Lundberg-Type Bounds for the Joint Distribution of Surplus Immediately Before and at Ruin Under the Sparre Andersen Model

Andrew C. Y. Ng* and Hailiang Yang†

Abstract

In this paper we consider the Sparre Andersen insurance risk model. Three cases are discussed: the ordinary renewal risk process, stationary renewal risk process, and *s*-delayed renewal risk process. In the first part of the paper we study the joint distribution of surplus immediately before and at ruin under the renewal insurance risk model. By constructing an exponential martingale, we obtain Lundberg-type upper bounds for the joint distribution. Consequently we obtain bounds for the distribution of the deficit at ruin and ruin probability. In the second part of the paper, we consider the special case of phase-type claims and rederive the closed-form expression for the distribution of the severity of ruin, obtained by Drekic et al. (2003, 2004). Finally, we present some numerical results to illustrate the tightness of the bounds obtained in this paper.

1. Introduction

Risk theory is one of the most important research areas in actuarial science. The Lundberg inequality provides an upper bound for the ruin probability under the classical insurance risk model and is one of the most celebrated results in risk theory. When considering a ruin model, the quantity of interest is the amount of surplus. To describe the dynamics of surplus, we need to model the claim payments, premiums collected, investment income, and expenses, along with any other items that impact the cash flow of the insurance company. Ruin probability has been investigated extensively under the classical insurance risk model and various extended models. Nowadays preferred methods in ruin theory are renewal theory and the martingale method. The former emerged from the work of Feller (1968, 1971), and the latter came from that of Gerber (1973, 1979).

Recently actuarial researchers have also started paying attention to the severity of ruin. The joint distribution of the surplus immediately before and at ruin has been investigated by many authors. Gerber, Goovaerts, and Kaas (1987) considered the distribution of the severity of ruin for the classical compound Poisson risk model. In their paper they obtained an integral equation satisfied by the distribution of the severity of ruin. In the case of claim sizes following a mixture of exponential distributions or a mixture of Gamma distributions, closed-form solutions were obtained. Later Dufresne and Gerber (1988) investigated the distribution of the surplus immediately before ruin in the compound Poisson model. Results similar to those in Gerber, Goovaerts, and Kaas (1987) were obtained in that paper. Dickson (1992) used a different method to deal with the distribution of surplus immediately

* Andrew C. Y. Ng is a postgraduate student in the Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong, h9916610@graduate.hku.hk.

† Hailiang Yang, ASA, PhD, is an Associate Professor in the Department of Statistics and Actuarial Science, University of Hong Kong, Pokfulam Road, Hong Kong, hlyang@hkusua.hku.hk.

prior to ruin. Dickson and Dos Reis (1994) extended the method of Dickson (1992) by using dual events to explain the relationship between the density of the surplus immediately prior to ruin, and the joint density of the surplus immediately prior to ruin and the severity of ruin. Gerber and Shiu (1997, 1998) examined the joint distribution of the time of ruin, the surplus immediately before ruin, and the deficit at ruin. They showed that the expected discounted penalty, considered as a function of the initial surplus, satisfies a renewal equation. They obtained explicit solutions for some special cases. Yang and Zhang (2001) considered the joint distribution of surplus immediately before and at ruin under the classical model with constant interest force.

For mathematical simplicity the classical model in risk theory is idealized. However, actuaries have made some efforts to extend the classical model to more general and realistic models. Recently the joint distribution of surplus immediately before and at ruin has been studied for more general insurance risk models. Cheng and Tang (2003) considered this problem for the Erlang(2) risk process; see also Gerber and Shiu (2003), Li (2003), and Lin (2003) for interesting discussions. Tsai and Sun (2004) extended the results in Cheng and Tang (2003) for the Erlang(2) risk process to the case when there is a positive discount factor. In the special cases of phase-type claims, Drekic et al. (2003, 2004) obtained a closed-form expression for the distribution of severity of ruin. Dickson and Drekic (2004) obtained expressions for the joint distribution of surplus immediately before and at ruin when the time elapsed between successive claims is phase-type(2) distributed and when claims are phase-type and time elapsed between successive claims is generalized Erlang(n) distributed. Gerber and Shiu (2005) studied the expected discounted penalty at ruin in a Sparre Andersen model and obtained an integro-differential equation; they gave the solution of this equation in terms of Laplace transforms. They obtained a closed-form expression for the discounted joint distribution of surplus immediately before and at ruin when the initial surplus is zero. Some other related recent papers are Li and Garrido (2004), Willmot (2004), and Willmot and Dickson (2003).

In most of the above-mentioned papers the analyses were based on a renewal equation. In this paper we use the method of Asmussen (2000) to construct an exponential martingale. We obtain Lundberg-type bounds for the joint distribution of surplus immediately before and at ruin. To the best of our knowledge, the upper bounds for the joint distribution of surplus process immediately before and at ruin under the renewal insurance risk model have not been discussed in the literature. The purpose of this paper is twofold: we provide new results on upper bounds for the joint distributions and demonstrate the power of the martingale approach. We also present an alternative derivation of the distribution of the severity of ruin first obtained by Drekic et al. (2003, 2004). Our derivation is very simple and short. Finally, we perform numerical studies to examine the tightness of the upper bounds. The numerical results show that our bounds are reasonable. The rest of the paper is organized as follows: we first describe the Sparre Andersen model in Section 2; then we construct a martingale and illustrate the idea of change of probability measure in Section 3. Section 4 deals with the main results of the paper. We first obtain the upper bounds for the joint distribution of surplus immediately before and at ruin; then, as corollaries, we provide bounds for the distribution of the deficit at ruin and bounds for the ruin probability. In Section 5, for the special case of phase-type claims, we give an alternative derivation of the distribution of the severity of ruin first obtained by Drekic et al. (2003, 2004). Section 6 provides some numerical studies. Finally, the Appendix provides proofs for some of the results stated in Section 3.

2. THE SPARRE ANDERSEN RISK MODEL

Consider the Sparre Andersen risk model with surplus process $\{U(t)\}_{t\geq 0}$ defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

and aggregate loss process $\{W(t)\}_{t\geq 0}$ defined by

$$W(t) = u - U(t) = \sum_{i=1}^{N(t)} X_i - ct.$$

Here $u = U(0) > 0$ is the initial surplus, and $c > 0$ is the constant premium rate. $\{N(t)\}$ is a renewal process, and X_i's are independent and identically distributed claim random variables with distribution B, moment-generating function $\hat{B}$, and mean μ_B. $\{N(t)\}$ and $\{X_i\}$ are independent. Let T_k denote the time when the kth claim occurs; then $V_1 = T_1, V_2 = T_2 - T_1, V_3 = T_3 - T_2, \ldots$ are independent random variables. $V_2, V_3, \ldots$ are the interclaim times, that is, the time elapsed between successive claims, and they have common distribution A, moment-generating function $\hat{A}$, and mean μ_A.

In the case of an ordinary renewal risk process, V_1 follows distribution A. But unlike the case for the compound Poisson risk model, the counting process of an ordinary renewal risk process does not have the stationary increment property unless V_1 is exponential. To make sure that the distribution of the number of claims in any time interval depends only on the length of the time interval, we have to let V_1 follow distribution A_0, which is the equilibrium distribution of A, that is, $A_0(x) = 1/\mu_A \int_0^x (1 - A(v))\, dv$. When V_1 follows distribution A_0, we have a stationary renewal risk process. Here we also consider the s-delayed renewal risk process with $V_1 = s$ deterministically to simplify some proofs. The ruin probabilities in these three cases are denoted by $\psi(u)$, $\psi^{(0)}(u)$, and $\psi_s(u)$, respectively, and we use such notations throughout this paper.

For these three cases,

$$\lim_{t\to\infty} \frac{U(t)}{t} = c - \lim_{t\to\infty} \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \frac{N(t)}{t} = c - \frac{\mu_B}{\mu_A}$$

with probability 1, and hence the condition of having a positive expected profit is

$$c - \frac{\mu_B}{\mu_A} > 0. \tag{2.1}$$

The reason behind equation (2.1) can be understood as follows: Since μ_A is the mean of the interclaim time random variable, $1/\mu_A$ is the expected claim frequency. Thus the expected net profit of the company per unit time after the first claim occurs is $c - \mu_B/\mu_A$. But the time when the first claim occurs has negligible effect in the long run, and thus $c - \mu_B/\mu_A$ has to be positive to make sure that the ruin is not a certain event.

For details on the Sparre Andersen risk model in general such as calculation of ruin probability, see, for example, Grandell (1991), Rolski et al. (1999), Asmussen (2000), and Willmot and Lin (2001) and references therein. For details on the Gerber-Shiu discounted penalty function of this model, see Willmot and Dickson (2003) and Gerber and Shiu (2005). For closed-form expressions of the severity of ruin and the joint distribution of surplus immediately before and at ruin in various special cases, see, for example, Cheng and Tang (2003), Tsai and Sun (2004), and Drekic et al. (2003, 2004).

3. Martingale and Change of Probability Measure

Gerber (1973, 1979) pioneered the use of martingales in ruin theory. Through combination with a change of probability measure, Lundberg bounds for the ruin probabilities for many insurance risk models, including the classical model, Sparre Andersen model, Markov-modulated risk model, and periodic risk model, can be obtained. In this and the next section we use the martingale approach to obtain Lundberg-type bounds for the joint distribution of surplus immediately before and at ruin and the distribution of the deficit at ruin in a manner similar to that used in ruin probabilities. Asmussen (2000) and Rolski et al. (1999) provided thorough discussion on likelihood ratios and change of probability measures. In this section we will discuss briefly how to use a martingale to construct a new probability measure and the use of the new probability measure in ruin theory. Readers who are interested in how the martingale is constructed can refer to the Appendix.

