function, justify the taking of *any* economic risk (asset returns, inflation, insolvency) at the scheme level. After explaining why economic asset-liability mismatches are desirable (and to whom), Owadally and Haberman might want to provide an economic rationale for their twin-smoothness criteria. Although most of the traditional actuarial literature points to smoothness as a desirable property of actuarial funding methods, as we move to integrate pension finance with the science of capital markets, we must develop anew any rationale for actuarial processes that smooth capital market returns. Until that rationale is proffered, I will iterate:

> "Volatility is a property of markets; it is not a disease for which . . . [actuarial smoothing] is the cure. The volatility of defined benefit plan funding status and cost is real, and it is generated primarily by the mismatch of assets and liabilities. Asset-liability matching can sharply curtail the volatility of financing gains and losses, and the purchase of deferred annuities can eliminate it" (Bader and Gold 2003, p. 12).

SECTIONS 2 AND 3

In Section 2, Owadally and Haberman's outline of a chain of work that includes Dufresne in addition to their own earlier work is clear and powerful and, within the portfolio selection and traditional actuarial disciplines, well worthy of kudos. In Section 3, similar remarks may be made with regard to their simulation work, which adds to the credibility and the robustness of their reduced form Section 2 model.

SECTION 4

Owadally and Haberman incorporate a basic asset return model (preceding equation 7) with an asset allocation control variable (y_t), a risk premium ($\alpha_t > 0$), and an associated variance, σ^2. They define a quadratic "cost" function (equation 8) reflecting a quadratic utility function.

Equations 9 and 10 combine the quadratic cost model with a time preference parameter (β) in order to develop what is described as an "objective criterion for the performance of the pension funding system. . . ." Equation 11 minimizes the cost of this system, solving for optimal control

[3] I agree with the authors that such default risk is beyond the general scope of the reduced form model discussed herein. Thus, I am not arguing about this aspect of the model but rather about their characterization of the purpose of funding, especially their view that smoothing employer cash flows is a significant economic reason to fund.

values for the annual contribution and the plan allocation to equities.

The adoption of a utility function and a time preference parameter may usually be justified in the context of portfolio construction for an individual investor whose preferences are thereby represented. In this instance, however, these parameters merely serve to substitute a cost of risk for the plan, which is different from the cost of risk for the very same assets and liabilities in the capital markets. The plan then develops an "optimal" exposure to capital market risks and passes these along to the shareholders of the plan sponsor. As such a shareholder, I would have to ask, "Why have you done this for (to) me? I have my own risk-reward and time preferences. To construct my own utility-maximizing portfolio, I must unwind what you have done. Furthermore, because investing in equities within a tax shelter is wasteful in the tax regimes of the United Kingdom, the United States, and Canada, I am left with an unnecessary tax burden, even after I unwind."

In short, because the plan assets are generally available to the shareholders of the sponsoring corporation on the same terms as they are to the plan, and are thus hedgeable, the entire exercise is wasteful from the perspective of the firm's owners. Under the reduced-form model, employees have no interest in the matter. Under a more comprehensive model (such as that alluded to in Section 5), the mismatch of plan assets and liabilities exposes employees to risk. This exposure will, in a transparent model, result in increased compensation costs that must, again, be borne by shareholders.

SECTION 5

Owadally and Haberman have considered methods for amortizing "unforeseen economic experience" in DB pension plans without justifying why anyone should ever voluntarily expose themselves to such experience. It seems that those who advocate equity investment in DB plans continue to tell us that it lowers the cost of defined benefits. This view, although still apparently held by a majority of practicing actuaries, is now all but bankrupt intellectually. Even though it remains the case that various regulatory regimes for statutory funding and corporate accounting reward such equity investment, economic science rejects it.

The Society of Actuaries' motto says "The work of science is to substitute facts for appearances and demonstrations for impressions." To this, we should now add, paraphrasing Exley (2003): It must further be the duty of scientists to apply tools that are appropriate to problems that are well defined.

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This paper contains some intuitive and illuminating solutions to a number of mathematical problems. The authors also investigate sensitivities to the choice of stochastic investment models, and have brought considerable mathematical ingenuity to bear. All this is commendable.

