On the Moments of the Time of Ruin with Applications to Phase-Type Claims

Steve Drekic* and Gordon E. Willmot†

Abstract

We describe an approach to the evaluation of the moments of the time of ruin in the classical Poisson risk model. The methodology employed involves the expression of these moments in terms of linear combinations of convolutions involving compound negative binomial distributions. We then adapt the results for use in the practically important case involving phase-type claim size distributions. We present numerical examples to illuminate the influence of claim size variability on the moments of the time of ruin.

1. Introduction

Ruin theory provides a systematic approach to the quantification of the solvency associated with a particular block of business. It explores the relationship between the initial capital required to support this block and the insolvency probability, or more generally, the probability that the associated surplus reaches a specified level. As such, the ideas inherent in its use are compatible with the use of various risk measures in insurance, including in particular "Value at Risk" (VaR).

Recent research in ruin theory is consistent with the use of what is generally viewed to be an improved risk measure over VaR, namely, "Conditional Tail Expectation" (CTE), also known as "Tail VaR." To be more specific, the analysis of the deficit at the time of ruin, or the severity of ruin, is of paramount importance in this regard, as the expected value of this deficit is the quantity of interest in connection with the CTE interpretation of the ruin problem. The impetus for this recent research was the seminal paper by Gerber and Shiu (1998), which quantified the relationship between the deficit at ruin, the time of ruin, and other related quantities. Sound financial risk management in this situation involves the use of both the time of and the deficit at ruin, and as such, knowledge of concomitant characteristics such as the associated density function and the moments is fundamentally important to this risk management process.

The time of ruin is a random variable that has proved to be somewhat difficult to analyze but is an integral component of the problem at hand, as discussed above. Included in the research of Gerber and Shiu (1998) is an expression for the Laplace transform of the time of ruin. In principle, this Laplace transform may be used to identify the *defective* density of the time of ruin as well as the moments. The technical details associated with the evaluation of both the density and the moments of the time of ruin are somewhat complicated, however. In the classical Poisson risk model, Drekic and Willmot (2003) derived expressions for these quantities in the case where the claim sizes are assumed to be exponentially distributed. Dickson and Willmot (2005) analytically inverted this Laplace transform to obtain an expression for the defective density of the time of ruin. This expression is complicated, however, and is not well suited for use in connection with the evaluation of the associated moments.

* Steve Drekic, PhD, is an Assistant Professor in the Department of Statistics and Actuarial Science, University of Waterloo, 200 University Ave. West, Waterloo, ON N2L 3G1, Canada, sdrekic@uwaterloo.ca.

† Gordon E. Willmot, FSA, PhD, is Munich Re Professor of Insurance in the Department of Statistics and Actuarial Science, University of Waterloo, 200 University Ave. West, Waterloo, ON N2L 3G1, Canada, gewillmo@uwaterloo.ca.

In this paper the focus is on the computational evaluation of these moments. This problem has been considered by various authors (e.g., Drekic, Stafford, and Willmot 2004, and references therein). This latter paper proposes an approach to the evaluation of the moments that is somewhat algebraically intensive, utilizing in particular nested integrals, which sometimes prove to be tedious and awkward to use. Our present paper, however, capitalizes on the fact that Drekic, Stafford, and Willmot (2004) provide an algorithm for the evaluation of a series of constants representing the moments of the time of ruin with no initial surplus. These constants may best be viewed as an integral component of the computation in the general case where the initial surplus can be an arbitrary non-negative quantity. In this paper our approach is to express these moments as a linear combination of convolution densities. To be precise, the convolution densities represent those of the sum of compound negative binomial random variables and what are sometimes referred to as *higher-order equilibrium* claim size densities, which we will describe. Recent advances in the evaluation of compound distributions suggest that this approach is well suited for numerical implementation for two reasons. First, recursive numerical evaluation of compound distributions is a standard approach to their evaluation (e.g., Bowers et al. 1997, chapter 12; Klugman, Panjer, and Willmot 2004, Sections 4.6 and 6.6). Second, the use of *phase-type* distributions (e.g., Asmussen 2000, and references therein) is becoming increasingly popular because they are quite flexible in terms of their possible shapes and because of their inherent mathematical and numerical tractability.

The organization of this paper is as follows. Section 2 presents the mathematical description of the classical Poisson surplus process, together with the necessary background to the particular problem as described above. Section 3 explains the methodology needed to compute the linear combination weights utilizing a procedure implicitly given in a characterization result of Willmot (2002) involving the moments. Section 4 considers the case involving phase-type claims, and Section 5 presents several numerical examples illustrating the flexibility of the present approach, as well as the influence of the claim size distribution on the moments of the time of ruin.

2. MATHEMATICAL PRELIMINARIES

In the classical insurance risk model, claims occur according to a Poisson process $\{N_t : t \geq 0\}$ with rate $\lambda > 0$. The individual claim sizes $\{X_1, X_2, \ldots\}$ are positive, independent, and identically distributed random variables with common distribution function (df) $P(x) = 1 - \bar{P}(x) = \Pr(X \leq x)$, where X is an arbitrary X_j. Let $\mu_n = \mathrm{E}[X^n] = \int_0^\infty x^n \, dP(x)$ denote the nth moment of this distribution. For later reference, we define recursively the nth-order equilibrium random variable $X_{(n)}$ (as long as $\mu_n < \infty$) having df

$$P_n(x) = 1 - \bar{P}_n(x) = \frac{\int_0^x \bar{P}_{n-1}(z)\, dz}{\int_0^\infty \bar{P}_{n-1}(z)\, dz}, \quad n = 1, 2, \ldots,$$

with $\bar{P}_0(x) = \bar{P}(x)$. Let $\tilde{p}_n(s) = \int_0^\infty e^{-sx}\, dP_n(x)$ and $p_n(x) = P_n'(x)$ be the associated Laplace-Stieltjes transform (LST) and probability density function (pdf), respectively. Premiums are paid continuously at rate $c = \lambda\mu_1(1 + \theta)$ per unit time, where $\mu_1 < \infty$ and $\theta > 0$ is the relative security loading. The surplus process $\{U_t : t \geq 0\}$ is defined by $U_t = u + ct - \sum_{i=1}^{N_t} X_i$, where $u \geq 0$ represents the initial surplus.