Let $1(G)$ be the indicator function of event G, which is equal to 1 if event G occurs and 0 otherwise. The following lemma plays a central role in the next section. Refer to chapter 2 of Asmussen (2000) for a proof.

Lemma 1

Let $\{M(t)\}_{t\geq 0}$ be a positive $\{\mathcal{F}_t\}$-martingale such that $\mathrm{E}[M(0)] = 1$. *Then* $\tilde{\mathrm{P}}$ *defined by* $\tilde{\mathrm{P}}(G) = \mathrm{E}[M(t)1(G)]$ *for all $G \in \mathcal{F}_t$ is a probability measure. Also, for any stopping time τ and event $G \in \mathcal{F}_\tau$, $G \subseteq \{\tau < \infty\}$, we have*

$$\mathrm{P}(G) = \tilde{\mathrm{E}}\left[\frac{1}{M(\tau)}1(G)\right].$$

Now suppose that we have constructed a positive martingale,

$$M(t) = e^{s[f(W(t))+g(J(t))]},$$

with $\mathrm{E}[M(0)] = 1$, based on the aggregate loss process $\{W(t)\}$, an auxiliary stochastic process $\{J(t)\}$, some functions f and g, and some real number s. Then for any event $G \in \mathcal{F}_T$ where $T = \inf\{t : W(t) > u\} = \inf\{t : U(t) < 0\}$ denotes the time of ruin with initial surplus u,

$$\mathrm{P}(G) = \tilde{\mathrm{E}}\left[\frac{1}{M(T)}1(G)\right] = \tilde{\mathrm{E}}\ [e^{-s[f(W(T))+g(J(T))]}1(G)].$$

By choosing a suitable s, one may be able to obtain a measure $\tilde{\mathrm{P}}$ such that ruin is certain, and then bound $\tilde{\mathrm{P}}(G)$ by finding the upper and lower bounds for the expectation by writing it as an integral. This, of course, requires the knowledge of the distribution of the counting process of the claim number and the claim sizes under the measure $\tilde{\mathrm{P}}$. This knowledge can be obtained by considering the moment-generating function or the Laplace transform under $\tilde{\mathrm{P}}$.

The reason to introduce the auxiliary process $\{J(t)\}$ is that it may be hard to construct a martingale as a function of $\{W(t)\}$ (or equivalently $\{U(t)\}$) only because $\{W(t)\}$ may not be Markov. For Markov processes, we can construct a martingale easily (see chapter 11 of Rolski et al. 1999, for example). If claims arrive according to a Poisson process, then the surplus process is Markov because the exponential distribution is memoryless and claims are independent. Using the martingale approach, many interesting results can be obtained; refer to Gerber and Shiu (1998) for a thorough discussion. But the surplus process of a renewal risk process is not Markov (unless $\{N(t)\}$ is a Poisson process) because the time since the last claim obviously provides information on the time until the next claim occurs. Hence, the most natural choice is to let $J(t) = T_{N(t)+1} - t$ be the time until the next claim occurs. Then the process $\{Y(t)\} = \{(J(t), W(t))\}$ will be Markov.

In the rest of the paper we will use P_s to denote the original probability measure when the surplus process $\{U(t)\}$ under consideration is s-delayed, that is, when $V_1 \equiv s$. The subscript in E_s carries the same meaning. With the assumption that the equation $\hat{B}(x)\hat{A}(-cx) = 1$ has a positive solution R, that is,

$$\hat{B}(R)\hat{A}(-cR) = 1, \tag{3.1}$$

the process $\{M(t)\}$ defined by

$$M(t) = e^{RW(t)-cR(J(t)-s)}$$

is a positive martingale with $\mathrm{E}_s[M(0)] = 1$. R is the adjustment coefficient of the s-delayed renewal risk process. Under the new probability measure in Lemma 1, we have the following results:

1. Ruin is certain.
2. The claims $X_1, X_2, \ldots$ and the interclaim times $V_2, V_3, \ldots$ are independent with distributions B_R and A_R, where

$$B_R(x) = \int_0^x \frac{e^{Rv}}{\hat{B}(R)}\, dB(v) \quad \text{and} \quad A_R(t) = \int_0^t \frac{e^{-cRs}}{\hat{A}(-cR)}\, dA(s). \tag{3.2}$$

(When the density function exists, the notation $dF(x)$ is the same as $f(x)\,dx$, where f is the derivative of F.)
3. The moment-generating function of A_R, denoted by $\hat{A}_R$, satisfies

$$\hat{A}_R(cR) = \hat{B}(R). \tag{3.3}$$

Readers interested in the proof of these properties can refer to the Appendix.

4. LUNDBERG BOUNDS FOR THE JOINT DISTRIBUTION OF SURPLUS IMMEDIATELY BEFORE AND AT RUIN

Now we derive exponential upper bounds for the joint distribution of surplus immediately before and at ruin. Let

$$a(R, v, y) = \frac{B(v + y) - B(v)}{\int_v^\infty e^{R(z-v)}\, dB(z)},$$

$$a_+(R, x, y) = \sup_{0\le v\le x} a(R, v, y) \quad \text{and} \quad a_-(R, x, y) = \inf_{0\le v\le x} a(R, v, y).$$

For the s-delayed case ($s > 0$), let

$$F_s(u, x, y) = \mathrm{P}_s(T < \infty,\, U(T-) \le x,\, |U(T)| \le y)$$

be the joint distribution of surplus immediately before and at ruin; then we have the following result.

Theorem 1

For $u \ge 0$, $s > 0$, we have

$$F_s(u, x, y) \le e^{-R(u+cs)}\, \hat{B}(R) a_+(R, x, y).$$

PROOF

Let $\mathrm{P}_{R;s}$ be the measure constructed from P_s by the martingale $\{M(t)\}$. By Lemma 1, we can recover the original probability measure P_s from $\mathrm{E}_{R;s}$, the expectation under $\mathrm{P}_{R;s}$. Noting that $W(T) = u + |U(T)|$ and that ruin is certain under $\mathrm{P}_{R;s}$,

$$\begin{aligned} F_s(u, x, y) &= \mathrm{E}_s[1(T < \infty, U(T-) \le x, |U(T)| \le y)] \\ &= \mathrm{E}_{R;s}\left[\frac{1}{M(T)} 1(U(T-) \le x, |U(T)| \le y)\right] \\ &= \mathrm{E}_{R;s}\left[\frac{e^{cR(J(T)-s)}}{e^{R(u+|U(T)|)}} 1(U(T-) \le x, |U(T)| \le y)\right] \\ &= e^{-R(u+cs)}\mathrm{E}_{R;s}\, [e^{-R|U(T)|}1(U(T-) \le x, |U(T)| \le y)]\hat{A}_R(cR), \end{aligned} \tag{4.1}$$

where the last equality follows from $J(T) = V_{L(u)+1}$, and $L(u)$ is the number of claims leading to ruin, which is independent of $U(T-)$ and $U(T)$.

Consider $T = t$ and $U(T-) = v$; the claim that causes ruin at time t has distribution

$$\frac{B_R(y) - B_R(v)}{1 - B_R(v)} \quad \text{for } y \geq v$$

under $P_{R;s}$. Hence

$$\begin{aligned}
&E_{R;s}\ [e^{-R|U(T)|}1(U(T-) \leq x,\ |U(T)| \leq y)|T = t,\ U(T-) = v] \\
&= 1(v \leq x) \int_v^{v+y} e^{-R(z-v)} \frac{dB_R(z)}{1 - B_R(v)} \\
&= 1(v \leq x)e^{Rv} \frac{\int_v^{v+y} e^{-Rz}e^{Rz}\, dB(z)}{\int_v^\infty e^{Rz}\, dB(z)} \\
&= 1(v \leq x)a(R, v, y).
\end{aligned}$$

Let $H(x, t) = P_{R;s}(T \leq t,\ U(T-) \leq x)$ be the joint distribution of the time of ruin and the surplus immediately prior to ruin. Then by

$$E_{R;s}\ [e^{-R|U(T)|}1(U(T-) \leq x,\ |U(T)| \leq y)] = \int_0^\infty \int_0^\infty 1(v \leq x)a(R, v, y)\ d^2H(v, t),$$

we have the following lower and upper bounds for the expectation:

$$\begin{aligned}
a_-(R, x, y)P_{R;s}(U(T-) \leq x) &\leq E_{R;s}\ [e^{-R|U(T)|}1(U(T-) \leq x,\ |U(T)| \leq y)] \\
&\leq a_+(R, x, y)P_{R;s}(U(T-) \leq x). \qquad (4.2)
\end{aligned}$$

Applying the second half of expression (4.2) to (4.1) and using equation (3.3), we obtain the result. □

For the ordinary renewal risk process and the stationary renewal risk process, let

$$F(u, x, y) = P(T < \infty,\ U(T-) \leq x,\ |U(T)| \leq y)$$

and

$$F^{(0)}(u, x, y) = P^{(0)}(T < \infty,\ U(T-) \leq x,\ |U(T)| \leq y)$$

be the joint distribution of surplus immediately before and at ruin. Then we have the following corollary.