We would welcome more discussion regarding the way corporate decisions are made and preferences defined. We are concerned that the authors' concept of efficiency (Section 2) overlooks important aspects of corporate decision making. Section 1.4 states that "The pension plan financial management is separate from that of the company sponsoring benefits . . ." The authors then proceed to ignore any correlation between the well-being of the pension plan and of the sponsor itself. Setting such correlations to zero is a heroic assumption which turns out to be much more fundamental to the authors' conclusions than is the choice of investment return distributions or amortization schedule. A growing number of U.K. pension professionals are convinced that the "plan-centric approach," ignoring all correlations with the outside world, is an unsound basis for decision making.

Consider first the variability in funding level. We agree with the authors' assertion in Section 1.4 that "one purpose of funding benefits in advance is to maximize the security of benefits." However, the proposed steady state model assumes that the plan continues to pay benefits forever, and therefore benefits are totally secure. This is an unpromising start for a default risk model.

A plan member's exposure to default consists of a sum at risk and a default frequency. The sum at risk is the difference between the funded sum and the cost to buy out promised benefits via deferred annuities. The funding surplus is a poor proxy, because the funding valuation basis takes advance credit for future risky asset returns (a consequence of the "unbiased" assumption in Section 1.4), unlike market annuity prices. Furthermore, a loss of benefits (ignoring the effect of any federal guarantee funds) only happens when a plan deficit coincides with corporate failure. A deficit alone does not trigger loss of benefits. Therefore, the covariance between plan strength and the sponsor's financial strength is critical to any assessment of benefit security. Indeed, the covariance effect dominates any effect arising from funding variability unless the plan is stochastically independent of the sponsor's fortunes. To say that plan management is separate from corporate management (Section 1.4) does not imply that the two entities are stochastically independent as the authors propose. For example, a corporation and its pension plan inevitably face the same level of interest rates and stock market prices. Few readers will accept the assertion (Section 1.4) that "variance [of the plan funding level] is a reasonable and tractable approximation [to plan security]."

We also agree with the suggestion in Section 1.4 that "Another motivation for pre-funding pension benefits is to spread the required future pension contributions and hence reduce strain on the sponsor's cash flows." Corporations seek to avoid large pension flows at the same time as tight cash flows elsewhere in the business. Corporations might also worry about the impact of pension risk on their cost of raising capital to finance their core business. Variability of pension flows themselves is a small part of the picture. Variable funding contributions might even be a good thing if they naturally hedged some other aspect of corporate cash flow. Wise corporate risk managers are not concerned with the variability of pension flows in themselves, but instead with the effect of a pension plan on an already variable corporate cash flow.

As with the funding level, a covariance is

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required, but the authors argue instead that "minimizing the variability of the contribution . . . is also a reasonable objective." Their assertion would be valid only if plan cash flow were independent of other corporate cash flows. The authors escape building a corporate model only by a heroic stochastic independence assumption regarding the cash flows of the pension plan and the sponsor.

It is surprisingly straightforward to analyze pensions even when a plan is correlated with the sponsor's fortunes. We characterize the emerging literature as "business-centric" rather than "plan-centric." Only arbitrage arguments and clear thought are required in order to understand the key conclusions. No calculations are needed. Sharpe (1976) and Treynor (1977) set out the main findings. Non-zero correlations have a profound impact—implying that the same risk controls routinely applied to corporate balance sheets are also appropriate for pension plans.

Some difficult questions relating to discretionary benefits, default risk, and tax require more detailed modeling. Clear thinking and model complexity need not exclude each other. For example on the question of amortization, Chapman et al. (2001, Section 6.9) deduce the following:

> "Shortening the [amortization] period [from 10 years] to 5 years increases the percentage of scenarios where the company goes into liquidation . . . Note that both shareholders and employees are winners as a result of this change. . . . The losers are the Government, which receives less tax, . . . and holders of debt, who suffer a higher rate of default."

Their approach is more useful for decisions than the corresponding findings of Dr. Owadally and Professor Haberman (see Section 2). "Dufresne (1988) concludes that . . . spreading gains and losses over a period between 1 and 10 years is efficient. . . . Owadally and Haberman (1999) conclude that the . . . practice of amortizing gains and losses over 5 years is also efficient."

The results of Chapman et al. (2001) are consistent with those of Sharpe (1976) and Treynor (1977), but conflict with the claims of Dr. Owadally and Professor Haberman. The current authors' refusal to cite the business-centric literature is all the more remarkable because a written discussion of Chapman et al. was published in the *British Actuarial Journal* (Haberman 2001).

The authors have presented some elegant mathematical results. Unfortunately, their usefulness is undermined by weaknesses in the assumptions. Their nemesis is the assumption of stochastic independence between the fortunes of a pension plan and its corporate sponsors. Had the authors investigated the effect of weakening this assumption, they would have recovered well-known results from the 1970s finance literature.