The time of ruin is defined as $T = \inf\{t : U_t < 0\}$, with $T = \infty$ if $U_t \geq 0$ for all $t \geq 0$. Let the probability of ruin, beginning with initial surplus u, be denoted by $\psi(u) = \Pr(T < \infty | U_0 = u)$. It is well known (e.g., Willmot and Lin 2001, and references therein) that $\psi(u) = \Pr(L > u)$, where L represents the maximal aggregate loss, and that $H_0(u) = 1 - \bar{H}_0(u) = \Pr(L \leq u)$ is a compound geometric df with LST

$$\tilde{h}_0(s) = \int_0^\infty e^{-sx}\, dH_0(x) = \frac{\theta}{1 + \theta - \tilde{p}_1(s)}. \tag{2.1}$$

Let $T_c = T|(T < \infty)$ be the time of ruin given that ruin occurs. To calculate moments of T_c, we introduce

$$\psi_k(u) = \mathrm{E}[T^k I\,(T < \infty)|U_0 = u] \tag{2.2}$$

as the kth (defective) moment of the time of ruin, with associated Laplace transform

$$\tilde{\psi}_k(s) = \int_0^\infty e^{-sx}\psi_k(x)\, dx, \quad k = 0, 1, \ldots. \tag{2.3}$$

In equation (2.2) $I\,(\mathcal{A})$ denotes the indicator function of the event $\mathcal{A}$, which is equal to 1 if the event occurs and 0 otherwise. Of course, $\psi_0(u) = \psi(u)$ is the ruin probability, and $\mathrm{E}[T_c^k|U_0 = u] = \psi_k(u)/\psi_0(u)$.

Drekic, Stafford, and Willmot (2004, Theorem 2.2) recently derived an explicit integral representation for $\psi_k(u)$ that holds for arbitrary u. In the special case when $u = 0$, however, further simplification is possible. As such, it is convenient to introduce

$$m_k(n) = \int_0^\infty x^n\psi_k(x)\, dx, \quad n = 0, 1, \ldots, \tag{2.4}$$

so that

$$\psi_{k+1}(0) = \frac{(k + 1)m_k(0)}{c}.$$

In particular, Drekic, Stafford, and Willmot showed in their Theorem 2.1 that $m_k(n)$ may be computed recursively using the formula

$$m_k(n) = \begin{cases} \dfrac{1}{n + 1}\,\mathrm{E}[L^{n+1}], & k = 0 \\ \dfrac{k}{(n + 1)\lambda\theta\mu_1}\displaystyle\sum_{j=0}^{n}\binom{n+1}{j+1} m_{k-1}(j + 1)\,\mathrm{E}[L^{n-j}], & k = 1, 2, \ldots, \end{cases} \tag{2.5}$$

where $\mathrm{E}[L^r]$, the rth moment of the maximal aggregate loss L, is itself given by the recursive formula

$$\mathrm{E}[L^r] = \frac{1}{(r + 1)\theta\mu_1}\sum_{j=0}^{r-1}\binom{r + 1}{j}\mathrm{E}[L^j]\mu_{r-j+1}, \quad r = 1, 2, \ldots.$$

While the results of Drekic, Stafford, and Willmot hold in general, the approach we describe in this paper is particularly well suited for use in connection with many claim size distributions of practical interest, including those from the important phase-type class, as is evident from the treatment of this issue in Section 4.

Our present approach, which also utilizes equation (2.5), begins with Willmot (2002, Theorem 6.1), who showed that if $\mu_{k+2} < \infty$, the Laplace transform (2.3) can be expressed as

$$\tilde{\psi}_k(s) = \beta_k\{\tilde{h}_0(s)\}^k\tilde{h}_{k+1}(s) + \sum_{n=1}^{k-1}\{\tilde{h}_0(s)\}^{n+1}\sum_{j=1}^{n}\sigma_{j,n,k}\tilde{p}_{j+1}(s), \quad k = 1, 2, \ldots, \tag{2.6}$$

where β_k and $\sigma_{j,n,k}$ are positive constants, and

$$\tilde{h}_{k+1}(s) = \tilde{h}_0(s)\sum_{j=1}^{k+1}\pi_{j,k+1}\tilde{p}_{j+1}(s), \tag{2.7}$$

where $\{\pi_{j,k+1}; j = 1, 2, \ldots, k + 1\}$ is a discrete probability measure for each $k = 1, 2, \ldots$. In equation

(2.6) and elsewhere, we adopt the notational convention that $\Sigma_{n=1}^{0} = 0$. Note that substituting equation (2.7) into equation (2.6) and combining terms yields the alternate and simpler representation

$$\tilde{\psi}_k(s) = \sum_{n=1}^{k} \sum_{j=1}^{n+1} \eta_{j,n,k} \{\tilde{h}_0(s)\}^{n+1} \tilde{p}_{j+1}(s), \quad k = 1, 2, \ldots, \tag{2.8}$$

where we define for $k = 1, 2, \ldots,$

$$\eta_{j,n,k} = \begin{cases} \sigma_{j,n,k}, & n < k; j < n+1 \\ 0, & n < k; j = n+1 \\ \beta_k \pi_{j,k+1}, & n = k; j \le n+1. \end{cases} \tag{2.9}$$

We remark that procedures for computing these various constants are not given explicitly in Willmot (2002), although implicit in the proof of Willmot's Theorem 6.1 are recursive formulae. In the next section we specify in full the recursive procedures necessary to calculate these constants.

If we now define the convolution LST $\tilde{b}_{j,n}(s) = \{\tilde{h}_0(s)\}^n \tilde{p}_j(s)$, then equation (2.8) becomes

$$\tilde{\psi}_k(s) = \sum_{n=1}^{k} \sum_{j=1}^{n+1} \eta_{j,n,k} \tilde{b}_{j+1,n+1}(s), \quad k = 1, 2, \ldots. \tag{2.10}$$

Since $\tilde{h}_0(s)$ is the compound geometric LST given by equation (2.1), it immediately follows that $\{\tilde{h}_0(s)\}^n$ is the LST of a compound negative binomial distribution with representation

$$\{\tilde{h}_0(s)\}^n = \sum_{i=0}^{\infty} \binom{n+i-1}{i} \left(\frac{\theta}{1+\theta}\right)^n \left(\frac{1}{1+\theta}\right)^i \{\tilde{p}_1(s)\}^i = \sum_{i=0}^{\infty} q_{i,n} \{\tilde{p}_1(s)\}^i. \tag{2.11}$$

Since $q_{0,n} = \{\theta/(1+\theta)\}^n$, however, it is possible to reexpress equation (2.11) as

$$\{\tilde{h}_0(s)\}^n = \left(\frac{\theta}{1+\theta}\right)^n + \tilde{r}_n(s), \tag{2.12}$$

where $\tilde{r}_n(s) = \Sigma_{i=1}^{\infty} q_{i,n} \{\tilde{p}_1(s)\}^i = \int_0^\infty e^{-sx} r_n(x)\, dx$, and $r_n(x)$ is the corresponding (defective) compound negative binomial pdf. Therefore, inverting both sides of equation (2.10) yields

$$\psi_k(u) = \sum_{n=1}^{k} \sum_{j=1}^{n+1} \eta_{j,n,k} b_{j+1,n+1}(u), \quad k = 1, 2, \ldots, \tag{2.13}$$

where

$$b_{j+1,n+1}(u) = \left(\frac{\theta}{1+\theta}\right)^{n+1} p_{j+1}(u) + \int_0^u r_{n+1}(u-x) p_{j+1}(x)\, dx \tag{2.14}$$

is the convolution pdf associated with the LST $\tilde{b}_{j+1,n+1}(s)$.

