

# THE TIME VALUE OF RUIN IN A SPARRE ANDERSEN MODEL

Hans U. Gerber\* and Elias S. W. Shiu†

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## ABSTRACT

This paper considers a Sparre Andersen collective risk model in which the distribution of the interclaim time is that of a sum of  $n$  independent exponential random variables; thus, the Erlang( $n$ ) model is a special case. The analysis is focused on the function  $\phi(u)$ , the expected discounted penalty at ruin, with  $u$  being the initial surplus. The penalty may depend on the deficit at ruin and possibly also on the surplus immediately before ruin. It is shown that the function  $\phi(u)$  satisfies a certain integro-differential equation and that this equation can be solved in terms of Laplace transforms, extending a result found in Lin (2003). As a consequence, a closed-form expression is obtained for the discounted joint probability density of the deficit at ruin and the surplus just before ruin, if the initial surplus is zero. For this formula and other results, the roots of Lundberg's fundamental equation in the right half of the complex plane play a central role. Also, it is shown that  $\phi(u)$  satisfies Li's (2003) *renewal equation*. Under the assumption that the penalty depends only on the deficit at ruin and that the individual claim amount density is a combination of exponential densities, a closed-form expression for  $\phi(u)$  is derived. In this context, known results of the Cauchy matrix are useful. Surprisingly, certain results are best expressed in terms of divided differences, a topic deleted from the actuarial examinations at the end of last century.

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## 1. INTRODUCTION

In the *classical model* of risk theory, premiums are received continuously at a constant rate  $c$ , and the aggregate claims constitute a compound Poisson process, with Poisson parameter  $\lambda$  and individual claim amount probability density function  $p(x)$ ,  $x \geq 0$ . For  $j = 1, 2, 3, \dots$ , let  $T_j$  denote the time when the  $j$ -th claim of amount  $X_j$  occurs. Also, let  $T_0$  denote the occurrence time of the last claim. Note that  $T_0 \leq 0$ , and at time 0 its value is assumed to be known. That the number of claims is a Poisson process means that the *interclaim time* random variables,

$$V_j = T_j - T_{j-1}, j = 1, 2, 3, \dots, \quad (1.1)$$

are independent with a common exponential distribution of mean  $1/\lambda$ . Because of the memoryless property of the exponential distribution, the random variables  $V_1$  and  $T_1$  have the same distribution.

The *Sparre Andersen model* is more general in that the common distribution of the i.i.d. random variables  $\{V_j\}$  is arbitrary (and, hence, the distribution of  $V_1$  is not necessarily the same as that of  $T_1$  unless a claim has occurred at time 0). The requirement of a positive *security loading* is that the expected premiums received between successive claims exceed the expected amount of a claim,

$$c \mathbb{E}[V_j] > \mathbb{E}[X_j]. \quad (1.2)$$

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\* Hans U. Gerber, ASA, PhD, is a Professor of Actuarial Science, Ecole des hautes études commerciales, Université de Lausanne, CH-1015 Lausanne, Switzerland, hgerber@unil.ch.

† Elias S. W. Shiu, ASA, PhD, is the Principal Financial Group Foundation Professor of Actuarial Science, Department of Statistics and Actuarial Science, University of Iowa, Iowa City, IA 52242-1409, and visiting Chair Professor of Actuarial and Investment Science, Department of Applied Mathematics, Hong Kong Polytechnic University, Hung Hom, Hong Kong, eshiu@stat.uiowa.edu.

We let  $f_V(t)$  and  $F_V(t)$ ,  $t \geq 0$ , denote the probability density function and distribution function of  $V_j$ , respectively. The Sparre Andersen model corresponds to a GI/G/1 queue in queuing theory, whereas the classical model corresponds to an M/G/1 queue. Although the model was proposed almost half a century ago, it remains an important area of research in actuarial science. Some very recent papers are Cai and Dickson (2003), Cheng and Tang (2003), Dickson and Drekić (2004), Li and Garrido (2004a, 2004b), Pavlova and Willmot (2004), Sun and Yang (2004), Tsai and Sun (2004), Willmot (2004), and Willmot and Dickson (2003).

Let  $\psi(u, \tau)$  denote the *probability of ultimate ruin*, where  $u \geq 0$  is the surplus at time 0 and  $\tau = -T_0$  is the length of time elapsed since the last claim. In particular,  $\psi(u, 0)$  is the probability of ultimate ruin if a claim has taken place at time 0, with  $u$  being the surplus immediately after this claim has been paid. We are interested in a somewhat more general problem, which is the evaluation of an *expected discounted penalty*

$$\phi(u, \tau) = \mathbb{E}[e^{-\delta T} \varpi(U(T-), |U(T)|) 1_{(T < \infty)} \mid T_0 = -\tau, U(0) = u], \quad u \geq 0, \tau \geq 0. \quad (1.3)$$