Corollary 1

For $u \geq 0$, the exponential upper bounds for $F(u, x, y)$ and $F^{(0)}(u, x, y)$ are

$$F(u, x, y) \leq e^{-Ru}a_+(R, x, y)$$

and

$$F^{(0)}(u, x, y) \leq e^{-Ru} \frac{\hat{B}(R) - 1}{cR\mu_A}\, a_+(R, x, y).$$

PROOF

From Theorem 1 and by letting s vary according to the interclaim distribution A,

$$F(u, x, y) = \int_0^\infty F_s(u, x, y)\ dA(s) \leq e^{-Ru}\hat{B}(R)a_+(R, x, y) \cdot \int_0^\infty e^{-cRs}\ dA(s).$$

Writing the integral as $\hat{A}(-cR)$ and applying equation (3.1), the first result is obtained. Similarly, by letting s vary according to the stationary distribution of the interclaim distribution defined by $A_0(x) = \int_0^x [1 - A(v)]/\mu_A \, dv$,

$$F^{(0)}(u, x, y) = \int_0^\infty F_s(u, x, y) \, dA_0(s)$$

$$\leq e^{-Ru} \hat{B}(R) a_+(R, x, y) \cdot \int_0^\infty e^{-cRs} \frac{1 - A(s)}{\mu_A} \, ds.$$

With the integral being $[1 - \hat{A}(-cR)]/(cR\mu_A)$ and by applying equation (3.1), we obtain the second result. □

We can also obtain two-sided Lundberg bounds for the deficit at ruin and severity of ruin for the Sparre Andersen model in a similar manner. Let

$$b(R, v, y) = \frac{1 - B(v + y)}{\int_v^\infty e^{R(z-v)} \, B(dz)},$$

$$b_-(R, y) = \inf_{v \geq 0} b(R, v, y) \text{ and } b_+(R, y) = \sup_{v \geq 0} b(R, v, y).$$

Corollary 2

The two-sided Lundberg bounds for $F(u, y)$ and $F^{(0)}(u, y)$ defined as the distribution of the deficit at ruin for the ordinary and stationary renewal risk processes are

$$e^{-Ru} a_-(R, \infty, y) \leq F(u, y) \leq e^{-Ru} a_+(R, \infty, y)$$

and

$$e^{-Ru} \frac{\hat{B}(R) - 1}{cR\mu_A} a_-(R, \infty, y) \leq F^{(0)}(u, y) \leq e^{-Ru} \frac{\hat{B}(R) - 1}{cR\mu_A} a_+(R, \infty, y).$$

The two-sided Lundberg bounds for the severity of ruin for the ordinary and stationary renewal risk processes are

$$e^{-Ru} b_-(R, y) \leq \mathrm{P}(T < \infty, |U(T)| > y) \leq e^{-Ru} b_+(R, y)$$

and

$$e^{-Ru} \frac{\hat{B}(R) - 1}{cR\mu_A} b_-(R, y) \leq \mathrm{P}^{(0)}(T < \infty, |U(T)| > y) \leq e^{-Ru} \frac{\hat{B}(R) - 1}{cR\mu_A} b_+(R, y).$$

PROOF

We can obtain the first set of results by letting x in expression (4.2) tend to infinity and proceed as in Corollary 1. For the second set of results, consider the s-delayed renewal risk model,

$$\begin{aligned} &\mathrm{P}_s(T < \infty, |U(T)| > y) \\ &= e^{-R(u+cs)} \hat{B}(R) \mathrm{E}_{R;s}[e^{-R|U(T)|} 1(|U(T)| > y)] \\ &= e^{-R(u+cs)} \hat{B}(R) \int_0^\infty \int_0^\infty \int_{v+y}^\infty e^{-R(z-v)} \frac{dB_R(z)}{1 - B_R(v)} \, d^2H(v, t) \\ &= e^{-R(u+cs)} \hat{B}(R) \int_0^\infty \int_0^\infty \frac{1 - B(v + y)}{\int_v^\infty e^{R(z-v)} B\,(dz)} \, d^2H(v, t) \end{aligned}$$

and hence

$$e^{-R(u+cs)} \hat{B}(R) b_-(R, y) \leq \mathrm{P}_s(T < \infty, |U(T)| > y) \leq e^{-R(u+cs)} \hat{B}(R) b_+(R, y).$$

Using the same method as in the proof of Corollary 1, we obtain the second set of results. □

Many authors have obtained two-sided Lundberg bounds for the ruin probabilities for the Sparre Andersen model. See, for example, theorem 6.5.2 of Rolski et al. (1999) for the ordinary case and equation (11.4.12) of Willmot and Lin (2001) for the stationary case. Here we use the bounds obtained in Corollary 2 to obtain the results.

Corollary 3

Let $\psi_s(u)$, $\psi(u)$, and $\psi^{(0)}(u)$ be the ruin probabilities with initial surplus u for the s-delayed, ordinary, and stationary renewal risk processes; then

$$e^{-R(u+cs)}\hat{B}(R)b_-(R, 0) \le \psi_s(u) \le e^{-R(u+cs)}\hat{B}(R)b_+(R, 0),$$

$$e^{-Ru}b_-(R, 0) \le \psi(u) \le e^{-Ru}b_+(R, 0),$$

and

$$e^{-Ru}\,\frac{\hat{B}(R) - 1}{cR\mu_A}\,b_-(R, 0) \le \psi^{(0)}(u) \le e^{-Ru}\,\frac{\hat{B}(R) - 1}{cR\mu_A}\,b_+(R, 0).$$

Finally, we obtain the corresponding exponential upper bound and two-sided Lundberg bounds for the joint distribution of surplus and the ruin probability for the compound Poisson model with claim arrival rate λ.

Corollary 4

For the compound Poisson model with claim arrival rate λ, assume that the equation

$$\lambda[\hat{B}(x) - 1] = cx \tag{4.3}$$

has a positive solution R. (If R exists, it is unique.) Then

$$F(u, x, y) \le e^{-Ru}a_+(R, x, y)$$

and

$$e^{-Ru}b_-(R, 0) \le \psi(u) \le e^{-Ru}b_+(R, 0).$$

PROOF

Note that equation (3.1) becomes

$$\hat{B}(R)\,\frac{\lambda}{\lambda + cR} = 1;$$

hence R is the unique positive root of the equation $\hat{B}(x) = 1 + cx/\lambda$. □

Of course, R here is the same as the adjustment coefficient in the classical model, and the bounds for $\psi(u)$ coincide with the results in Asmussen (2000) and Rolski et al. (1999). For readers familiar with the Society of Actuaries textbook Bowers et al. (1997), equation (4.3) is the same as equation (13.4.2) of Bowers et al. It is easy to see that

$$b_+(R, 0) \le \sup_{v\ge 0} \frac{1 - B(v)}{\int_0^\infty dB(z)} = 1 - B(0) = 1.$$

Hence the inequality can be written as $\psi(u) \le e^{-Ru}$, which is equation (13.4.5) in Bowers et al.

5. CLOSED-FORM SOLUTION FOR THE SEVERITY OF RUIN WITH PHASE-TYPE CLAIMS

A positive random variable is of phase-type if it is the absorption time of a continuous-time Markov chain with finitely many states, one of which is absorbing and all others being transient. Neuts (1981) first used the class of phase-type distributions intensively in a queuing theory context. In ruin theory it is the only class of claim size distributions for which we can obtain explicit results for the ruin probability under the classical compound Poisson risk model. A phase-type distribution inherits their special structure from the Markov property of the underlying continuous-time Markov chain, and it is this unique property that makes many derivations possible. Another added advantage of phase-type distributions is that all formulae can be written in a compact matrix form by extensive use of matrix calculus and Kronecker products. Computations involved can thus be programmed easily using mathematical languages such as MATLAB and Mathematica.

Let $\{C(t)\}_{t\geq 0}$ be a continuous-time Markov chain with $d + 1$ states, one of which is absorbing. The state space of $\{C(t)\}$ is $\{1, 2, \ldots, d, 0\} = E \cup \{0\}$ and the initial distribution is $(\alpha_1, \alpha_2, \ldots, \alpha_d, 0) = (\boldsymbol{\alpha}, 0)$. The generator of $\{C(t)\}$ has the structure

$$\begin{bmatrix} \boldsymbol{T} & \boldsymbol{t} \\ \boldsymbol{0}' & 0 \end{bmatrix},$$

where the $d \times d$ matrix $\boldsymbol{T} = [t_{ij}]$ is a subintensity matrix, meaning that

1. $t_{ii} \leq 0$ for any $i \in E$
2. $t_{ij} \geq 0$ for any $i, j \in E$ with $i \neq j$
3. $\Sigma_{j\in E}\, t_{ij} \leq 0$ for any $i \in E$.

Since each row of a generator has to sum up to 0, we have $\boldsymbol{t} = (t_1, t_2, \ldots, t_d)' = -\boldsymbol{Te}$, where $\boldsymbol{e}$ is the $d \times 1$ column vector of 1. By property (3), $t_i = -\Sigma_{j\in E}\, t_{ij} \geq 0$. It is obvious that state 0 is the absorbing state. From this we can see that $\boldsymbol{t}$ is the exit rate vector as t_i represents the rate from state i to enter the absorbing state. We also assume that at least one of the t_i values is greater than zero since otherwise the chain may never visit state 0.

The time for the chain to enter the absorbing state

$$\zeta = \inf\{t \geq 0 : C(t) = 0\}$$

is said to follow phase-type distribution PH$(E, \boldsymbol{\alpha}, \boldsymbol{T})$. From the transition probability of a continuous-time Markov chain, we can see that the distribution function and the density function of a phase-type distribution are $F(x) = \mathrm{P}(\zeta \leq x) = 1 - \boldsymbol{\alpha} e^{x\boldsymbol{T}}\boldsymbol{e}$ and $f(x) = F'(x) = \boldsymbol{\alpha} e^{x\boldsymbol{T}}\boldsymbol{t}$. The mean of the distribution is $-\boldsymbol{\alpha}\boldsymbol{T}^{-1}\boldsymbol{e}$, which can be obtained by integrating the tail of the distribution function $1 - F(x)$. In addition, a phase-type distribution has two important characteristics. First, it is closed under mixture and convolution. Second, for any distribution F on the positive real line, a sequence of phase-type distributions exists that converges in distribution to F.

The usefulness of the class of phase-type distributions is that if a problem can be solved in a particular case of exponential distribution, then very often the problem also can be solved in the case of a phase-type distribution numerically, and finally the problem with a general distribution can be approximated by using a sequence of phase-type distributions that converges to the desired probability distribution. Many methods to find a good approximating sequence have been proposed. Refer to, for example, Schmickler (1992), using moment matching, and Asmussen, Nerman, and Olsson (1996), using an EM algorithm.