In general, it is quite difficult to identify the distribution defined by equation (2.14). When individual claim sizes have a phase-type distribution, however, it is possible to identify this distribution in an explicit, tractable manner. As a result, we will be able to use equation (2.13) to compute moments of the time of ruin. This will be the focus of Section 4.

3. Computation of the Constants

To use equation (2.13) we must be able to compute the constants $\eta_{j,n,k}$ defined by equation (2.9). Note that several pieces are involved here, namely, the constants $\sigma_{j,n,k}$, β_k, and $\pi_{j,k+1}$. We begin with the following theorem, which specifies how the probabilities $\pi_{j,k+1}$ are determined.

Theorem 3.1

For $k = 1, 2, \ldots,$

$$\pi_{1,k+1} = \frac{\mu_2}{\mu_2 + 2\theta\mu_1\Sigma_{i=1}^{k} \pi_{i,k}\mu_{i+2}/\{(i+2)\mu_{i+1}\}} \tag{3.1}$$

and

$$\pi_{j,k+1} = \frac{2\theta\mu_1\pi_{j-1,k}\mu_{j+1}/\{(j+1)\mu_j\}}{\mu_2 + 2\theta\mu_1 \Sigma_{i=1}^{k} \pi_{i,k}\mu_{i+2}/\{(i+2)\mu_{i+1}\}}, \quad j = 2, 3, \ldots, k+1. \tag{3.2}$$

PROOF

Since $\tilde{h}_0(s)$ is of the form (2.1), it follows that $\pi_{j,k+1}$ may be computed recursively via the procedure in the proof of Corollary 4.1 of Willmot (2002), by setting $\phi = (1+\theta)^{-1}$ and $\tilde{f}(s) = \tilde{p}_1(s)$. In particular, beginning with $\pi_{1,1} = 1$, we therefore obtain

$$\pi_{1,k+1} = \frac{\mathrm{E}[X_{(1)}]}{\mathrm{E}[X_{(1)}] + \theta\Sigma_{i=1}^{k} \pi_{i,k} \int_0^\infty \bar{P}_{i+1}(t)\, dt} \tag{3.3}$$

and

$$\pi_{j,k+1} = \frac{\theta\pi_{j-1,k} \int_0^\infty \bar{P}_j(t)\, dt}{\mathrm{E}[X_{(1)}] + \theta \Sigma_{i=1}^{k} \pi_{i,k} \int_0^\infty \bar{P}_{i+1}(t)\, dt}, \quad j = 2, 3, \ldots, k+1. \tag{3.4}$$

Since $\int_0^\infty \bar{P}_i(t)\, dt = \mathrm{E}[X_{(i)}] = \mu_{i+1}/\{(i+1)\mu_i\}$ (e.g., Hesselager, Wang, and Willmot 1998, p. 129), however, equations (3.3) and (3.4) simplify further to yield equations (3.1) and (3.2), respectively. □

We next proceed to determine the constants β_k and $\sigma_{j,n,k}$, which arise in the formula for $\eta_{j,n,k}$. To do so, we begin by reconciling formulae (6.5) and (6.11) of Willmot (2002, pp. 338–39). Recalling that $m_k(0) = \tilde{\psi}_k(0)$ from equations (2.3) and (2.4), the result of this exercise leads to

$$\beta_k = m_k(0) \prod_{i=1}^{k} \tau_{i,i-1,k-i+1} \tag{3.5}$$

and

$$\sigma_{j,n,k} = m_k(0)\tau_{j,n,k-n} \prod_{i=1}^{n} \tau_{i,i-1,k-i+1}, \tag{3.6}$$

where $\{\tau_{j,n,r};\, j = 1, 2, \ldots, n+1\}$ is another discrete probability measure for each $r = 1, 2, \ldots$. These probabilities, in turn, arise out of Willmot's equation (6.9), namely,

$$\tilde{w}_{r,n}(s) = \tilde{h}_0(s) \left\{ \sum_{j=1}^{n} \tau_{j,n,r}\tilde{p}_{j+1}(s) + \tau_{n+1,n,r}\tilde{w}_{r-1,n+1}(s) \right\}, \tag{3.7}$$

where $\tilde{w}_{r,n}(s) = \int_0^\infty e^{-sx}\, dW_{r,n}(x)$ is the LST of the nth-order equilibrium random variable having df

$$W_{r,n}(x) = 1 - \bar{W}_{r,n}(x) = \frac{\int_0^x \bar{W}_{r,n-1}(t)\, dt}{\int_0^\infty \bar{W}_{r,n-1}(t)\, dt}, \quad n = 1, 2, \ldots, \tag{3.8}$$

with

$$\bar{W}_{r,0}(x) = \frac{\int_x^\infty \psi_r(t)\, dt}{\int_0^\infty \psi_r(t)\, dt}. \tag{3.9}$$

The constants $m_k(0)$ can be calculated using the recursive procedure defined by equation (2.5). Therefore, to employ equations (3.5) and (3.6), we must be able to compute the discrete probability measure $\{\tau_{j,n,r};\ j = 1, 2, \ldots, n + 1\}$ for each $r = 1, 2, \ldots$. The following Theorem concerns the determination of these probabilities.

Theorem 3.2

For $r = 1, 2, \ldots,$

$$\tau_{1,n,r} = \frac{\mu_2}{M_{r,n-1}}, \tag{3.10}$$

$$\tau_{j,n,r} = \frac{2\theta\mu_1\tau_{j-1,n-1,r}\mu_{j+1}/\{(j+1)\mu_j\}}{M_{r,n-1}}, \quad j = 2, 3, \ldots, n, \tag{3.11}$$

and

$$\tau_{n+1,n,r} = \frac{2\theta\mu_1\tau_{n,n-1,r}m_{r-1}(n+1)}{(n+1)m_{r-1}(n)M_{r,n-1}}, \tag{3.12}$$

where

$$M_{r,n-1} = \mu_2 + 2\theta\mu_1\sum_{i=1}^{n-1}\frac{\tau_{i,n-1,r}\mu_{i+2}}{(i+2)\mu_{i+1}} + \frac{2\theta\mu_1\tau_{n,n-1,r}m_{r-1}(n+1)}{(n+1)m_{r-1}(n)}. \tag{3.13}$$