Here, the positive parameter  $\delta$  can be interpreted as a force of interest,  $U(t)$  is the surplus at time  $t$ ,  $T$  is the *time of ruin*,  $\varpi(U(T-), |U(T)|)$  is the penalty at ruin, and  $1_{(\cdot)}$  is the indicator function. For simplicity, we write

$$\phi(u) = \phi(u, 0). \quad (1.4)$$

Then, by conditioning on  $T_1$ , the time of the first claim, and on  $X_1$ , the amount of the first claim, we obtain

$$\begin{aligned} \phi(u, \tau) &= \int_0^\infty e^{-\delta t} \left[ \int_0^{u+ct} \phi(u + ct - x) p(x) dx \right] \frac{f_V(\tau + t)}{1 - F_V(\tau)} dt \\ &\quad + \int_0^\infty e^{-\delta t} \left[ \int_{u+ct}^\infty \varpi(u + ct, x - u - ct) p(x) dx \right] \frac{f_V(\tau + t)}{1 - F_V(\tau)} dt. \end{aligned} \quad (1.5)$$

A main goal in this paper is to obtain more specific results under the assumption that each of the interclaim time random variables  $\{V_j\}$  is the sum of  $n$  independent, exponentially distributed random variables, say, with parameters  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Note that the Erlang( $n$ ) model is the special case where  $\lambda_1 = \lambda_2 = \dots = \lambda_n$ .

We conclude this introduction by explaining a result that is essentially attributable to Bruno de Finetti. Suppose  $T_0 = 0$ . Let  $U_k$  denote the surplus immediately after the payment of the  $k$ -th claim. Thus,  $U_0 = u$ , and for  $k = 1, 2, 3, \dots$ ,

$$U_k = u + \sum_{j=1}^k (cV_j - X_j). \quad (1.6)$$

We are interested in coefficients  $\xi$  such that the sequence of random variables,

$$\left\{ \exp\left(-\delta \sum_{j=1}^k V_j + \xi U_k\right); k = 0, 1, 2, \dots \right\}, \quad (1.7)$$

is a *martingale*. In (1.7), the term for  $k = 0$  is the constant  $e^{\xi u}$ . The martingale condition is

$$\hat{f}_V(\delta - c\xi) \hat{p}(\xi) = 1, \quad (1.8)$$

where  $\hat{f}_V$  and  $\hat{p}$  are the *Laplace transforms* of the probability density functions  $f_V$  and  $p$ , respectively. Suppose that equation (1.8), which is a generalization of *Lundberg's fundamental equation*, has a solution  $-R < 0$ . ( $R$  can be called the *adjustment coefficient*.) Then, with  $\xi = -R$ , the sequence (1.7) is a bounded martingale before ruin. By applying the *optional sampling theorem*, we see that, with  $\varpi(x, y) = e^{Ry}$ ,

$$\phi(u) = e^{-Ru}. \quad (1.9)$$

## 2. A RENEWAL EQUATION FOR $\phi(u)$

We now determine a (defective) renewal equation for  $\phi(u)$  by probabilistic reasoning. For  $T_0 = 0$  and  $U(0) = u \geq 0$ , let  $f(x, y, t|u)$  denote the (defective) joint probability density function of  $U(T-)$ ,  $|U(T)|$ , and  $T$ . Also, define

$$f(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u) dt. \tag{2.1}$$

Consider the first time the surplus falls below the initial level. For  $T_0 = 0$  and  $U(0) = u \geq 0$ , the probability that this event occurs between time  $t$  and time  $t + dt$ , with

$$u + x \leq U(t-) \leq u + x + dx$$

and

$$u - y - dy \leq U(t) \leq u - y,$$

is

$$f(x, y, t|0) dx dy dt.$$

Also, the occurrence  $y > u$  means that ruin takes place with this claim. Thus,

$$\begin{aligned} \phi(u) &= \int_0^u \int_0^\infty \int_0^\infty e^{-\delta t} \phi(u - y) f(x, y, t|0) dt dx dy \\ &\quad + \int_u^\infty \int_0^\infty \int_0^\infty e^{-\delta t} \varpi(x + u, y - u) f(x, y, t|0) dt dx dy \\ &= \int_0^u \int_0^\infty \phi(u - y) f(x, y|0) dx dy \\ &\quad + \int_u^\infty \int_0^\infty \varpi(x + u, y - u) f(x, y|0) dx dy, \end{aligned} \tag{2.2}$$

which is the desired renewal equation.

It remains to determine  $f(x, y|0)$ , the joint “discounted” probability density function of  $U(T-)$  and  $U(T)$ , given that  $T_0 = 0$  and  $U(0) = 0$ . Setting  $u = 0$  in (2.2), we have

$$\phi(0) = 0 + \int_0^\infty \int_0^\infty \varpi(x, y) f(x, y|0) dx dy. \tag{2.3}$$

If we can determine  $\hat{\phi}$ , the Laplace transform of  $\phi$ , then we have another formula for  $\phi(0)$  by means of the *initial value theorem* (Spiegel 1965, p. 5),

$$\phi(0) = \lim_{\xi \rightarrow \infty} \xi \hat{\phi}(\xi). \tag{2.4}$$

We shall see in Section 8 that, by comparing formulas (2.3) and (2.4), we obtain an explicit formula for  $f(x, y|0)$ , which is a key result in this paper.