Drekic et al. (2004) and Asmussen (2000) provided comprehensive and detailed descriptions of the use of phase-type claims in ruin theory. Asmussen (2000) considered the calculation of ruin probabilities for the ordinary renewal risk model and stationary renewal risk model in the case of phase-type claims. Let $B \sim$ PH$(E, \boldsymbol{\alpha}, \boldsymbol{T})$. Define m_x as the phase state of the claim when $\{W(t)\}$ first upcrosses x for $x \geq 0$. Propositions VIII.4.1, 4.2, and 4.3 of Asmussen (2000) (with a scale change of time axis) stated that $\{m_x\}_{x\geq 0}$ is a terminating continuous-time Markov chain with generator $\boldsymbol{Q}$, where

$$\boldsymbol{Q} = \boldsymbol{T} + \boldsymbol{t}\boldsymbol{\alpha}_+$$

and $\boldsymbol{\alpha}_+$ satisfies the nonlinear matrix equation

$$\boldsymbol{\alpha}_+ = \boldsymbol{\alpha} \int_0^\infty e^{cy(\boldsymbol{T}+\boldsymbol{t\alpha}_+)}\, dA(y).$$

Moreover, when the claim arrival process is an ordinary renewal process, the initial distribution of m_x is $\boldsymbol{\alpha}_+$, and when the claim arrival process is a stationary renewal process, the initial distribution of m_x is

$$\boldsymbol{\alpha}^{(0)} = -\frac{1}{c\mu_A}\boldsymbol{\alpha T}^{-1}. \tag{5.1}$$

A method to compute $\boldsymbol{\alpha}_+$ was also given in proposition VIII.4.4 of Asmussen (2000), which stated that $\boldsymbol{\alpha}_+$ is the unique solution of the nonlinear matrix equation

$$\boldsymbol{\alpha}_+ = \varphi(\boldsymbol{\alpha}_+) \quad \text{where } \varphi(\boldsymbol{\alpha}_+) = \boldsymbol{\alpha} \int_0^\infty e^{cy(\boldsymbol{T}+\boldsymbol{t\alpha}_+)}\, dA(y), \tag{5.2}$$

and $\boldsymbol{\alpha}_+$ can be computed by iterating equation (5.2), that is, by

$$\boldsymbol{\alpha}_+^{(1)} = \mathbf{0}, \qquad \boldsymbol{\alpha}_+^{(n+1)} = \varphi(\boldsymbol{\alpha}_+^{(n)}), \quad n = 1, 2, \ldots .$$

The sequence $\{\boldsymbol{\alpha}_+^{(n)}\}$ converges to $\boldsymbol{\alpha}_+$.

Drekic et al. (2003, 2004) obtained the closed-form expression of $F(u, y)$ by first deriving the distribution of the deficit at ruin given that ruin occurs and then obtained $F^{(0)}(u, y)$. Here we adopt a more direct approach without finding the distribution of the deficit at ruin. We condition on the phase state of the claim when ruin occurs.

Proposition 1

The distributions of the severity of ruin in the ordinary and stationary renewal risk model with phase-type claims are

$$F(u, y) = \psi(u) - \boldsymbol{\alpha}_+ e^{u\boldsymbol{Q}} e^{y\boldsymbol{T}} \boldsymbol{e},$$

$$F^{(0)}(u, y) = \psi^{(0)}(u) - \boldsymbol{\alpha}^{(0)} e^{u\boldsymbol{Q}} e^{y\boldsymbol{T}} \boldsymbol{e}.$$

PROOF

Let $\psi(u, j) = \mathrm{P}(T < \infty, m_u = j)$ and $\psi^{(0)}(u, j) = \mathrm{P}^{(0)}(T < \infty, m_u = j)$. Then by the results above,

$$\psi(u, j) = \boldsymbol{\alpha}_+ e^{u\boldsymbol{Q}} \boldsymbol{e}_j \quad \text{and} \quad \psi^{(0)}(u, j) = \boldsymbol{\alpha}^{(0)} e^{u\boldsymbol{Q}} \boldsymbol{e}_j,$$

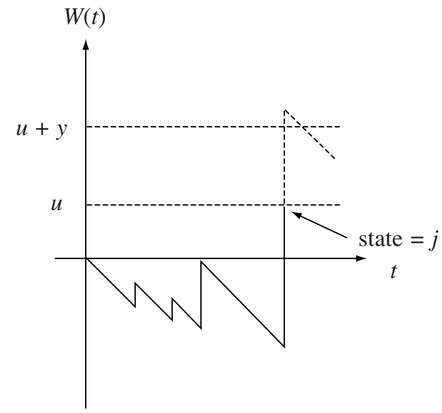
where $\boldsymbol{e}_i$ is the ith unit column vector. Summing over all j,

$$\psi(u) = \boldsymbol{\alpha}_+ e^{u\boldsymbol{Q}} \boldsymbol{e} \quad \text{and} \quad \psi^{(0)}(u) = \boldsymbol{\alpha}^{(0)} e^{u\boldsymbol{Q}} \boldsymbol{e}.$$

To find the distribution of the deficit at ruin, consider the event that ruin occurs, and the deficit is greater than y; that is, the overshoot of $W(t)$ above level u is greater than y. This is the same as the event that at time T, $W(T) > u + y$ (see Figure 1). To calculate the probability of this event, we further partition this event into E disjoint events by considering the phase state of the claim when $W(t)$ first upcrosses level u as follows:

$$\begin{aligned}
\mathrm{P}(T < \infty, |U(T)| > y) &= \mathrm{P}(T < \infty, W(T) > u + y) \\
&= \sum_{j\in E} \mathrm{P}(T < \infty, m_u = j, W(T) > u + y) \\
&= \sum_{j\in E} \psi(u, j)\mathrm{P}(W(T) > u + y | m_u = j) \\
&= \sum_{j\in E} \boldsymbol{\alpha}_+ e^{u\boldsymbol{Q}} \boldsymbol{e}_j \boldsymbol{e}_j' e^{y\boldsymbol{T}} \boldsymbol{e} \\
&= \boldsymbol{\alpha}_+ e^{u\boldsymbol{Q}} e^{y\boldsymbol{T}} \boldsymbol{e},
\end{aligned}$$

Figure 1
Ruin Occurring with $m_u = j$ and $U(T) > y$



which is the same as the first assertion. Similarly,

$$\mathrm{P}^{(0)}(T < \infty, |U(T)| > y) = \boldsymbol{\alpha}^{(0)} e^{u\boldsymbol{Q}} e^{y\boldsymbol{T}} \boldsymbol{e},$$

which yields the required result. □

6. NUMERICAL ILLUSTRATION

In this section we perform a simulation study to examine the tightness of the bounds obtained in Section 4.

Consider an ordinary renewal risk process. The time elapsed between successive claims follows a phase-type distribution with

$$\boldsymbol{\alpha} = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & 0 \end{bmatrix}, \qquad \boldsymbol{T} = \begin{bmatrix} -4/5 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -5/2 \end{bmatrix}.$$

The moment-generating function of the arrival distribution is

$$\hat{A}(s) = \frac{3}{5}\frac{4}{4 - 5s} + \frac{2}{5}\frac{1}{1 - s}\frac{5}{5 - 2s} \quad \text{for } s < \frac{4}{5}$$

and $\mu_A = 131/100$. The distribution is in fact a mixture of exponential and generalized Erlang(2) distribution.

Claims follow an inverse Gaussian distribution IG(8,15). Here we use the notation IG(μ, θ) for an inverse Gaussian distribution with density function

$$b(x) = \sqrt{\frac{\theta}{2\pi}}\frac{1}{x^{3/2}}\exp\left\{-\frac{\theta}{2\mu^2 x}(x - \mu)^2\right\}$$

and cumulative distribution function

$$B(x) = \Phi\left(\frac{x - \mu}{\mu}\sqrt{\frac{\theta}{x}}\right) + e^{2\theta/\mu}\Phi\left(-\frac{x + \mu}{\mu}\sqrt{\frac{\theta}{x}}\right) \quad \text{for } x > 0,$$

where Φ is the distribution function of the standard normal random variable. The moment-generating function of the claim size distribution is

$$\hat{B}(s) = \exp\left\{\frac{15}{8}\left[1 - \sqrt{1 - \frac{128s}{15}}\right]\right\} \quad \text{for } s < \frac{15}{128}$$

and $\mu_B = 8$.

We choose $c = 1{,}080/131$ corresponding to a relative safety loading of 0.35. The unique positive solution of equation (3.1) is $R = 0.047368$ (to six decimal places of accuracy).

It follows from a straightforward integration that

$$a(R, v, y) = \frac{e^{0.047368v}[B(v + y) - B(v)]}{e^{0.4237319}[1 - B^*(v)]},$$

where B^* is the distribution function of IG(10.3643546,15).

To obtain the simulated values of $F(u, x, y)$, we generate 100,000 sample paths for a finite time horizon of 0 to 10,000 (large enough, but not infinite). The simulated $F(u, x, y)$ could be slightly smaller than the true one, but the experiment should indicate the tightness of the upper bound. We show the simulated values of the joint distribution $F(u, x, y)$ and the corresponding upper bound for various combinations of u, x, and y in Table 1.

Except for small values of x, the ratios of the upper bound to the simulated joint distribution are less than 4. The reason for the overestimation for small x is that we use the inequality $P_{R;s}(U(T-) \le x) \le 1$ in expression (4.2) in the proof of Theorem 1. Numerical studies with other claim size distributions and arrival processes show similar results. We can see that the upper bounds are reasonable except for small values of x.