PROOF

Since $\tilde{w}_{r,n}(s)$ exhibits the form (3.7), it follows that $\tau_{j,n,r}$ may be computed recursively via equations (4.3)–(4.6) given in the proof of Proposition 4.1 of Willmot (2002), by setting $\phi = (1+\theta)^{-1}$, $\tilde{f}(s) = \tilde{p}_1(s)$, and $\tilde{a}(s) = \tilde{w}_{r-1,1}(s)$. Starting with $\tau_{1,0,r} = 1$ for each $r = 1, 2, \ldots$, Willmot's equations (4.4)–(4.6) yield, after some algebraic manipulation, equations (3.10), (3.11), and

$$\tau_{n+1,n,r} = \frac{2\theta\mu_1\tau_{n,n-1,r}\int_0^\infty \bar{W}_{r-1,n}(t)\,dt}{M_{r,n-1}}, \tag{3.14}$$

where $M_{r,n-1}$ is obtained via Willmot's equation (4.3), namely,

$$M_{r,n-1} = \mu_2 + 2\theta\mu_1\sum_{i=1}^{n-1}\frac{\tau_{i,n-1,r}\mu_{i+2}}{(i+2)\mu_{i+1}} + 2\theta\mu_1\tau_{n,n-1,r}\int_0^\infty \bar{W}_{r-1,n}(t)\,dt. \tag{3.15}$$

If we define $E_{r,n} = \int_0^\infty \bar{W}_{r,n}(t)\,dt$, it only remains to show how to compute $E_{r,n}$. First, it follows from relation (3.8) that

$$E_{r,n} = \frac{\int_0^\infty\int_t^\infty \bar{W}_{r,n-1}(x)\,dx\,dt}{E_{r,n-1}}.$$

By interchanging the order of integration in the numerator, we obtain

$$E_{r,n} = \frac{\int_0^\infty\int_0^x \bar{W}_{r,n-1}(x)\,dt\,dx}{E_{r,n-1}} = \frac{\int_0^\infty x\bar{W}_{r,n-1}(x)\,dx}{E_{r,n-1}}. \tag{3.16}$$

If we now replace $\bar{W}_{r,n-1}(x)$ in the numerator of equation (3.16) by its definition via equation (3.8), interchange the order of integration, and then continue inductively in this fashion, it is a straightforward exercise to show that

$$E_{r,n} = \frac{\int_0^\infty x^n\bar{W}_{r,0}(x)\,dx}{n!\prod_{j=0}^{n-1}E_{r,j}}.$$

However, note that

$$E_{r,1} = \frac{\int_0^\infty x\bar{W}_{r,0}(x)\ dx}{E_{r,0}} = \frac{\int_0^\infty x\bar{W}_{r,0}(x)\ dx}{\int_0^\infty \bar{W}_{r,0}(x)\ dx},$$

$$E_{r,2} = \frac{\int_0^\infty x^2\bar{W}_{r,0}(x)\ dx}{2!E_{r,1}E_{r,0}} = \frac{\int_0^\infty x^2\bar{W}_{r,0}(x)\ dx}{2\int_0^\infty x\bar{W}_{r,0}(x)\ dx},$$

$$E_{r,3} = \frac{\int_0^\infty x^3\bar{W}_{r,0}(x)\ dx}{3!E_{r,2}E_{r,1}E_{r,0}} = \frac{\int_0^\infty x^3\bar{W}_{r,0}(x)\ dx}{3\int_0^\infty x^2\bar{W}_{r,0}(x)\ dx},$$

and continuing inductively, we get

$$E_{r,n} = \frac{\int_0^\infty x^n\bar{W}_{r,0}(x)\ dx}{n\int_0^\infty x^{n-1}\bar{W}_{r,0}(x)\ dx}. \tag{3.17}$$

Since $\bar{W}_{r,0}(x) = \int_x^\infty \psi_r(t)\ dt/\int_0^\infty \psi_r(t)\ dt$ from equation (3.9), it follows that equation (3.17) simplifies to give

$$E_{r,n} = \frac{\int_0^\infty \int_x^\infty x^n\psi_r(t)\ dt\ dx}{n\int_0^\infty \int_x^\infty x^{n-1}\psi_r(t)\ dt\ dx}.$$

By interchanging the order of integration in both numerator and denominator above, as well as making use of equation (2.4), we obtain

$$E_{r,n} = \frac{\int_0^\infty x^{n+1}\psi_r(x)\ dx}{(n+1)\int_0^\infty x^n\psi_r(x)\ dx} = \frac{m_r(n+1)}{(n+1)m_r(n)}.$$

With this result, equations (3.14) and (3.15) can therefore be reexpressed as equations (3.12) and (3.13) respectively. □

Again, we remark that the recursive procedure defined by equation (2.5) can be used to compute the coefficients $m_{r-1}(n+1)$ and $m_{r-1}(n)$, which appear in equations (3.12) and (3.13).

4. PHASE-TYPE CLAIM SIZES

In this section we consider the situation in which the individual claim sizes $\{X_1, X_2, \ldots\}$ have a phase-type distribution. Phase-type distributions are one of the most general classes of distributions permitting a Markovian interpretation and include combinations and mixtures of exponential and Erlang distributions as special cases. Moreover, phase-type distributions are dense in the set of all distributions, so that, in principle, one can replace any non-phase-type distribution by a suitable phase-type approximation.

Here we assume that X has a *continuous* phase-type distribution with representation $[\boldsymbol{\nu}, S]$ of dimension m, denoted by $X \sim PH_m[\boldsymbol{\nu}, S]$. We note that S is an $m \times m$ transition rate matrix, and $\boldsymbol{\nu}$ is a $1 \times m$ row vector of probabilities satisfying $\boldsymbol{\nu}e_m = 1$, where e_m denotes an $m \times 1$ column vector of ones. With this representation, X has df of the form

$$P(x) = 1 - \boldsymbol{\nu}\exp\{xS\}e_m, \quad x \geq 0,$$

and pdf of the form

$$P'(x) = \boldsymbol{\nu}\exp\{xS\}\,\boldsymbol{S}^o, \quad x > 0,$$

where $\exp\{xS\} = \Sigma_{i=0}^\infty x^iS^i/i!$ and $\boldsymbol{S}^o = -Se_m$. All moments of X are finite with the nth moment given by $\mu_n = (-1)^n n!\boldsymbol{\nu}S^{-n}e_m$. We refer the reader to Neuts (1981), Latouche and Ramaswami (1999), Rolski et al. (1999), and Asmussen (2000) for a detailed treatment of phase-type distributions.