For  $T_0 = 0$  and  $U(0) = u \geq 0$ , we define the marginal discounted probability density functions

$$f_{U(T-)}(x|u) = \int_0^\infty f(x, y|u) dy \tag{2.5}$$

and

$$f_{|U(T)|}(y|u) = \int_0^\infty f(x, y|u) dx. \tag{2.6}$$

With the definition

$$g(y) = f_{|U(T)|}(y|0), \quad (2.7)$$

the first integral on the right-hand side of (2.2) is the convolution  $(\phi * g)(u)$ . The differential  $g(y) dy$  can be interpreted as the discounted probability that the surplus will ever fall below its initial value  $u$ , and will be between  $u - y$  and  $u - y - dy$  when it happens for the first time.

**REMARK**

It follows from the conditional probability formula

$$\Pr(A \cap B) = \Pr(A) \Pr(B|A)$$

that  $f(x, y, t|u)$  is the joint probability density function of  $U(T-)$  and  $T$  at the point  $(x, t)$  multiplied by the conditional probability density function of  $|U(T)|$  at  $y$ , given that  $U(T-) = x$  and  $T = t$ . The latter is  $p(x + y) / \int_0^\infty p(x + z) dz$ . Hence, with  $P$  denoting the distribution function of the individual claim amounts, we have

$$f(x, y, t|u) = \left[ \int_0^\infty f(x, z, t|u) dz \right] \frac{p(x + y)}{1 - P(x)} \quad (2.8)$$

and

$$f(x, y|u) = f_{U(T-)}(x|u) \frac{p(x + y)}{1 - P(x)}. \quad (2.9)$$

### 3. A SPECIFIC ASSUMPTION ON INTERCLAIM TIME

In the rest of this paper, we assume that each of the interclaim time random variables  $\{V_j\}$  is the sum of  $n$  independent, exponentially distributed random variables. Write  $V = V_1$ . Then

$$V = W_1 + W_2 + \cdots + W_n, \quad (3.1)$$

where  $\{W_i\}$  are  $n$  independent exponentially distributed random variables with  $\mathbb{E}[W_i] = 1/\lambda_i$ . It follows that

$$\mathbb{E}[e^{-\xi V}] = \hat{f}_V(\xi) = \prod_{i=1}^n \frac{\lambda_i}{\lambda_i + \xi}. \quad (3.2)$$

Note that the random variables  $\{W_i\}$  may not be observed individually.

At time 0, we know that  $T_0 = -\tau$  or, equivalently,  $V > \tau$ ,  $\tau \geq 0$ . The event  $\{V > \tau\}$  is the union of the following  $n$  mutually exclusive events:

$$A_0 = \{W_1 > \tau\},$$

and, for  $j = 1, 2, \dots, n - 1$ ,

$$A_j = \{W_1 + \cdots + W_j \leq \tau < W_1 + \cdots + W_{j+1}\}. \quad (3.3)$$

For  $j = 0, 1, 2, \dots, n - 1$ , define the conditional expectation

$$\phi_j(u) = \mathbb{E}[e^{-\delta T} \varpi(U(T-), |U(T)|) \mathbf{1}_{(T < \infty)} | A_j, U(0) = u], u \geq 0. \quad (3.4)$$

Then

$$\phi(u, \tau) = \sum_{j=0}^{n-1} \phi_j(u) \Pr(A_j | V > \tau) = \sum_{j=0}^{n-1} \phi_j(u) \frac{\Pr(A_j)}{\Pr(V > \tau)}. \quad (3.5)$$

We shall study the functions  $\phi_j(u)$  in Section 5. To determine the probabilities on the right-hand side of (3.5), we note that

$$\Pr(A_j) = \Pr(W_1 + \dots + W_{j+1} > \tau) - \Pr(W_1 + \dots + W_j > \tau), \tag{3.6}$$

and that the moment-generating function of  $W_1 + \dots + W_k$  is

$$\prod_{i=1}^k \frac{\lambda_i}{\lambda_i - z}.$$

By applying the method of partial fractions to the moment-generating function, we can identify the distribution. Two cases are particularly simple. The first is the Erlang( $n$ ) case

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda.$$

Then

$$\Pr(W_1 + \dots + W_k > \tau) = \int_{\tau}^{\infty} \frac{\lambda^k s^{k-1}}{(k-1)!} e^{-\lambda s} ds = \sum_{i=0}^{k-1} \frac{\lambda^i \tau^i}{i!} e^{-\lambda \tau}, \tag{3.7}$$

and

$$\Pr(A_j) = \frac{\lambda^j \tau^j}{j!} e^{-\lambda \tau}. \tag{3.8}$$

Formula (3.7), with  $k = n$ , gives  $\Pr(V > \tau)$ . The second is the case of distinct parameters  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Then

$$\prod_{i=1}^k \frac{\lambda_i}{\lambda_i - z} = \sum_{m=1}^k \left( \prod_{i=1, i \neq m}^k \frac{\lambda_i}{\lambda_i - \lambda_m} \right) \frac{\lambda_m}{\lambda_m - z},$$

and, hence,

$$\Pr(W_1 + \dots + W_k > \tau) = \sum_{m=1}^k \left( \prod_{i=1, i \neq m}^k \frac{\lambda_i}{\lambda_i - \lambda_m} \right) e^{-\lambda_m \tau}. \tag{3.9}$$