APPENDIX

In this Appendix we provide the detailed construction of the martingale based on Asmussen's approach. Asmussen (2000) constructed a positive martingale for the s-delayed renewal risk process by using a forward Markovization technique and the theory of Markov additive process. For a general theory of a Markov additive process, refer to Asmussen (2003). Let $J(t) = T_{N(t)+1} - t$ be the time until the next claim occurs; thus $J(0) = T_1 = V_1 = s$. Then the process $\{Y(t)\} = \{(J(t), W(t))\}$ is a Markov additive process with infinite state space on the positive real line. The condition for the stochastic process $\{M(t)\}_{t\ge0}$ defined by

Table 1

Simulated Values of $F(u, x, y)$ and Corresponding Upper Bounds

u	$F(u, 10, 20)$	Upper Bound	$F(u, 30, 50)$	Upper Bound
4	0.3702	0.5848	0.5849	0.6036
8	0.2245	0.4839	0.4765	0.4994
16	0.1242	0.3312	0.3171	0.3419
32	0.0575	0.1552	0.1388	0.1602
64	0.0123	0.0341	0.0301	0.0352
x	$F(5, x, 20)$	Upper Bound	$F(20, x, 10)$	Upper Bound
4	0.0503	0.5569	0.0202	0.2358
8	0.2441	0.5577	0.0662	0.2365
16	0.4642	0.5577	0.1427	0.2365
32	0.5378	0.5577	0.2128	0.2365
y	$F(5, 10, y)$	Upper Bound	$F(20, 10, y)$	Upper Bound
4	0.1752	0.3001	0.0545	0.1475
8	0.2605	0.4417	0.0812	0.2170
16	0.3190	0.5413	0.1009	0.2660
32	0.3381	0.5730	0.1064	0.2816

$$M(t) = \frac{h(J(t))}{h(J(0))} e^{\alpha W(t) - t\kappa(\alpha)}$$

for some function h and constant $\kappa(\alpha)$ (both depending on α) to be a martingale is

$$\mathrm{E}_s\, [e^{\alpha W(t)} h(J(t))] = e^{t\kappa(\alpha)} h(s).$$

Ignoring higher-order terms and assuming differentiability of h,

$$\mathrm{E}_s\, [e^{\alpha W(dt)}\, h(J(dt))] = h(s - dt)e^{-c\alpha dt} = h(s) - [h(s)c\alpha + h'(s)]\, dt$$

and

$$e^{\kappa(\alpha)dt}\, h(s) = h(s) + \kappa(\alpha)h(s)\, dt.$$

Thus a choice of h which has derivatives of all orders is obtained by solving the differential equation

$$\frac{h'(s)}{h(s)} = -\kappa(\alpha) - c\alpha,$$

yielding

$$h(s) = e^{-[c\alpha + \kappa(\alpha)]s}.$$

To find $\kappa(\alpha)$, consider an ordinary renewal risk process. Since $h(0) = 1$,

$$\mathrm{E}_0\, [e^{\alpha W(dt)}\, h(J(dt))] = 1. \tag{A.1}$$

For an ordinary renewal risk process, a claim has just occurred, and hence the left-hand side of equation (A.1) is $\mathrm{E}[e^{\alpha X} h(V)]$, where X follows distribution B, V follows distribution A, and X and V are independent. By the choice of h, we get the condition

$$\hat{B}(\alpha)\hat{A}[-c\alpha - \kappa(\alpha)] = 1, \tag{A.2}$$

where $\hat{B}$ and $\hat{A}$ are the moment-generating functions of distributions B and A, respectively.

It suffices to consider $\alpha \geq 0$. Rewrite condition (A.2) as

$$\hat{A}[-c\alpha - \kappa(\alpha)] = \frac{1}{\hat{B}(\alpha)}. \tag{A.3}$$

It is obvious that for each $\alpha \geq 0$, a unique $\kappa(\alpha)$ exists such that equation (A.3) holds by the monotonicity of the moment-generating function. This concludes the construction of the martingale. For another approach to obtaining the martingale $\{M(t)\}$, refer to Section 11.5 of Rolski et al. (1999).

The positive martingale

$$M(t) = \exp\{\alpha W(t) - t\kappa(\alpha) - [c\alpha + \kappa(\alpha)](J(t) - s)\}$$

satisfies $\mathrm{E}_s[M(0)] = 1$ because $J(0) = s$, and thus, for each α, we can define a new probability measure $\mathrm{P}_{\alpha;s}$ by

$$\mathrm{P}_{\alpha;s}(G) = \mathrm{E}_s\, [M(t)1(G)]$$

for all events $G \in \sigma(\{Y(v)\}_{0\leq v\leq t})$. In the notation $\mathrm{P}_{\alpha;s}$, the subscript α means that the new measure is constructed by the martingale $\{M(t)\}$ with parameter α. The expectation under $\mathrm{P}_{\alpha;s}$ is denoted by $\mathrm{E}_{\alpha;s}$. Note that the filtration $\sigma(\{Y(v)\}_{0\leq v\leq t})$ is different from the natural filtration $\sigma(\{U(v)\}_{0\leq v\leq t})$ generated by $\{U(t)\}$ since $\{J(t)\}$ is not observable in reality for $t \geq s$ under the probability measure $\mathrm{P}_{\alpha;s}$.

Under this new probability measure, $\mathrm{P}_{\alpha;s}(J(0) = s) = \mathrm{E}_s[M(0)1(J(0) = s)] = 1$. To find the distribution of the claims and interclaim times, let

$$M(b_1, b_2, \ldots, b_n; c_2, c_3, \ldots, c_{n+1}) = \mathrm{E}_{\alpha;s}\left[\exp\left\{\sum_{i=1}^{n} b_i X_i + \sum_{i=2}^{n+1} c_i V_i\right\}\right], \quad n \geq 2$$

be the joint moment-generating function of $X_1, X_2, \ldots, X_n, V_2, V_3, \ldots, V_{n+1}$. The random variables $X_1, X_2, \ldots, X_n, V_2, V_3, \ldots, V_{n+1}$ are jointly measurable with respect to $\sigma(\{Y(v)\}_{0\leq v\leq T_n})$ since $J(T_n) = V_{n+1}$. Hence,

$$M(b_1, b_2, \ldots, b_n; c_2, c_3, \ldots, c_{n+1}) = \mathrm{E}_s\left[\exp\left\{\sum_{i=1}^{n} b_i X_i + \sum_{i=2}^{n+1} c_i V_i\right\} M(T_n)\right]. \tag{A.4}$$

Because $T_n = \Sigma_{i=1}^{n} V_i = s + \Sigma_{i=2}^{n} V_i$, we can write the exponent of $M(T_n)$ as

$$\alpha\left(\sum_{i=1}^{n} X_i - cs - c\sum_{i=2}^{n} V_i\right) - \kappa(\alpha)\left(s + \sum_{i=2}^{n} V_i\right) - [c\alpha + \kappa(\alpha)](V_{n+1} - s)$$

$$= \alpha\sum_{i=1}^{n} X_i - [c\alpha + \kappa(\alpha)]\sum_{i=2}^{n+1} V_i.$$

Thus, the right-hand side of equation (A.4) is

$$\mathrm{E}_s\left[\exp\left\{\sum_{i=1}^{n} (b_i + \alpha)X_i\right\}\exp\left\{\sum_{i=2}^{n+1} [c_i - c\alpha - \kappa(\alpha)]V_i\right\}\right]$$

$$= \prod_{i=1}^{n} \hat{B}(b_i + \alpha)\prod_{i=2}^{n+1} \hat{A}[c_i - c\alpha - \kappa(\alpha)].$$

By condition (A.2), we can write the above as

$$\prod_{i=1}^{n} \frac{\hat{B}(b_i + \alpha)}{\hat{B}(\alpha)}\prod_{i=2}^{n+1} \frac{\hat{A}[c_i - c\alpha - \kappa(\alpha)]}{\hat{A}[-c\alpha - \kappa(\alpha)]} = \prod_{i=1}^{n} \hat{B}_\alpha(b_i)\prod_{i=2}^{n+1} \hat{A}_\alpha(c_i),$$

where

$$B_\alpha(x) = \int_0^x \frac{e^{\alpha v}}{\hat{B}(\alpha)}\,dB(v) \quad \text{and} \quad A_\alpha(t) = \int_0^t \frac{e^{-[c\alpha+\kappa(\alpha)]s}}{\hat{A}[-c\alpha - \kappa(\alpha)]}\,dA(s). \tag{A.5}$$

Consequently, under the changed measure, the claims $X_1, X_2, \ldots$ and the interclaim times $V_2, V_3, \ldots$ are independent with distributions B_α and A_α, respectively. The distributions are, in fact, the Esscher transforms of B and A. Therefore, we still have an s-delayed renewal risk process with changed interclaim time distribution A_α and claim distribution B_α. The premium rate remains c, since the set of trajectories of $\{U(t)\}$ is the same under P_s and $\mathrm{P}_{\alpha;s}$.

Under $\mathrm{P}_{\alpha;s}$, the means of V and X are

$$\mu_{\alpha,A} = \frac{\hat{A}'[-c\alpha - \kappa(\alpha)]}{\hat{A}[-c\alpha - \kappa(\alpha)]} \quad \text{and} \quad \mu_{\alpha,B} = \frac{\hat{B}'(\alpha)}{\hat{B}(\alpha)},$$

respectively.