To use equation (2.13) to compute moments of the time of ruin, we must be able to identify the distribution defined by the pdf (2.14), or equivalently, the convolution LST that appears in equation (2.10), namely,

$$\tilde{b}_{j+1,n+1}(s) = \{\tilde{h}_0(s)\}^{n+1}\tilde{p}_{j+1}(s) = \{\tilde{h}_0(s)\}^{n}\{\tilde{h}_0(s)\tilde{p}_{j+1}(s)\}. \tag{4.1}$$

In equation (4.1) we note that $\{\tilde{h}_0(s)\}^n$ is the compound negative binomial LST given by equation (2.11). However, it is well known (e.g., Neuts 1981) that the negative binomial distribution defined by $q_{i,n}$ in equation (2.11) belongs to the *discrete* phase-type family of distributions, satisfying $q_{i,n} = \boldsymbol{\alpha}A^{i-1}\boldsymbol{a}$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is the $1 \times n$ row vector with

$$\alpha_i = \binom{n}{i-1}\left(\frac{\theta}{1+\theta}\right)^{i-1}\left(\frac{1}{1+\theta}\right)^{n-i+1}, \quad i = 1, 2, \ldots, n,$$

A is the $n \times n$ transition probability matrix given by

$$A = \begin{bmatrix} \frac{1}{1+\theta} & \frac{\theta}{1+\theta} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{1+\theta} & \frac{\theta}{1+\theta} & \ddots & 0 & 0 \\ 0 & 0 & \frac{1}{1+\theta} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \frac{1}{1+\theta} & \frac{\theta}{1+\theta} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{1+\theta} \end{bmatrix},$$

and $\boldsymbol{a}$ is the $n \times 1$ column vector given by

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{\theta}{1+\theta} \end{bmatrix}.$$

Furthermore, since $X \sim PH_m[\boldsymbol{\nu}, S]$, we have that $X_{(1)} \sim PH_m[-\boldsymbol{\nu}S^{-1}/\mu_1, S]$ (e.g., Rolski et al. 1999, Lemma 8.3.1). Therefore, by Theorem 2.2.5 of Neuts (1981), the compound negative binomial distribution characterized by equation (2.11) can be represented as a continuous phase-type distribution of dimension mn, with representation $[\boldsymbol{\delta}, D]$, where

$$\boldsymbol{\delta} = (-\boldsymbol{\nu}S^{-1}/\mu_1) \otimes \boldsymbol{\alpha} \tag{4.2}$$

and

$$D = (S \otimes I_n) + [(-\boldsymbol{S}^o\boldsymbol{\nu}S^{-1}/\mu_1) \otimes A]. \tag{4.3}$$

In equations (4.2) and (4.3) I_n denotes an $n \times n$ identity matrix, and $\otimes$ represents the *Kronecker product* (see Asmussen 2000, pp. 346–47, for a detailed discussion of Kronecker products).

We next turn our attention to the quantity $\tilde{h}_0(s)\tilde{p}_{j+1}(s)$, which also appears in equation (4.1). Clearly $\tilde{h}_0(s)\tilde{p}_{j+1}(s)$ is the LST of the sum of L and $X_{(j+1)}$. For notational convenience, define the random variable $Z_{j+1} = X_{(j+1)} + L$ having df $G_{j+1}(z) = \Pr(Z_{j+1} \leq z)$. Since $X \sim PH_m[\boldsymbol{\nu}, S]$, it follows from Willmot, Drekic, and Cai (2005, Theorem 4.2) that $X_{(j+1)} \sim PH_m[\boldsymbol{\nu}_{j+1}, S]$, where $\boldsymbol{\nu}_{j+1} = (-1)^{j+1}(j+1)!\boldsymbol{\nu}S^{-(j+1)}/\mu_{j+1}$. Moreover, from Theorem 8.3.1 in Rolski et al. (1999), we have that $L \sim PH_m[\phi\boldsymbol{\nu}_1, S + \phi\boldsymbol{S}^o\boldsymbol{\nu}_1]$, where $\phi = (1+\theta)^{-1}$.

Since $X_{(j+1)}$ and L both have continuous phase-type distributions, it follows from Neuts (1981, Theorem 2.2.2) that the convolution of these distributions is again continuous phase-type distributed with representation $[(\boldsymbol{\nu}_{j+1}, \mathbf{0}_m), C]$ of dimension $2m$, where $\mathbf{0}_m$ denotes a $1 \times m$ row vector of zeros, $(\boldsymbol{\nu}_{j+1}, \mathbf{0}_m)$ denotes the concatenated row vector of dimension $2m$, and C is the $2m \times 2m$ matrix written in block form as

$$C = \begin{bmatrix} S & \phi \boldsymbol{S}^o \boldsymbol{\nu}_1 \\ \mathbf{0}_{m,m} & S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1 \end{bmatrix}. \tag{4.4}$$

In equation (4.4) $\mathbf{0}_{m,m}$ denotes an $m \times m$ matrix of zeros. Therefore, we have

$$G_{j+1}(z) = 1 - (\boldsymbol{\nu}_{j+1}, \mathbf{0}_m) \exp\{zC\}\mathbf{e}_{2m}, \quad z \geq 0. \tag{4.5}$$

We now show, however, that Z_{j+1} has a simpler phase-type representation than $[(\boldsymbol{\nu}_{j+1}, \mathbf{0}_m), C]$. In particular, it is not difficult to show by induction that

$$C^i = \begin{bmatrix} S^i & (S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)^i - S^i \\ \mathbf{0}_{m,m} & (S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)^i \end{bmatrix}, \quad i = 0, 1, \ldots .$$

Therefore, it immediately follows that

$$\begin{aligned} \exp\{zC\} &= \sum_{i=0}^{\infty} \frac{z^i}{i!} \begin{bmatrix} S^i & (S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)^i - S^i \\ \mathbf{0}_{m,m} & (S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)^i \end{bmatrix} \\ &= \begin{bmatrix} \exp\{zS\} & \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} - \exp\{zS\} \\ \mathbf{0}_{m,m} & \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} \end{bmatrix}. \end{aligned}$$

Substituting this result into equation (4.5) yields

$$\begin{aligned} G_{j+1}(z) &= 1 - (\boldsymbol{\nu}_{j+1}, \mathbf{0}_m) \begin{bmatrix} \exp\{zS\} & \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} - \exp\{zS\} \\ \mathbf{0}_{m,m} & \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} \end{bmatrix} e_{2m} \\ &= 1 - (\boldsymbol{\nu}_{j+1} \exp\{zS\}, \boldsymbol{\nu}_{j+1} \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} - \boldsymbol{\nu}_{j+1} \exp\{zS\}) \begin{bmatrix} e_m \\ e_m \end{bmatrix} \\ &= 1 - \boldsymbol{\nu}_{j+1} \exp\{z(S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1)\} e_m, \end{aligned}$$

which implies that $Z_{j+1} \sim PH_m[\boldsymbol{\nu}_{j+1}, S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1]$, a representation that is simpler and half the dimensionality of the original one.