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The paper provides a closely argued and detailed consideration of how funding deficits and surpluses should be dealt with. However, by taking a plan-centric view of a pension plan, without incorporating the economic relationship of the plan to the owners of the sponsor (typically company shareholders) and the plan members, the paper is, therefore, unable to consider the economic impact of sponsor insolvency on members and contribution volatility on shareholder value. These features are of prime importance when considering pension plan funding.

The authors set out to find an efficient way of spreading gains and optimal funding. They adopt the objective of minimizing some combination of the variability of the deviation from the funding target and the contribution rate. The economic justification for choosing this combined objective is not made clear. Moreover, the two most impor-

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tant controls in determining plan funding, namely, the funding target and investment strategy, are assumed to be specified exogenously and, therefore, omitted from the analysis. For instance, the optimal solution of investing in hedging assets and paying the future service cost is excluded without further consideration.

The authors made an implicit assumption that the sponsor can never fail. This approach assumes away the possibility of any serious economic analysis because the main justification for providing advance pension plan funding via a separate trust (as opposed to the sponsor making advance provision via an unassigned savings vehicle) is to provide security in the event of corporate insolvency (see, e.g., McLeish and Stewart 1987).

The objective that appears to relate most closely to security is "minimizing the volatility of the funding level or of the unfunded liability." However, this seems unlikely to be economically meaningful as:

- an unfunded plan would meet the funding volatility minimization criterion but clearly fail to provide the security of a funded plan, and
- a plan constantly funded at 50% clearly provides less security than a plan with a funding level that fluctuates between 120% and 100%.

Accordingly, we disagree with the authors that "variance is a reasonable . . . approximation" to "an ideal measure of security."

In Section 4, the authors introduce the idea of a cost function. This function is not justified in terms of how it affects the members or the ultimate providers of the pension, the shareholders (and other capital providers) of the sponsor. Indeed, using a Modigliani-Miller (1958) framework, where there is no insolvency, the cost to shareholders is fixed and so the cost function has no meaning for shareholders. Members will have no concerns over the funding level because the sponsoring company can always meet any deficit and, therefore, it has no meaning for them either. There is no attempt to reconcile the cost to the market value of instruments that could be used to achieve the desired goal and, given the structure of the model, it is difficult to see how this could be achieved.

In Section 4.3, we are provided with the result that "the optimal proportion invested in the risky asset decreases as the f_t [the market value of the assets] increases." Let us consider the converse. As the value of the assets decreases, the proportion of risky assets increases. From the members' point of view, this says that the worse the security the more risk should be taken. It is difficult to reconcile this with the members' interest in the security of their benefits. Economic feedback of an insecure pension promise into employee remuneration will not compensate members who have large accrued benefits or who are no longer employees.

The security aspect also affects the contributions. A plan in deficit effectively has a debt owed from the sponsor. The longer the period taken to remove the deficit (i.e., the smaller the adjustment contribution), the longer the members are at risk of sponsor default. Hence, the two objectives are contradictory.

The plan-centric view means that this paper is unable to address satisfactorily the issues raised by surpluses, deficits, and funding policy. Two possibilities arise: (1) that members and shareholders of the sponsor are essentially in conflict—a gain to one means a loss to the other—or (2) they can collaborate to realize value from a third party. In either case the approach of the paper is unable to help.

The points discussed above are a cause for concern, but even if they could be addressed, the most serious concern about the paper would remain: The approach adopted in this paper depends on a Markowitz (1952) style of portfolio selection. While portfolio theory is an important branch of finance, it is not appropriate to the problem being addressed in the paper. The area of finance theory that is applicable to pension funds and their sponsors relates to the theory of the firm as advanced by Modigliani and Miller (1958) and applied to pensions by Sharpe (1976), Black (1980), and Tepper (1981) and more recently Exley et al. (1997).

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AUTHORS' REPLY

The contributors to the discussion of our paper have outlined a so-called "business-centric" approach to defined-benefit pension funding. They claim that there is a fundamental division between this and the "plan-centric" approach. The truth about this division is more pedestrian. The plan-centric approach is embedded in the current regulatory and statutory reality of pension funding. Our paper confines itself to this reality. On the other hand, the contributors reiterate the "extended balance sheet" argument (which integrates the balance sheet of the pension plan into that of the sponsor) and would like to overhaul current pension legislation, regulation, and standards of practice. Their arguments ignore funding requirements, employee contributions, trust law, and discretionary benefits.