It follows from (3.6) and (3.9) that

$$\Pr(A_j) = \left( \prod_{i=1}^j \lambda_i \right) \sum_{m=1}^{j+1} \frac{e^{-\lambda_m \tau}}{\prod_{i=1, i \neq m}^{j+1} (\lambda_i - \lambda_m)}. \tag{3.10}$$

Differentiating (3.9) with respect to  $\tau$ , we see that the probability density function of  $W_1 + \dots + W_k$  is

$$\lambda_1 \lambda_2 \dots \lambda_k \sum_{m=1}^k \frac{e^{-\lambda_m \tau}}{\prod_{i=1, i \neq m}^k (\lambda_i - \lambda_m)}, \quad \tau > 0. \tag{3.11}$$

**REMARK**

The sum in (3.11) may remind some actuaries of a  $(k-1)$ -th *divided difference*. For a function  $h(s)$ , its divided differences, with respect to distinct numbers  $r_1, r_2, r_3, \dots$ , can be defined recursively as follows:

$$h(s) = h(r_1) + (s - r_1)h[r_1, s], \tag{3.12a}$$

$$h[r_1, s] = h[r_1, r_2] + (s - r_2)h[r_1, r_2, s], \tag{3.12b}$$

$$h[r_1, r_2, s] = h[r_1, r_2, r_3] + (s - r_3)h[r_1, r_2, r_3, s], \tag{3.12c}$$

and so on. We have the following formula for the  $(k-1)$ -th divided difference

$$h[r_1, r_2, \dots, r_k] = \sum_{j=1}^k \frac{h(r_j)}{\prod_{i=1, i \neq j}^k (r_j - r_i)} \tag{3.13}$$

(Freeman 1960, p. 41; Steffensen 1950, p. 15). With the definition

$$\varepsilon_\tau(\lambda) = e^{-\lambda\tau}, \quad (3.14)$$

it follows from (3.13) that (3.11) can be rewritten as

$$\lambda_1 \lambda_2 \dots \lambda_k (-1)^{k-1} \varepsilon_\tau[\lambda_1, \lambda_2, \dots, \lambda_k], \quad \tau > 0. \quad (3.15)$$

Formula (3.15) is meaningful even if some of the  $\lambda$ 's coincide, because a divided difference with *repeated points of collocation* can be evaluated as a derivative. For example, if  $a, b$ , and  $c$  are three distinct numbers, then

$$h[a, a, a, b, b, c] = \frac{1}{(3-1)!} \frac{1}{(2-1)!} \frac{\partial^2}{\partial a^2} \frac{\partial}{\partial b} h[a, b, c]. \quad (3.16)$$

A contour-integration proof of (3.16) can be found in Shiu (1983). Divided differences play an important role in this paper.

#### 4. SOLUTIONS OF LUNDBERG'S FUNDAMENTAL EQUATION

By (3.2), the reciprocal of  $\hat{f}_V(\delta - c\xi)$  is the  $n$ -th degree polynomial

$$\gamma(\xi) = \prod_{j=1}^n \left[ \left( 1 + \frac{\delta}{\lambda_j} \right) - \frac{c}{\lambda_j} \xi \right]. \quad (4.1)$$

Thus, the fundamental equation (1.8) can be written as

$$\gamma(\xi) - \hat{p}(\xi) = 0. \quad (4.2)$$

In the right half of the complex plane, equation (4.2) has exactly  $n$  solutions. To see this, consider a domain that is a half disk centered at 0, lying in the right half of the complex plane, and with a sufficiently large radius. For  $\text{Re } \xi \geq 0$ , we have  $|\hat{p}(\xi)| \leq 1$ . Because  $\gamma(\xi)$  has exactly  $n$  zeros and they are positive, our claim follows from *Rouché's theorem* if we can show that  $|\gamma(\xi)| > 1$  on the boundary of such a half disk. It is obvious from (4.1) that  $|\gamma(\xi)| > 1$  for  $|\xi|$  sufficiently large. Now, for  $\xi$  on the imaginary axis,  $\text{Re } \xi = 0$ , we have

$$|\gamma(\xi)| \geq \prod_{j=1}^n \frac{\lambda_j + \delta}{\lambda_j} > 1$$

also. We denote these  $n$  roots of (4.2) in the right half of the complex plane as  $\rho_1, \rho_2, \dots, \rho_n$ .