Based on our above observations and the form of equation (4.1), $\tilde{b}_{j+1,n+1}(s)$ therefore can be viewed as the LST of the sum of a continuous $PH_{mn}[\boldsymbol{\delta}, D]$ random variable and an independent continuous $PH_m[\boldsymbol{\nu}_{j+1}, S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1]$ random variable. By Theorem 2.2.2 of Neuts (1981) again, the convolution of these distributions is continuous phase-type distributed with representation $[\boldsymbol{\zeta}_{j+1}, V]$ of dimension $m(n + 1)$, where $\boldsymbol{\zeta}_{j+1} = (\boldsymbol{\nu}_{j+1}, \mathbf{0}_{mn})$ and

$$V = \begin{bmatrix} S + \phi \boldsymbol{S}^o \boldsymbol{\nu}_1 & (1 - \phi) \boldsymbol{S}^o \boldsymbol{\delta} \\ \mathbf{0}_{mn,m} & D \end{bmatrix}.$$

If we further define

$$\boldsymbol{V}^o = -Ve_{m(n+1)} = \begin{bmatrix} (1 - \phi)^{n+1} \boldsymbol{S}^o \\ \boldsymbol{S}^o \otimes \boldsymbol{\alpha} \end{bmatrix},$$

it therefore follows that the pdf $b_{j+1,n+1}(u) = \boldsymbol{\zeta}_{j+1} \exp\{uV\} \boldsymbol{V}^o$, so that equation (2.13) finally becomes

$$\psi_k(u) = \sum_{n=1}^{k} \sum_{j=1}^{n+1} \eta_{j,n,k} (\boldsymbol{\zeta}_{j+1} \exp\{uV\} \boldsymbol{V}^o), \; k = 1, 2, \ldots . \tag{4.6}$$

5. NUMERICAL EXAMPLES

In this section we present Table 1, which displays the values (rounded to two decimal places) of the first four standardized moments of the time of ruin. These moment-based quantities, which are routinely used in statistical applications to describe the location, variability, skewness, and heavy-tailedness of a probability distribution (e.g., Stuart and Ord 1998), are as follows:

1. Mean: $\mathrm{E}[T_c|U_0 = u]$
2. Coefficient of variation: $\dfrac{\mathrm{E}[[T_c - \mathrm{E}[T_c|U_0 = u]]^2|U_0 = u]^{1/2}}{\mathrm{E}[T_c|U_0 = u]}$
3. Coefficient of skewness: $\dfrac{\mathrm{E}[[T_c - \mathrm{E}[T_c|U_0 = u]]^3|U_0 = u]}{\mathrm{E}[[T_c - \mathrm{E}[T_c|U_0 = u]]^2|U_0 = u]^{3/2}}$
4. Coefficient of kurtosis: $\dfrac{\mathrm{E}[[\mathrm{T}_c - \mathrm{E}[T_c|U_0 = u]]^4|U_0 = u]}{\mathrm{E}[[T_c - \mathrm{E}[T_c|U_0 = u]]^2|U_0 = u]^2}.$

In Table 1 the above quantities are denoted by *mean*, *cv*, *skew*, and *kurt*, respectively. Without loss of

Table 1
Standardized Moments of T_c

u	$\theta = 0.1$				$\theta = 0.25$				$\theta = 0.5$			
	mean	*cv*	*skew*	*kurt*	*mean*	*cv*	*skew*	*kurt*	*mean*	*cv*	*skew*	*kurt*
Exponential Claim Size Distribution												
0	10.00	4.58	13.74	317.57	4.00	3.00	8.96	137.00	2.00	2.24	6.62	76.20
10	100.91	1.47	4.24	32.92	36.00	1.05	2.86	16.61	15.33	0.87	2.23	11.25
20	191.82	1.07	3.07	18.70	68.00	0.76	2.08	10.17	28.67	0.64	1.63	7.37
30	282.73	0.88	2.53	13.65	100.00	0.63	1.71	7.87	42.00	0.53	1.34	5.97
40	373.64	0.77	2.20	11.05	132.00	0.55	1.49	6.68	55.33	0.46	1.17	5.25
50	464.55	0.69	1.97	9.48	164.00	0.49	1.33	5.96	68.67	0.42	1.05	4.81
Erlang-6 Claim Size Distribution												
0	5.83	4.53	13.74	317.66	2.33	2.92	8.97	137.18	1.17	2.13	6.63	76.45
10	97.37	1.13	3.29	21.04	35.54	0.79	2.19	11.00	15.67	0.63	1.68	7.67
20	190.44	0.81	2.35	12.22	69.43	0.56	1.57	7.09	30.57	0.45	1.20	5.39
30	283.51	0.66	1.93	9.19	103.33	0.46	1.29	5.75	45.46	0.37	0.98	4.61
40	376.58	0.58	1.67	7.66	137.23	0.40	1.12	5.07	60.36	0.32	0.85	4.21
50	469.65	0.52	1.50	6.74	171.12	0.36	1.00	4.66	75.26	0.29	0.76	3.97
Combination of Two Exponentials Claim Size Distribution												
0	8.61	4.58	13.74	317.59	3.44	2.99	8.96	137.04	1.72	2.22	6.62	76.26
10	99.51	1.38	3.96	29.10	35.53	0.98	2.67	14.85	15.14	0.81	2.08	10.17
20	190.64	0.99	2.86	16.60	67.71	0.71	1.93	9.19	28.60	0.60	1.51	6.76
30	281.77	0.82	2.35	12.20	99.89	0.58	1.59	7.19	42.07	0.49	1.24	5.55
40	372.90	0.71	2.04	9.95	132.07	0.51	1.38	6.17	55.54	0.43	1.08	4.93
50	464.04	0.64	1.83	8.58	164.25	0.46	1.24	5.55	69.00	0.38	0.97	4.55
Mixture of Two Erlangs Claim Size Distribution												
0	21.11	4.59	13.73	317.40	8.44	3.01	8.95	136.62	4.22	2.25	6.58	75.55
10	112.28	2.03	5.85	60.00	40.37	1.44	3.95	28.91	17.48	1.18	3.07	18.62
20	202.95	1.51	4.34	34.43	72.29	1.08	2.94	17.36	30.87	0.90	2.30	11.74
30	293.62	1.26	3.61	24.69	104.21	0.90	2.44	12.93	44.26	0.75	1.92	9.07
40	384.30	1.10	3.15	19.56	136.13	0.79	2.14	10.59	57.65	0.66	1.68	7.65
50	474.97	0.99	2.84	16.40	168.05	0.71	1.92	9.15	71.04	0.59	1.51	6.76
Mixture of Three Exponentials Claim Size Distribution												
0	215.99	4.95	13.79	319.35	86.39	3.54	9.12	140.50	43.20	2.92	6.94	81.47
10	525.58	3.13	8.95	136.67	209.01	2.21	5.98	62.60	103.94	1.80	4.55	37.45
20	704.10	2.69	7.78	103.81	277.52	1.89	5.24	48.91	136.58	1.54	4.04	30.20
30	831.40	2.47	7.16	88.51	321.88	1.75	4.88	42.78	155.07	1.44	3.81	27.22
40	932.69	2.34	6.75	79.02	354.09	1.68	4.65	39.00	166.91	1.39	3.67	25.42
50	1021.35	2.24	6.44	72.11	380.79	1.63	4.47	36.22	176.08	1.36	3.56	24.07

generality, all examples were carried out with $\lambda = \mu_1 = 1$ so that we might examine the impact of the remaining parameters (namely, θ and u) on the distribution of T_c. The specific phase-type claim size distributions we chose to examine were the following:

Example 1

Exponential

$$m = 1, \qquad \boldsymbol{\nu} = (1), \qquad S = [-1],$$
$$P(x) = 1 - e^{-x}, \quad x \geq 0.$$

Example 2

Erlang-6

$$m = 6, \qquad \boldsymbol{\nu} = (1, 0, 0, 0, 0, 0), \qquad S = \begin{bmatrix} -6 & 6 & 0 & 0 & 0 & 0 \\ 0 & -6 & 6 & 0 & 0 & 0 \\ 0 & 0 & -6 & 6 & 0 & 0 \\ 0 & 0 & 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix},$$
$$P(x) = 1 - e^{-6x} \sum_{j=0}^{5} \frac{(6x)^j}{j!}, \quad x \geq 0.$$

Example 3

Combination of two exponentials

$$m = 2, \qquad \boldsymbol{\nu} = (1, 0), \qquad S = \begin{bmatrix} -6 & 6 \\ 0 & -6/5 \end{bmatrix},$$
$$P(x) = 1 + \frac{1}{4} e^{-6x} - \frac{5}{4} e^{-6x/5}, \quad x \geq 0.$$

Example 4

Mixture of two Erlangs

$$m = 4, \qquad \boldsymbol{\nu} = (1/4, 0, 3/4, 0), \qquad S = \begin{bmatrix} -3/5 & 3/5 & 0 & 0 \\ 0 & -3/5 & 0 & 0 \\ 0 & 0 & -9 & 9 \\ 0 & 0 & 0 & -9 \end{bmatrix},$$
$$P(x) = 1 - \frac{3}{4} e^{-9x} - \frac{1}{4} e^{-3x/5} - \frac{27}{4} x e^{-9x} - \frac{3}{20} x e^{-3x/5}, \quad x \geq 0.$$

Example 5

Mixture of three exponentials

$$m = 3, \qquad \boldsymbol{\nu} = (0.003979, 0.1078392, 0.8881815),$$
$$S = \begin{bmatrix} -0.014631 & 0 & 0 \\ 0 & -0.190206 & 0 \\ 0 & 0 & -5.514588 \end{bmatrix},$$
$$P(x) = 1 - 0.003979 e^{-0.014631x} - 0.1078392 e^{-0.190206x} - 0.8881815 e^{-5.514588x}, \quad x \geq 0.$$

We note that the shapes of these five claim size distributions are quite varied. While they all have mean equal to 1, they possess different amounts of variability. In particular, if we view the claim size distribution in Example 1 (having coefficient of variation equal to 1) as a benchmark, the coefficients of variation corresponding to the claim size distributions in Examples 2–5 are (rounded to two decimal places) 0.41, 0.85, 1.80, and 6.50, respectively. Clearly the Erlang-6 distribution is the least variable, while the mixture of three exponentials distribution is the most variable. This is not surprising, however, as the claim size distribution in Example 5 (first introduced by Wikstad 1971) is extremely skewed.

Table 1 was generated by implementing equation (4.6), in conjunction with the pertinent formulae in Sections 3 and 4, into the computational software package Mathematica. In particular, Figure 1 presents the exact Mathematica code employed to calculate $E[T_c^k | U_0 = u]$ for the mixture of two Er-

Figure 1

Mathematica Program Used to Calculate Moments of T_c for Mixture of Two Erlangs Claim Size Example

Enter the PH Parameters m, v, and S of the Claim Size Distribution

m=4;

v={{1/4,0,3/4,0}};

S={{−3/5,3/5,0,0},{0,−3/5,0,0},{0,0,−9,9},{0,0,0,−9}};

S0={{0},{3/5},{0},{9}};

$e_{r\_Integer}$:=Table[1,{i,1,r},{j,1,1}]

$\mu_{r\_Integer}$:=$(-1)^r$r!(v.Inverse[MatrixPower[S,r]].e_m)[[1,1]]

Computation of the Constants

$\pi_{1,1} = 1;$

$$\pi_{1,k\_Integer} := \pi_{1,k} = \frac{\mu_2}{\mu_2 + 2\theta\mu_1 \sum_{j=1}^{k-1} \frac{\pi_{j,k-1}\ \mu_{j+2}}{(j+2)\ \mu_{j+1}}}$$

$$\pi_{j\_Integer,k\_Integer} := \pi_{j,k} = \frac{2\theta\mu_1\ \pi_{j-1,k-1}\mu_{j+1}}{(j+1)\ \mu_j \left(\mu_2 + 2\theta\mu_1 \sum_{i=1}^{k-1} \frac{\pi_{i,k-1}\mu_{i+2}}{(i+2)\ \mu_{i+1}}\right)}$$

$$m_0[n\_\text{Integer}] := m_0[n] = \frac{EL_{n+1}}{n+1}$$

$$m_{k\_Integer}[n\_\text{Integer}] := m_k[n] = \frac{k}{(n+1)\ \lambda\theta\mu_1} \sum_{j=0}^{n} \text{Binomial}[n+1, j+1]\ m_{k-1}[j+1]\ EL_{n-j}$$