By the implicit function theorem, the function $\alpha \to \kappa(\alpha)$ is differentiable, and by differentiating condition (A.2) with respect to α,

$$\hat{B}'(\alpha)\hat{A}[-c\alpha - \kappa(\alpha)] + \hat{B}(\alpha)\hat{A}'[-c\alpha - \kappa(\alpha)][-c - \kappa'(\alpha)] = 0.$$

Hence

$$c - \frac{\mu_{\alpha,B}}{\mu_{\alpha,A}} = -\kappa'(\alpha). \tag{A.6}$$

Lemma 11.5.1 of Rolski et al. (1999) shows that the function $\alpha \to \kappa(\alpha)$ is strictly convex assuming that not both X_i and V_i are deterministic. This is a reasonable assumption, since otherwise $\{U(t)\}$ would be deterministic. By condition (A.2), $\kappa(0) = 0$. By condition (2.1), we have $\kappa'(0) < 0$, and a unique $R > 0$ may exist such that $\kappa(R) = 0$. Such R, if it exists, is the adjustment coefficient. By the convexity of $\kappa(\alpha)$, $\kappa'(R) > 0$. By equation (A.6), the expected net profit is negative under $P_{R;s}$. Hence ruin is almost sure under $P_{R;s}$.

In the paper we assume that the adjustment coefficient R exists. With the choice $\alpha = R$, we have

$$\hat{B}(R)\hat{A}(-cR) = 1 \qquad \text{and} \qquad M(t) = e^{RW(t)-cR(J(t)-s)}. \tag{A.7}$$

Also, by equations (A.5) and (A.7),

$$\hat{A}_R(cR) = \int_0^\infty e^{cRz}\, dA_R(z) = \int_0^\infty e^{cRz} \frac{e^{-cRz}}{\hat{A}(-cR)}\, dA(z) = \frac{1}{\hat{A}(-cR)} = \hat{B}(R).$$

ACKNOWLEDGMENT

This paper was partially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU 7239/04H).

REFERENCES

ASMUSSEN, SØREN. 2000. *Ruin Probabilities*. Singapore: World Scientific.

———. 2003. *Applied Probability and Queues*. 2nd ed. New York: Springer-Verlag.

ASMUSSEN, SØREN, OLLE NERMAN, AND MORITA OLSSON. 1996. Fitting Phase-Type Distributions via the EM Algorithm. *Scandinavian Journal of Statistics* 30: 365–72.

BOWERS, NEWTON L., JR., HANS U. GERBER, JAMES C. HICKMAN, DONALD A. JONES, AND CECIL J. NESBITT. 1997. *Actuarial Mathematics*. 2nd ed. Schaumburg, Ill.: Society of Actuaries.

CHENG, YEBIN, AND QIHE TANG. 2003. Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process. *North American Actuarial Journal* 7(1): 1–12.

DICKSON, DAVID C. M., 1992. On the Distribution of the Surplus prior to Ruin. *Insurance: Mathematics and Economics* 11: 191–207.

DICKSON, DAVID C. M., AND ALFREDO EGIDIO DOS REIS. 1994. Ruin Problems and Dual Events. *Insurance: Mathematics and Economics* 14: 51–60.

DICKSON, DAVID C. M., AND STEVE DREKIC. 2004. The Joint Distribution of the Surplus Prior to Ruin and the Deficit at Ruin in Some Sparre Andersen Models. *Insurance: Mathematics and Economics* 34: 97–108.

DREKIC, STEVE, DAVID C. M. DICKSON, DAVID A. STANFORD, AND GORDON E. WILLMOT. 2003. The Deficit at Ruin in the Stationary Renewal Risk Model. Working paper. Department of Statistics and Actuarial Science, University of Waterloo.

———. 2004. On the Distribution of the Deficit at Ruin When Claims are Phase-Type. *Scandinavian Actuarial Journal*: 105–20.

DUFRESNE, FRANÇOIS, AND HANS U. GERBER. 1988. The Surpluses immediately before and at Ruin, and the Amount of the Claim Causing Ruin. *Insurance: Mathematics and Economics* 7: 193–99.

FELLER, WILLIAM. 1968. *An Introduction to Probability Theory and Its Applications*. Vol. 1. 3rd ed. New York: John Wiley.

———. 1971. *An Introduction to Probability Theory and Its Applications*. Vol. 2. 2nd ed. New York: John Wiley.

GERBER, HANS U. 1973. Martingales in Risk Theory. *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* 205–16.

———. 1979. *An Introduction to Mathematical Risk Theory*. S. S. Huebner Foundation Monograph Series 8. Homewood, Ill.: Richard Irwin.

GERBER, HANS U., MARC J. GOOVAERTS, AND ROB KAAS. 1987. On the Probability and Severity of Ruin. *ASTIN Bulletin* 17: 151–63.

GERBER, HANS U., AND ELIAS S. W. SHIU. 1997. The Joint Distribution of the Time of Ruin, the Surplus immediately before Ruin, and the Deficit at Ruin. *Insurance: Mathematics and Economics* 21: 129–37.

———. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48–72.

———. 2003. Discussion of Yebin Cheng and Qihe Tang, Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process. *North American Actuarial Journal* 7(3): 117–19, 7(4): 96–101.

———. 2005. The Time Value of Ruin in a Sparre Andersen Model. *North American Actuarial Journal* 9(2): 49–69; Discussions: 69–84.

GRANDELL, JAN. 1991. *Aspects of Risk Theory*. New York: Springer-Verlag.

LI, SHUANMING. 2003. Discussion of Yebin Cheng and Qihe Tang, Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process. *North American Actuarial Journal* 7(3): 119–22.

LI, SHUANMING, AND JOSE GARRIDO. 2004. On Ruin for the Erlang(n) Risk Process. *Insurance: Mathematics and Economics* 34: 391–408.

LIN, SHELDON X. 2003. Discussion of Yebin Cheng and Qihe Tang, Moments of the Surplus before Ruin and the Deficit at Ruin in the Erlang(2) Risk Process. *North American Actuarial Journal* 7(3): 122–24.

NEUTS, MARCEL F. 1981. *Matrix-Geometric Solutions in Stochastic Models*. Baltimore: Johns Hopkins University Press.

ROLSKI, TOMASZ, HANSPETER SCHMIDLI, VOLKER SCHMIDT, AND JOZEF TEUGELS. 1999. *Stochastic Processes for Insurance and Finance*. New York: John Wiley.

SCHMICKLER, LEONHARD 1992. MEDA: Mixed Erlang Distributions as Phase-Type Representations of Empirical Distribution Functions. *Stochastic Models* 6: 131–56.

TSAI, CARY CHI-LIANG, AND LI-JUAN SUN. 2004. On the Discounted Distribution Functions for the Erlang(2) Risk Process. *Insurance: Mathematics and Economics* 35: 5–19.

WILLMOT, GORDON E. 2004. A Note on a Class of Delayed Renewal Risk Process. *Insurance: Mathematics and Economics* 34: 251–57.

WILLMOT, GORDON E., AND DAVID C. M. DICKSON. 2003. The Gerber-Shiu Discounted Penalty Function in the Stationary Renewal Risk Model. *Insurance: Mathematics and Economics* 32: 403–11.

WILLMOT, GORDON E., AND X. SHELDON LIN. 2001. *Lundberg Approximations for Compound Distributions with Insurance Applications*. New York: Springer-Verlag.

YANG, HAILIANG, AND LIHONG ZHANG. 2001. The Joint Distribution of Surplus immediately before Ruin and the Deficit at Ruin under Interest Force. *North American Actuarial Journal* 5(3): 92–103.

DISCUSSIONS

HANS U. GERBER* AND ELIAS S. W. SHIU†

The authors are to be congratulated for their elegant method in bounding the joint distribution of the surplus immediately before ruin and the deficit at ruin for a Sparre Andersen model. Their main results are given in Section 4. A key step of their approach is to seek an ''Esscher transform'' of the probability measure, with respect to which ruin is certain. The Radon-Nikodym derivative for changing the measure is the positive martingale $\{M(t)\}$ defined near the end of Section 3,

$$M(t) = e^{R\{W(t)-c[J(t)-s]\}}. \tag{D1}$$

Three useful properties are listed at the end of Section 3, with their proof given in the Appendix. The focus of this discussion is on the last paragraph in Section 3 of this interesting paper.

Because

$$W(t) = X_1 + \cdots + X_{N(t)} - ct,$$
$$J(t) = T_{N(t)+1} - t$$

and

$$s = T_1,$$

we have

$$W(t) - c[J(t) - s] = X_1 + \cdots + X_{N(t)} - c[T_{N(t)+1} - T_1].$$

Since

$$T_{N(t)+1} - T_1 = V_2 + \cdots + V_{N(t)+1},$$

equation (D1) can be written as

* Hans U. Gerber, ASA, PhD is a Professor of Actuarial Science, École des hautes études commerciales, Université de Lausanne, CH-1015 Lausanne, Switzerland, hgerber@unil.ch.

† Elias S. W. Shiu, ASA, PhD is the Principal Financial Group Foundation Professor of Actuarial Science, Department of Statistics and Actuarial Science, University of Iowa, Iowa City, IA 52242-1409, and visiting Chair Professor of Actuarial and Investment Science, Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong, eshiu@stat.uiowa.edu.

$$M(t) = \exp\left[R \sum_{j=1}^{N(t)} (X_j - cV_{j+1})\right], \tag{D2}$$

with $M(t) = 1$ when $N(t) = 0$. Since $X_1, X_2, \ldots$ and $V_2, V_3, \ldots$ are two sequences of i.i.d. random variables and they are also independent of each other, the condition for $\{M(t)\}$ to be a martingale is

$$\begin{aligned} 1 &= \mathrm{E}[e^{R(X-cV)}] \\ &= \mathrm{E}[e^{RX}]\mathrm{E}[e^{(-cR)V}], \end{aligned} \tag{D3}$$

which is formula (3.1) of the paper and formula (1.8), with $\delta = 0$, of Gerber and Shiu (2005). Here X is a representative random variable of $X_1, X_2, \ldots$, and V is a representative of $V_2, V_3, \ldots$ It may be instructive to compare the martingale defined by equation (D2) with the discrete process (1.7), for $\delta = 0$, in Gerber and Shiu (2005), while noting that, in the present context, the random variable V_1, which is the same as T_1, is constant.