##### Example: Combination of Exponential Densities

Let us assume that the claim amount distribution is a *combination of exponential distributions*, say, with probability density function

$$p(x) = \sum_{i=1}^m A_i \beta_i e^{-\beta_i x}, \quad x > 0, \quad (4.3)$$

where  $0 < \beta_1 < \beta_2 < \dots < \beta_m$  and the  $A_i$ 's are nonzero numbers. Some of the  $A_i$ 's may be negative, as long as  $p(x) \geq 0$  for all  $x > 0$ . Thus,

$$\hat{p}(\xi) = \sum_{i=1}^m \frac{A_i \beta_i}{\beta_i + \xi}, \quad \text{Re } \xi > -\beta_1. \quad (4.4)$$

Although  $\hat{p}(\xi)$  does not exist for  $\text{Re } \xi \leq -\beta_1$ , the fundamental equation (4.2) can be analytically extended as

$$\prod_{j=1}^n \left[ \left( 1 + \frac{\delta}{\lambda_j} \right) - \frac{c}{\lambda_j} \xi \right] - \sum_{i=1}^m \frac{A_i \beta_i}{\beta_i + \xi} = 0 \quad (4.5)$$

for  $\xi$  in the complex plane with the points  $-\beta_1, -\beta_2, \dots, -\beta_n$  removed. As equation (4.5) is equivalent to a polynomial equation of degree  $n + m$ , it must have  $n + m$  roots. Thus, in the left side of the complex plane it has  $m$  roots, which will also be denoted as  $\rho$ 's. For  $p(x) \geq 0$ , we must have  $A_1 > 0$ . Hence, there is a root  $\rho_{n+1}$ , which is  $-R$  introduced at the end of Section 1, between  $-\beta_1$  and 0.

If  $A_1, \dots, A_m$  are positive numbers, that is, if  $p(x)$  is a *mixture* of exponential distributions, then these  $m$  roots are real and interweave with the negative of the exponential parameters,

$$-\beta_m < \rho_{n+m} < -\beta_{m-1} < \rho_{n+m-1} < \dots < -\beta_2 < \rho_{n+2} < -\beta_1 < \rho_{n+1} < 0. \tag{4.6}$$

See also Figure 13.6.2 of Bowers et al. (1997).

### 5. INTEGRO-DIFFERENTIAL EQUATION FOR $\phi(u)$

A main purpose of this section is to derive an integro-differential equation for the function  $\phi(u)$ , which is defined by (1.4). The function  $\phi(u)$  is the same as  $\phi_0(u)$  defined by (3.4), because of the memoryless property of the exponential distribution of  $W_1$ . With this integro-differential equation we can readily determine the Laplace transform of  $\phi(u)$ , which we shall do in Section 7.

Given the event  $A_j$  defined by (3.3), we calculate the conditional expectation of the discounted penalty by conditioning on the value of  $W_1 + \dots + W_{j+1} - \tau$ . Because the conditional distribution of this random variable is identical to the exponential distribution of  $W_{j+1}$ , we see that

$$\phi_j(u) = \lambda_{j+1} \int_0^\infty e^{-(\lambda_{j+1} + \delta)t} \phi_{j+1}(u + ct) dt, \quad j = 0, 1, 2, \dots, n - 2. \tag{5.1}$$

The reader might find it helpful to consult Figure 1 which depicts a justification of formula (5.1). Similarly,

$$\begin{aligned} \phi_{n-1}(u) &= \lambda_n \int_0^\infty e^{-(\lambda_n + \delta)t} \left[ \int_0^{u+ct} \phi(u + ct - x) p(x) dx \right] dt \\ &\quad + \lambda_n \int_0^\infty e^{-(\lambda_n + \delta)t} \left[ \int_{u+ct}^\infty \tau \omega(u + ct, x - u - ct) p(x) dx \right] dt, \end{aligned} \tag{5.2}$$

which should be compared with (1.5). With  $z = u + ct$ , equations (5.1) and (5.2) become

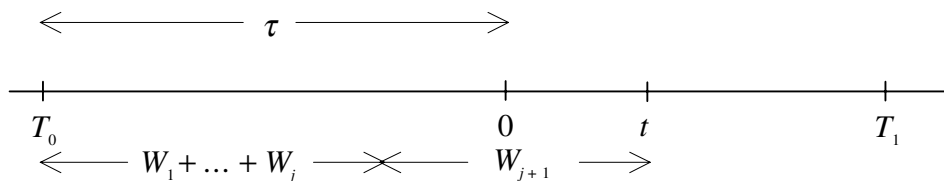
$$\phi_j(u) = \frac{\lambda_{j+1}}{c} \int_u^\infty e^{-(\lambda_{j+1} + \delta)(z-u)/c} \phi_{j+1}(z) dz \tag{5.3}$$

and

$$\begin{aligned} \phi_{n-1}(u) &= \frac{\lambda_n}{c} \int_u^\infty e^{-(\lambda_n + \delta)(z-u)/c} \left[ \int_0^z \phi(z - x) p(x) dx \right] dz \\ &\quad + \frac{\lambda_n}{c} \int_u^\infty e^{-(\lambda_n + \delta)(z-u)/c} \left[ \int_z^\infty \tau \omega(z, x - z) p(x) dx \right] dz, \end{aligned} \tag{5.4}$$

respectively. Differentiating (5.3) and (5.4) with respect to  $u$  yields

Figure 1  
Justification of Formula (5.1)



$$\phi_j'(u) = \frac{\lambda_{j+1} + \delta}{c} \phi_j(u) - \frac{\lambda_{j+1}}{c} \phi_{j+1}(u) \quad (5.5)$$

and

$$\phi_{n-1}'(u) = \frac{\lambda_n + \delta}{c} \phi_{n-1}(u) - \frac{\lambda_n}{c} \int_0^u \phi(u-x)p(x)dx - \frac{\lambda_n}{c} \int_u^\infty \varpi(u, x-u)p(x)dx, \quad (5.6)$$

respectively. Thus, we have a system of  $n$  equations for the  $n$  functions  $\phi(u), \phi_1(u), \dots, \phi_{n-1}(u)$ .