We address, in this paper, a purely practical issue concerning funding methods and amortization mechanisms and we suggest that spreading is better than amortization. The point of the paper is not to examine the purpose of pension funding and the overall suitability of actuarial funding methods. Nor are we promoting one asset allocation strategy over another. We are pointing out that, under the usual (and rather limited) objectives of actuarial funding methods, which are to minimize the volatility of contribution rates and funding levels around a normal cost and actuarial liability respectively, a contrarian investment strategy follows (which is expected), together with an optimal funding strategy involving spreading (which is new).

The discussions misrepresent a number of points. We emphasize that the first moment of the unfunded liability is zero, which enables us to look at volatility through the second moment. (We point out in the paper that the quadratic criterion is simplistic.) The funding valuation basis does not have to take advance credit for the risk premium in risky assets. It is quite possible to value liabilities at a suitable discount rate while calculating contributions and making funding decisions based on a higher expected rate of return on plan assets (Owadally 2003). Finally, our use of the term "efficient" does not mean that we simplify pension funding into a one-period portfolio selection problem.

The discussants assume that the pension plan is not a financial entity distinct from the corporation. This view turns accrued benefits into bonds and plan members into bondholders. It is difficult to see what level of funding is optimal under this scenario. Disregarding statutory funding requirements, an unfunded book-reserve system may even be optimal for the firm.

Several of the discussants refer to the "irrelevancy proposition" of Modigliani and Miller (1958). Under this theorem, the purchase or sale of financial assets by a company makes no difference to the shareholders of the company. This theorem is derived and hence applies under a series of strict conditions:

(i) all shareholders can buy or sell financial assets on the same terms as the company;
(ii) all shareholders have optimised their private portfolios.

Condition (ii) requires that all the shareholders have efficient portfolios in the Markowitz sense. In order to apply the theorem to the financial strategy of a defined benefit pension plan sponsored by the company, we must add the following two further conditions:

(iii) the assets of the pension fund belong to the company;
(iv) the benefits provided by the pension plan are marketable financial assets.

There are serious difficulties when we attempt to apply the Modigliani-Miller theorem to the financial strategy of the company pension plan. Is a company really in a similar financial position if it invests its pension fund entirely in bonds while simultaneously issuing a matching quantity of its own bonds, as the discussants appear to believe? If the assets of the pension fund did belong to the company, one could argue that the net gearing of the company would be unchanged. Pension fund assets, however, cannot be used to redeem the company's loan stock if it gets into financial difficulties, so that condition (iii) does not hold. It follows that a financial strategy for the pension plan that results in higher gearing for the company would reduce the security of the retirement benefits.

Thus, a defined benefit pension plan is not in reality simply an extension to the balance sheet of the sponsor. It is by design independent, involving an independent fund, managed independently in the best interests of the plan members and, as such, the plan forms part of the remuneration package of the members. The investment of the funds is the responsibility, in the United Kingdom, of the trustees who are required to invest on behalf of the members and beneficiaries, as a prudent person would do.

Turning to the members of the pension plan, it is clear that the Modigliani-Miller conditions are not satisfied. These individuals have large amounts of their wealth tied up in defined-benefit pension assets that cannot be traded, so they cannot change their private portfolios in ways that could nullify changes to their pension benefits – thus, assumption (iv) does not hold. We argue that the financial strategy adopted by the pension plan affects the balance between the security of the termination benefit and the security of the retirement benefit. There are no market transactions members can perform that would alter the balance between these risks. Consequently, changes to the financial strategy of the pension plan will have real "first-order" consequences for the diversification of risk within each member's private portfolio.

Our view is that the assumption of independence between the sponsor and the plan is crucial and the erosion of this assumption is partly the cause of the decline in defined-benefit pension provision. It must be recognized that defined-benefit pension risk is not born by stockholders alone, but is also borne by plan beneficiaries because of uncertain discretionary benefit enhancement and job mobility. In our view, a "consumer-centric" approach must be restored, whereby the defined-benefit promise is simplified and made explicit so that both employees and stockholders can cost it under standardized valuation models and identify the risks they take. This should then improve the quality of information available to investors and employees when they value company stock and employee contracts, respectively. This may involve the introduction of novel designs for defined benefit pension plans, such as variable benefit accrual rates (Khorasanee and Ng 2000).

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