Property 1 at the end of Section 3 is that ruin is certain under the changed measure. This is the case, if we can show that

$$\lim_{t\to\infty} \frac{1}{t} \mathrm{E}^*[U(t)] < 0,$$

where the asterisk signifies that the expectation is taken with respect to the changed probability measure. Thus, we now prove that

$$\mathrm{E}^*[cV - X] < 0. \tag{D4}$$

Observe that

$$\mathrm{E}^*[cV - X] = \frac{\mathrm{E}[(cV - X)e^{R(X-cV)}]}{\mathrm{E}[e^{R(X-cV)}]} = \mathrm{E}[(cV - X)e^{R(X-cV)}] \tag{D5}$$

because of equation (D3). Let

$$g(\alpha) = \mathrm{E}[e^{\alpha(X-cV)}] \tag{D6}$$

be the moment-generating function of the random variable $(X - cV)$. Then, equation (D5) is

$$\mathrm{E}^*[cV - X] = -g'(R). \tag{D7}$$

The function g is strictly convex unless $(X - cV)$ is zero. Also, $g(0) = 1$, and $g(R) = 1$ because of equation (D3). With $0 < R$, we must have $g'(R) > 0$, and hence, by equation (D7), the inequality (D4) is proved. A similar analysis can be found in Section 11 of Gerber and Shiu (1996); see also Morales (2004, p. 84).

Property 2 at the end of Section 3 states that, under the changed measure, the random variables X_1, $X_2, \ldots, V_2, V_3, \ldots$ remain independent; the common distribution of $X_1, X_2, \ldots$ is the Esscher transform of their original distribution with respect to parameter R; the common distribution of V_1, $V_2, \ldots$ is the Esscher transform of their original distribution with respect to parameter $-cR$. This property is a consequence of the following observation: Given two independent random variables Y and Z, we have

$$\frac{\mathrm{E}[e^{bY+cZ}e^{\alpha(Y+Z)}]}{\mathrm{E}[e^{\alpha(Y+Z)}]} = \frac{\mathrm{E}[e^{bY}e^{\alpha Y}]}{\mathrm{E}[e^{\alpha Y}]} \frac{\mathrm{E}[e^{cZ}e^{\alpha Z}]}{\mathrm{E}[e^{\alpha Z}]},$$

which shows the factorization property of the joint moment-generating function under an Esscher transform.

Property 3 at the end of Section 3 is the formula

$$\frac{\mathrm{E}[e^{cRV}e^{-cRV}]}{\mathrm{E}[e^{-cRV}]} = \frac{1}{\mathrm{E}[e^{-cRV}]} = \mathrm{E}[e^{RX}].$$

The last equality is due to the martingale condition (D3).

REFERENCES

GERBER, HANS, U., AND ELIAS S. W. SHIU. 1996. Actuarial Bridges to Dynamic Hedging and Option Pricing. *Insurance: Mathematics and Economics* 18: 183–218.

———. 2005. The Time Value of Ruin in a Sparre Andersen Model. *North American Actuarial Journal* 9(2): 49–69; Discussions: 69–84.

MORALES, MANUEL. 2004. On a Surplus Process under a Periodic Environment: A Simulation Approach. *North American Actuarial Journal* 8(4): 76–89.

X. SHELDON LIN* AND XIAOMING LIU†

In recent years, there has been increasing interest in analyzing the Sparre Andersen insurance risk (or aggregate claims) model, where the number of claims process follows a general renewal process. An obvious motivation is that the classical compound Poisson risk model is too restrictive, assuming that interclaim times are exponentially distributed. However, it is often very difficult to analyze the Sparre Andersen model without any specific distributional assumption on interclaim times. Hence, most research papers on the Sparre Andersen model, including this interesting paper by Mr. Ng and Dr. Yang, either focus on certain quantitative aspects of the model or assume that the interclaim times and/or the severity of claims are phase-type distributed as described in Section 5 of their paper. See Section 1, of this paper, Gerber and Shiu (2005), and the references therein.

A main advantage of using phase-type distributions for insurance risk models is that the model and its related ruin problems become mathematically tractable. Very often closed-form expressions exist for many quantities of interest, such as the distribution of the surplus before ruin and the deficit at ruin, as well as their moments. See Asmussen (2000), Asmussen and Rolski (1992), Dickson and Drekic (2004), Drekic et al. (2004), and the references therein. On the other hand, from a practical viewpoint if we are to use the Sparre Andersen model to model claims arising from an insurance portfolio, the interclaim time distribution and the claim severity distribution should be estimated from its claim database. The resulting risk model is thus a general Sparre Andersen model and mathematically less tractable. Hence, there is seemingly a gap between the practical use of the Sparre Andersen model and the mathematical analysis on the model. However, this gap can be filled by approximating the interclaim time distribution and/or the claim severity distribution with phase-type distributions. This is due to the fact that a continuous positive distribution can be approximated by a phase-type distribution to a given accuracy. See Asmussen (2000) and Tijms (1994).

The purpose of this discussion is to draw readers' attention to a useful computer program called EMpht. The program was developed by Professor Soren Asmussen and his collaborators; it enables us to fit a phase-type distribution either to a data set or to a given positive continuous distribution. As a result, we are able to compute ruin-related quantities for a general Sparre Andersen model. We also wish to point out that the program can have other actuarial applications. For example, one may use it for fitting a phase-type distribution to mortality data.

In the following we recall briefly the phase-type distribution and the EMpht program, and then present some numerical examples.

* X. Sheldon Lin, ASA, PhD is a Professor of Actuarial Science in Department of Statistics at the University of Toronto, Toronto, ON M5S 3G3, Canada, sheldon@utstat.utoronto.ca.

† Xiaoming Liu is a PhD candidate in Department of Statistics at University of Toronto, Toronto, ON M5S 3G3, Canada, xiaoming@utstat.utoronto.ca.

PHASE-TYPE DISTRIBUTION AND EMPHT PROGRAM

As described in Ng and Yang's paper, a phase-type distribution of order p is the distribution of the absorbing time Y of a continuous-time $(p + 1)$–state Markov chain J_t, where the state 0 is absorbing and the states 1, 2, . . . , p are transient. The Markov chain J_t is characterized by its initial distribution α (a p-dimensional row vector) and its $p \times p$ dimensional intensity matrix $\mathbf{T}$. Fitting a phase-type distribution involves the estimation of parameters in the phase-type representation $(\boldsymbol{\alpha}, \mathbf{T})$.

The EMpht program is based on the expectation maximization (EM) algorithm, which is a general iteration-optimization method for maximum likelihood estimation for incomplete data (see Dempster, Laird, and Rubin 1977). The claim data or the data generated from an underlying interclaim time distribution or claim severity distribution are treated as the incomplete data of the Markov chain J_t. The details of the program description and its implementation can be found in Asmussen, Nerman, and Olsson (1996) and Olsson (1998). A remarkable feature of using the EM algorithm is that the structure of zeros in $(\boldsymbol{\alpha}, \mathbf{T})$ is preserved during iterations. Thus we can specify a special class of phase-type distributions for fitting (mixtures of exponentials, mixtures of Erlangs, Coxian distributions, etc.).

NUMERICAL EXAMPLES

In Section 6 of Ng and Yang's paper, a renewal risk process is considered with a relative security loading 0.35, where the interclaim times follow the phase-type distribution with

$$\boldsymbol{\alpha} = [3/5,\ 2/5,\ 0], \qquad \mathbf{T} = \begin{bmatrix} -4/5 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -5/2 \end{bmatrix},$$

and the claim severity follows the inverse Gaussian distribution IG(8, 15). In that case a closed-form expression for the joint distribution of the surplus before ruin and the deficit at ruin is not available and hence a simulation scheme is performed to obtain exact values. In the following we use a phase-type distribution of order 2 to approximate the interclaim time distribution and a phase-type distribution of order 3 to approximate the claim severity distribution using the EMpht program. As shown in Dickson and Drekic (2004), a closed-form expression for the joint distribution of the surplus before ruin and the deficit at ruin is now available. As a result, we obtain very good numerical approximations for the joint distribution.

Running the EMpht program, we obtain the phase-type(2) approximation for the interclaim time distribution with the following parameter values:

$$\boldsymbol{\alpha} = [1,\ 0], \qquad \mathbf{T} = \begin{bmatrix} -0.883935 & 0.476023 \\ 0 & -3.007728 \end{bmatrix}.$$

Similarly, for the inverse Gaussian claim severity distribution, we obtain the following phase-type(3) approximation with

$$\boldsymbol{\beta} = [1,\ 0,\ 0], \qquad \mathbf{B} = \begin{bmatrix} -0.676866 & 0.676866 & 0 \\ 0 & -0.204189 & 0.204189 \\ 0 & 0 & -0.676866 \end{bmatrix}.$$

The densities and distribution functions of the approximating distributions as well as the original distributions are given in Figures 1 and 2. As shown, these approximations are satisfactory.

We remark that for the claim severity, the approximation is not limited to a phase-type distribution of order 3. A closed form expression for the joint distribution is available for any phase-type claim severity, as shown in Dickson and Drekic (2004). One may use a phase-type distribution of higher order for claim severity, as the higher the order is, the better its approximation is. In Figure 3 we use a phase-type(6) to fit the same inverse Gaussian distribution, and the result is more satisfactory.