We can rewrite (5.5) as

$$\lambda_{j+1}\phi_{j+1}(u) = (\lambda_{j+1} + \delta)\phi_j(u) - c\phi_j'(u) = [(\lambda_{j+1} + \delta)I - cD]\phi_j(u),$$

or

$$\phi_{j+1}(u) = \left[ \left(1 + \frac{\delta}{\lambda_{j+1}}\right)I - \frac{c}{\lambda_{j+1}}D \right] \phi_j(u), \quad j = 0, 1, \dots, n-2, \quad (5.7)$$

where  $I$  and  $D$  denote the *identity operator* and *differentiation operator*, respectively.

It follows from successive substitution that, for  $0 \leq h < k \leq n-1$ ,

$$\phi_k(u) = \left\{ \prod_{j=h}^{k-1} \left[ \left(1 + \frac{\delta}{\lambda_{j+1}}\right)I - \frac{c}{\lambda_{j+1}}D \right] \right\} \phi_h(u). \quad (5.8)$$

Applying (5.8), with  $h = 0$  and  $k = n-1$ , to equation (5.6), we obtain the desired integro-differential equation:

$$\gamma(D)\phi(u) = \int_0^u \phi(u-x)p(x)dx + \int_u^\infty \varpi(u, x-u)p(x)dx, \quad u > 0, \quad (5.9)$$

where the  $n$ -th degree polynomial  $\gamma$  is defined by (4.1). As in Gerber and Shiu (1998), we let

$$\omega(u) = \int_u^\infty \varpi(u, x-u)p(x)dx = \int_0^\infty \varpi(u, y)p(u+y)dy. \quad (5.10)$$

Then the integro-differential equation (5.9) can be written as

$$\gamma(D)\phi = \phi * p + \omega. \quad (5.11)$$

With  $n = 2$  and  $\lambda_1 = \lambda_2 = \beta$ , equation (5.11) yields equation (2.2) of Cheng and Tang (2003) and equation (2.1) of Dickson and Hipp (2001), among others, as special cases. With  $\lambda_1 = \lambda_2 = \dots = \lambda_n = \beta$ , it also yields equation (6.2) of Dickson (1998) as a special case. In queuing theory, interarrival times have also been modeled as a sum of independent and exponentially distributed random variables. Hence, formulas similar to (5.11) can be found in the literature of queuing theory; see, for example, Cox and Smith (1961, p. 122, eq. 15) and Willmot (1999).

### Example Continued: Penalty as a Function of the Deficit Only

Here, we solve the integro-differential equation (5.11) for the case where the penalty at ruin is independent of the surplus immediately before ruin,

$$\varpi(x, y) = \varpi(y), \quad (5.12)$$

under the assumption that the claim amount distribution is a combination of exponential distributions. Recall that the assumption has been discussed in the example of the last section. With (5.12) and (4.3), we can evaluate (5.10),

$$\omega(u) = \int_0^\infty \varpi(y)p(u+y)dy = \sum_{i=1}^m A_i \beta_i \hat{\varpi}(\beta_i) e^{-\beta_i u}. \quad (5.13)$$

Thus, (5.11) becomes

$$\gamma(D)\phi(u) = (\phi * p)(u) + \sum_{i=1}^m A_i \beta_i \hat{\omega}(\beta_i) e^{-\beta_i u}. \quad (5.14)$$

Observe that, for an integrable function  $\varphi$ ,

$$D(\varphi(u) * e^{-\beta u}) = -\beta(\varphi(u) * e^{-\beta u}) + \varphi(u),$$

or

$$(\beta I + D)(\varphi(u) * e^{-\beta u}) = \varphi(u). \quad (5.15)$$

Define the  $m$ -th degree polynomial

$$\kappa(\xi) = \prod_{j=1}^m (\beta_j + \xi) \quad (5.16)$$

and consider the operator  $\kappa(D)$ . It follows from (5.15) and (4.3) that

$$\kappa(D)(\varphi * p) = \sum_{i=1}^m A_i \beta_i \left[ \prod_{j=1, j \neq i}^m (\beta_j I + D) \right] \varphi. \quad (5.17)$$

Thus, applying the operator  $\kappa(D)$  to (5.14) yields

$$\kappa(D)\gamma(D)\phi = \sum_{i=1}^m A_i \beta_i \left[ \prod_{j=1, j \neq i}^m (\beta_j I + D) \right] \phi + 0. \quad (5.18)$$