$EL_0 = 1;$

$$EL_{r\_Integer} := EL_r = \frac{1}{(r+1)\ \theta\mu_1} \sum_{j=0}^{r-1} \text{Binomial}[r+1, j]\ EL_j\ \mu_{r-j+1}$$

$$\tau_{1,0,r\_Integer} := \tau_{1,0,r} = 1$$

$$\tau_{1,n\_Integer,r\_Integer} := \tau_{1,n,r} = \frac{\mu_2}{\mathcal{M}_{r,n-1}}$$

$$\tau_{j\_Integer,n\_Integer,r\_Integer} := \tau_{j,n,r} = \text{If}\left[j = n+1, \frac{2\theta\mu_1\ \tau_{n,n-1,r}m_{r-1}[n+1]}{(n+1)\ m_{r-1}[n]\ \mathcal{M}_{r,n-1}}, \frac{2\theta\mu_1\ \tau_{j-1,n-1,r}\mu_{j+1}}{(j+1)\ \mu_j\mathcal{M}_{r,n-1}}\right]$$

$$\mathcal{M}_{r\_Integer,n\_Integer} := \mathcal{M}_{r,n} = \mu_2 + 2\theta\mu_1 \sum_{i=1}^{n} \frac{\tau_{i,n,r}\mu_{i+2}}{(i+2)\ \mu_{i+1}} + \frac{2\theta\mu_1\ \tau_{n+1,n,r}m_{r-1}[n+2]}{(n+2)\ m_{r-1}[n+1]}$$

$$\beta_{k\_Integer} := \beta_k = m_k[0] \prod_{i=1}^{k} \tau_{i,i-1,k-i+1}$$

$$\sigma_{j\_Integer,n\_Integer,k\_Integer} := \sigma_{j,n,k} = m_k[0]\ \tau_{j,n,k-n} \prod_{i=1}^{n} \tau_{i,i-1,k-i+1}$$

$\eta_{j\_Integer,n\_Integer,k\_Integer} :=$

$$\eta_{j,n,k} = \text{Which}\ [(n < k)\ \&\&\ (j < n+1), \sigma_{j,n,k}, (n < k)\ \&\&\ (j = n+1), 0, (n == k)\ \&\&\ (j \le n+1), \beta_k\ \pi_{j,k+1}]$$

Figure 1
(continued)

Phase-Type Calculations

temp1[x_,y_]:=If[Mod[x,y]=0,Floor[x/y],Floor[x/y]+1]

temp2[x_,y_]:=If[Mod[x,y]=0,y,Mod[x,y]]

$\alpha_{\text{n\_Integer}} := \alpha_n = \text{Table[Binomial[n, j} - 1] \left(\frac{\theta}{1+\theta}\right)^{j-1} \left(\frac{1}{1+\theta}\right)^{n-j+1}, \{i, 1, 1\}, \{j, 1, n\}]$

$A_{\text{n\_Integer}} := A_n = \text{Table[Switch}\left[i - j, -1, \frac{\theta}{1+\theta}, 0, \frac{1}{1+\theta}, 1, 0, \_, 0\right], \{i, n\}, \{j, n\}]$

$A0_{\text{n\_Integer}} := A0_n = \text{Table[If}\left[i = n, \frac{\theta}{1+\theta}, 0\right], \{i, 1, n\}, \{j, 1, 1\}]$

$v_{\text{r\_Integer}} := \frac{(-1)^r \text{ r!v.MatrixPower[Inverse[S],r]}}{\mu_r}$

$B = S + \frac{1}{1+\theta} \text{ S0.}v_1;$

$D_{\text{n\_Integer}} :=$
$D_n =$
$\text{Table[(Outer[Times, S, IdentityMatrix[n]] + Outer[Times, S0.}v_1, A_n])[[\text{temp1[i, n], temp1[j, n], temp2[i, n], temp2[j, n]}]], \{i, 1, mn\}, \{j, 1, mn\}]$

$\delta_{\text{n\_Integer}} := \delta_n = \text{Table[Outer[Times, }v_1, \alpha_n][[\text{temp1[i, n], temp1[j, n], temp2[i, n], temp2[j, n]}]], \{i, 1, 1\}, \{j, 1, mn\}]$

$\xi_{\text{n\_Integer,r\_Integer}} := \xi_{n,r} = \text{Table[If[}j \leq m, v_r[[1, j]], 0], \{i, 1, 1\}, \{j, 1, m\ (n + 1)\}]$

$\nu_{\text{n\_Integer}} :=$
$\nu_n = \text{Table[Which[}(i \leq m) \ \&\&\ (j \leq m), B[[i, j]], (i > m) \ \&\&\ (j > m), D_n[[i - m, j - m]], (i \leq m) \ \&\&\ (j > m),$
$\frac{\theta}{1+\theta} (\text{S0.}\delta_n) [[i, j - m]], \text{True}, 0], \{i, m(n + 1)\}, \{j, m\ (n + 1)\}]$

$\nu 0_{\text{n\_Integer}} :=$
$\nu 0_n = \text{Table[Flatten[Append[Flatten}\left[\left(\frac{\theta}{1+\theta}\right)^{n+1} \text{S0}\right], \text{Flatten[Outer[Times, S0, } A0_n]]]][[i]], \{i, 1, m\ (n + 1)\},$
$\{j, 1, 1\}]$

$\text{dens}_{\text{n\_Integer,r\_Integer}}[u\_] := \text{dens}_{n,r}[u] = (\xi_{n,r}.\text{MatrixExp}[\nu_n\ u]\ .\nu 0_n)[[1, 1]]$

Desired Quantity

$\text{DefMom}_{\text{k\_Integer}}[u\_] := \text{DefMom}_k[u] = \sum_{n=1}^{k} \sum_{j=1}^{n+1} \eta_{j,n,k} \text{dens}_{n,j+1}[u]$

$\text{RuinProb}[u\_] := \left(\frac{1}{1+\theta} v_1.\text{MatrixExp[Bu].}e_m\right)[[1, 1]]$

$\text{ProMom}_0[u\_] := 1$

$\text{ProMom}_{\text{k\_Integer}}[u\_] := \text{ProMom}_k[u] = \text{DefMom}_k[u]/\text{RuinProb}[u]$

langs claim size distribution (Example 4). For each of the above examples, we computed the first four standardized moments for several combinations of θ and u. Specifically, we considered $\theta = 0.1, 0.25, 0.5$ and $u = 0, 10, 20, 30, 40, 50$. For any given combination, our Mathematica program was able to compute the results in mere seconds.

Table 1 reveals that the distribution of T_c is positively skewed, particularly for smaller values of u. This characteristic is consistent with the numerical examples contained in Cardoso and Egídio dos Reis (2002), Dickson and Waters (2002), Drekic and Willmot (2003), and Drekic, Stafford, and Willmot (2004). However, note that as θ or u increases, Table 1 suggests that the distribution of T_c becomes less and less skewed. In fact, the values of *skew* seem to be approaching 0. This observation is also in agreement with previous findings. In particular, Segerdahl (1955) has shown that the distribution of T_c becomes normal as $u \to \infty$.

It is worthwhile to note that T_c in Example 5 is clearly the most variable. Moreover, for any given combination of θ and u, we observe that the ranking of examples based on their values of *mean* or *cv* is identical to the ranking of individual claim size distributions based simply on their coefficient of variation values. This seems to suggest that the variability of the underlying claim size distribution plays a major role in the shape of the distribution of T_c.

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