Figure 1
Interclaim Distribution Fitted by a Phase-Type(2)

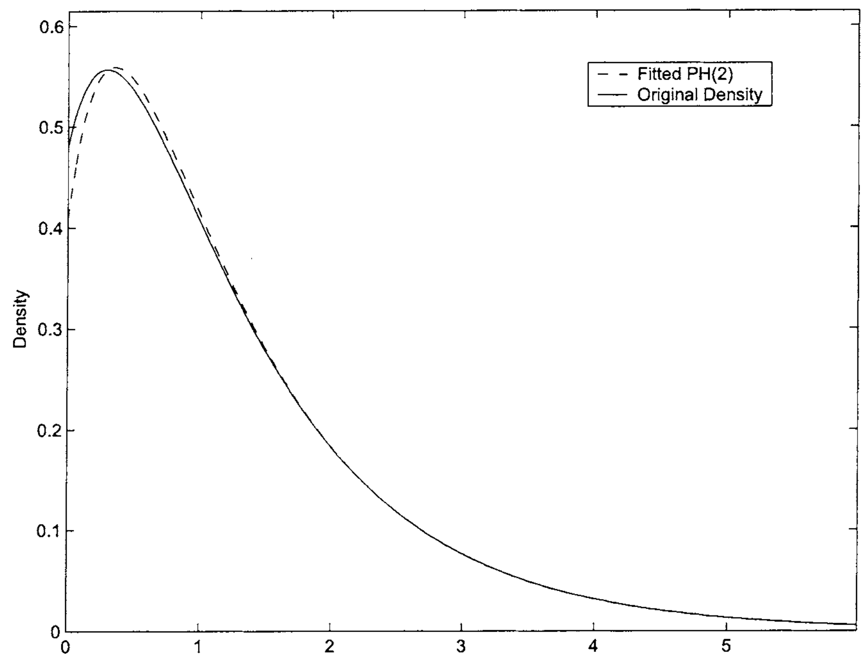


Figure 2
Claim Severity Distribution Fitted by a Phase-Type(3)

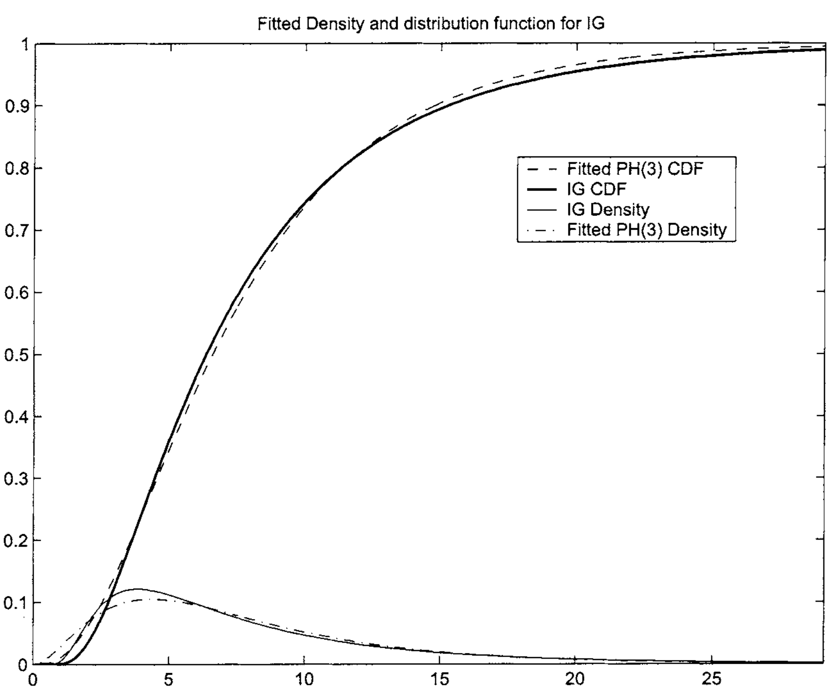
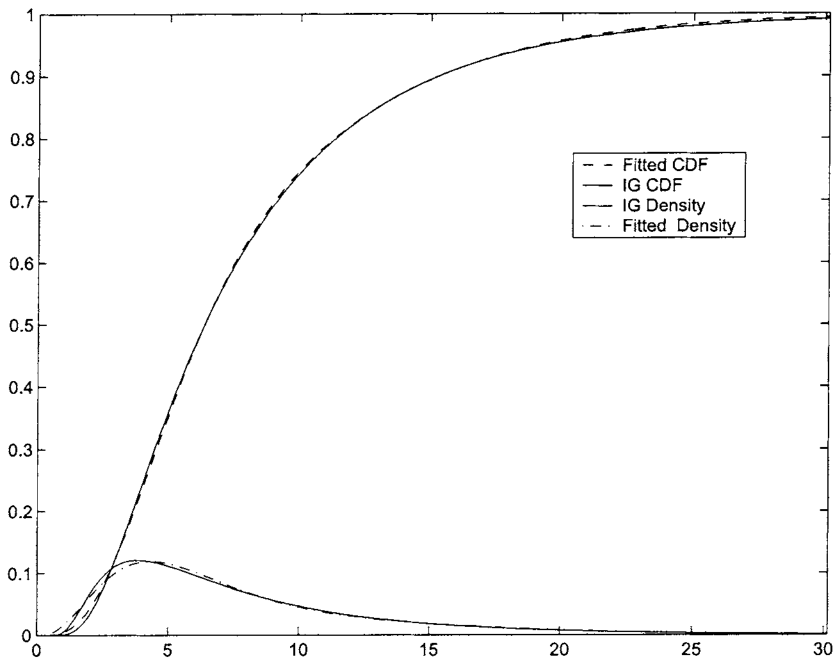


Figure 3
Claim Severity Distribution Fitted by a Phase-Type(6)



We now provide numerical approximations to the joint distribution $F(u, x, y)$ of the surplus before ruin and the deficit at ruin, where u is the initial surplus, x is the surplus before ruin, and y is deficit at ruin. We compare these numerical values with those in Ng and Yang's. Again, the results are based on the closed-form expression obtained by Dickson and Drekic (2004). Tables 1 and 2 show numerical approximations for various initial surplus values, while the values of the surplus before ruin and the deficit at ruin are fixed. Similarly, in Tables 3 and 4 we fix the initial surplus and the deficit at ruin, and in Tables 5 and 6 we fix the initial surplus and the surplus before ruin. Our numerical results show that the phase-type approximating procedure is very effective and accurate.

Table 1
The Joint Distribution with $x = 10$ and $y = 20$

u	$F(u, 10, 20)$	Approximated F	Upper Bound
4	0.3702	0.3871752853	0.5848
8	0.2245	0.2373902922	0.4839
16	0.1242	0.1360395714	0.3312
32	0.0575	0.0577635972	0.1552
64	0.0123	0.0104038485	0.0341

Table 2
The Joint Distribution with $x = 30$ and $y = 50$

u	$F(u, 30, 50)$	Approximated F	Upper Bound
4	0.5849	0.5831989704	0.6036
8	0.4765	0.4707296353	0.4994
16	0.3171	0.3039737318	0.3419
32	0.1388	0.1257380808	0.1602
64	0.0301	0.0226500508	0.0352

Table 3
The Joint Distribution with $u = 5$ and $y = 20$

x	$F(5, x, 20)$	Approximated F	Upper Bound
4	0.0503	0.0498448734	0.5569
8	0.2441	0.2541730597	0.5577
16	0.4642	0.4862168267	0.5577
32	0.5378	0.5444350849	0.5577

Table 4
The Joint Distribution with $u = 20$ and $y = 10$

x	$F(20, x, 10)$	Approximated F	Upper Bound
4	0.0202	0.0198965731	0.2358
8	0.0662	0.0697004049	0.2365
16	0.1427	0.1551401488	0.2365
32	0.2128	0.2119435305	0.2365

Table 5
The Joint Distribution with $u = 5$ and $x = 10$

y	$F(5, 10, y)$	Approximated F	Upper Bound
4	0.1752	0.1833295043	0.3001
8	0.2605	0.2762428998	0.4417
16	0.3190	0.3374390078	0.5413
32	0.3381	0.3518030268	0.5730

Table 6
The Joint Distribution with $u = 20$ and $x = 10$

y	$F(20, 10, y)$	Approximated F	Upper Bound
4	0.0545	0.05744478976	0.1475
8	0.0812	0.08733128107	0.2170
16	0.1009	0.1071506044	0.2660
32	0.1064	0.1118084625	0.2816

REFERENCES

ASMUSSEN, SOREN. 2000. *Ruin Probabilities.* Singapore: World Scientific.

ASMUSSEN, SOREN, OLLE NERMAN, AND MARITA OLSSON. 1996. Fitting Phase-Type Distributions via the EM Algorithm. *Scandinavian Journal of Statistics* 23: 419–41.

ASMUSSEN, SOREN, AND TOMASZ ROLSKI. 1992. Computational Methods in Risk Theory: A Matrix-Algorithmic Approach. *Insurance: Mathematics and Economics* 10(4): 259–74.

DEMPSTER, ARTHUR P., NAN M. LAIRD, AND DONALD B. RUBIN. 1977. Maximum Likelihood from Incomplete Data via the EM Algorithm. *Journal of the Royal Statistical Society,* Series B (Methodological) 39(1): 1–38.

DICKSON, DAVID C. M., AND STEVE DREKIC. 2004. The Joint Distribution of the Surplus prior to Ruin and the Deficit at Ruin in some Sparre Andersen Models. *Insurance: Mathematics and Economics* 34(1): 97–107.

DREKIC, STEVE, DAVID C. M. DICKSON, DAVID A. STANFORD, AND GORDON E. WILLMOT. 2004. On the Distribution of the Deficit at Ruin when Claims are Phase-Type. *Scandinavian Actuarial Journal* 105–120.

GERBER, HANS U., AND ELIAS S. W. SHIU. 2005. On the Time Value of Ruin in a Sparre Anderson Model. *North American Actuarial Journal* 9(2): 49–69; Discussions: 69–84.

OLSSEN, MARITA. 1998. The EMpht-Programme. Technical report. Department of Mathematics, Chalmers University of Technology, Goteborg, Sweden. Online at home.imf.au.dk/asmus.

TIJMS, HENK C. 1994. *Stochastic Models: An Algorithmic Approach.* Chichester: John Wiley.

Additional discussions on this paper can be submitted until October 1, 2005. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.