The polynomial  $\kappa(\xi)\gamma(\xi)$  is of degree  $n + m$ . Thus, (5.18) is a linear homogeneous differential equation (with constant coefficients) of degree  $n + m$  for the function  $\phi(u)$ . Consequently,  $\phi(u)$  is of the form

$$\phi(u) = \sum_{k=1}^{n+m} c_k e^{\xi_k u}, \quad (5.19)$$

where the  $\xi_k$ 's are the solutions of the *characteristic equation*

$$\kappa(\xi)\gamma(\xi) = \sum_{i=1}^m A_i \beta_i \prod_{j=1, j \neq i}^m (\beta_j + \xi), \quad (5.20)$$

or equivalently, the solutions of (4.5). (In writing (5.19), we have made the simplifying assumption that the  $\xi_k$ 's are distinct.) Hence, we may set  $\xi_k = \rho_k$ ,  $k = 1, 2, \dots, n + m$ . It remains to determine the  $c_k$ 's. Because  $\phi(u) \rightarrow 0$  for  $u \rightarrow \infty$ , the first  $n$  coefficients are zero, and (5.19) simplifies as

$$\phi(u) = \sum_{k=n+1}^{n+m} c_k e^{\rho_k u}. \quad (5.21)$$

Now we substitute (5.21) in equation (5.14). Because

$$\phi(u) * e^{-\beta u} = \sum_{k=n+1}^{n+m} c_k \frac{1}{\beta + \rho_k} (e^{\rho_k u} - e^{-\beta u}), \quad (5.22)$$

the substitution results in an equation that is linear with respect to the functions  $e^{\rho_{n+1}u}, \dots, e^{\rho_{n+m}u}, e^{-\beta_1 u}, \dots, e^{-\beta_m u}$ . These  $2m$  exponential functions are linearly independent. The coefficients of  $e^{\rho_{n+j}u}$  in this equation yield a condition that is automatically satisfied, because  $\rho_{n+j}$  is a solution of (4.5). From the coefficients of  $e^{-\beta_i u}$  we obtain

$$\sum_{k=n+1}^{n+m} \frac{1}{\beta_i + \rho_k} c_k = \hat{\omega}(\beta_i), \quad i = 1, 2, \dots, m,$$

or, more concisely,

$$C\mathbf{c} = \hat{\mathbf{w}}. \quad (5.23)$$

Here,  $C$  is an  $m$ -by- $m$  matrix with entry  $1/(\beta_i + \rho_{n+j})$  in its  $i$ - $j$  position, and  $\mathbf{c}$  and  $\hat{\mathbf{w}}$  are  $m$ -dimensional column vectors,

$$\mathbf{c} = (c_{n+1}, c_{n+2}, \dots, c_{n+m})^t, \quad (5.24)$$

$$\hat{\mathbf{w}} = (\hat{\omega}(\beta_1), \hat{\omega}(\beta_2), \dots, \hat{\omega}(\beta_m))^t. \quad (5.25)$$

It is remarkable that the structure of this system of equations is the same as in the classical case where  $n = 1$  and  $\delta = 0$ . The matrix  $C$  is called a *Cauchy matrix*. The appendix provides a review of some results needed in this paper. There,  $a_i$  is  $\beta_i$  and  $b_j$  is  $\rho_{n+j}$ .

If the  $2m$  numbers  $\rho_{n+1}, \rho_{n+2}, \dots, \rho_{n+m}, -\beta_1, -\beta_2, \dots, -\beta_m$  are distinct, which, by (4.6), is always the case when  $p(x)$  is a mixture of the exponential probability densities, then the matrix  $C$  exists and it is invertible. Hence,

$$\mathbf{c} = C^{-1} \hat{\mathbf{w}}, \quad (5.26)$$

and (5.21) becomes

$$\phi(\mathbf{u}) = (e^{\rho_{n+1}u}, e^{\rho_{n+2}u}, \dots, e^{\rho_{n+m}u}) C^{-1} \hat{\mathbf{w}}. \quad (5.27)$$

By (A2) in the appendix, the  $i$ - $j$  entry of  $C^{-1}$  is

$$\left( \prod_{h=1, h \neq i}^m \frac{\rho_{n+h} + \beta_j}{\rho_{n+h} - \rho_{n+i}} \right) (\beta_j + \rho_{n+i}) \left( \prod_{k=1, k \neq j}^m \frac{\beta_k + \rho_{n+i}}{\beta_k - \beta_j} \right), \quad (5.28)$$

applying which to (5.27) yields a closed-form expression for  $\phi(\mathbf{u})$ . Furthermore, it follows from formula (A10) that

$$\phi(0) = \sum_{k=1}^m \hat{\omega}(\beta_k) (\beta_k + \rho_{n+k}) \prod_{j=1, j \neq k}^m \frac{\beta_k + \rho_{n+j}}{\beta_k - \beta_j}. \quad (5.29)$$

Suppose that  $\omega(y)$  is the *Dirac delta function* with respect to  $y = y_0$ . Then  $\hat{\omega}(\beta_i) = e^{-\beta_i y_0}$ ,  $i = 1, 2, \dots, m$ . By (5.27) and (5.28) we obtain a closed-form expression for

$$\phi(\mathbf{u}) = f_{|U(T)|}(y_0|\mathbf{u}), \quad (5.30)$$

which is defined by (2.6). By (5.27),  $f_{|U(T)|}(y|\mathbf{u})$  is a sum of  $m^2$  terms. However, the special cases,  $f_{|U(T)|}(y|0)$  (which is also denoted as  $g(y)$  by [2.7]) and  $f_{|U(T)|}(0|\mathbf{u})$ , can be expressed with  $m$  terms. The function  $f_{|U(T)|}(y|0)$  [or  $g(y)$ ] is given by (5.29) with  $\hat{\omega}(\beta_k)$  replaced by  $e^{-\beta_k y}$ . Similarly, we have

$$f_{|U(T)|}(0|\mathbf{u}) = \sum_{k=1}^m e^{\rho_{n+k}u} (\beta_k + \rho_{n+k}) \prod_{j=1, j \neq k}^m \frac{\rho_{n+k} + \beta_j}{\rho_{n+k} - \rho_{n+j}}. \quad (5.31)$$

With  $u = y = 0$ , we can apply (A12) to obtain

$$g(0) = f_{|U(T)|}(0|0) = \sum_{j=1}^m (\beta_j + \rho_{n+j}), \quad (5.32)$$

which is quite an elegant formula.

Finally, consider the special case of a penalty function of the form

$$\omega(y) = e^{-\rho y}, \quad y > 0, \quad (5.33)$$

where  $\rho > -\beta_1$ . Then  $\hat{\omega}(\beta_i) = 1/(\beta_i + \rho)$ ,  $i = 1, 2, \dots, m$ , and

$$\phi(u) = \mathbb{E}[e^{-\delta T + \rho U(T)} | U(0) = u]. \tag{5.34}$$

It follows from (5.26) and (A3) that the coefficient  $c_{n+i}$  in (5.21) is

$$c_{n+i} = \left( \prod_{h=1, h \neq i}^m \frac{\rho_{n+h} - \rho}{\rho_{n+h} - \rho_{n+i}} \right) \left( \prod_{k=1}^m \frac{\beta_k + \rho_{n+i}}{\beta_k + \rho} \right). \tag{5.35}$$

The value of  $\phi(0)$  can be determined by summing (5.35) over  $i$  or, more simply, by applying the right-hand side of (A14). Let us consider two particular values of  $\rho$ :

(a) With  $\rho = 0$ , (5.34) is

$$\phi(u) = \mathbb{E}[e^{-\delta T} 1_{(T < \infty)} | U(0) = u],$$

the Laplace transform of  $T$ . The limiting case  $\delta \rightarrow 0$  gives the probability of ruin. Then, for  $n = 1$ , formula (5.35) can be found in Täcklind (1942); an elegant proof has been given by Chan (1990).

(b) With  $\rho = \rho_{n+1} = -R$ , formula (5.35) yields  $c_{n+1} = 1$  and  $c_{n+i} = 0$  for  $i = 2, 3, \dots, m$ . Hence,

$$\phi(u) = \mathbb{E}[e^{-\delta T - RU(T)} | U(0) = u] = e^{-Ru},$$

which is formula (1.9).

### 6. SYMMETRY OF $\phi(u, \tau)$

Unless  $\lambda_1 = \lambda_2 = \dots = \lambda_n$  or  $j = 0$ , the expectations  $\phi_j(u)$  and the probabilities  $\Pr(A_j)$  in formula (3.5) are not invariant with respect to permutations of  $\lambda_1, \lambda_2, \dots, \lambda_n$ . But of course the function  $\phi(u, \tau)$  must be invariant under these permutations. It is instructive to rewrite (3.5) in the form

$$\phi(u, \tau) = \phi(u) + \frac{1}{\Pr(V > \tau)} \sum_{j=1}^{n-1} [\phi_j(u) - \phi(u)] \Pr(A_j). \tag{6.1}$$

This shows that  $\phi(u, \tau)$  is  $\phi(u)$  plus a correction term that is a function of  $\tau$ . By (5.8), the sum on the right-hand side of (6.1) is

$$\sum_{j=1}^{n-1} \Pr(A_j) \left\{ \prod_{h=1}^j \left[ \left( 1 + \frac{\delta}{\lambda_h} \right) I - \frac{c}{\lambda_h} D \right] - I \right\} \phi(u), \tag{6.2}$$

which is a linear combination of  $\phi(u)$  and its derivatives up to order  $n - 1$ , with coefficients that are functions of  $\tau$ .

For an illustration, consider the case  $n = 2$ . Suppose  $\lambda_1 \neq \lambda_2$ . It follows from (3.6) and (3.9) that

$$\begin{aligned} \Pr(A_1) &= \Pr(W_1 + W_2 > \tau) - \Pr(W_1 > \tau) \\ &= \left( \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 \tau} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 \tau} \right) - e^{-\lambda_1 \tau} \\ &= \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_1 \tau} + \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{-\lambda_2 \tau}. \end{aligned} \tag{6.3}$$

Hence, (6.2) becomes

$$\left( \frac{e^{-\lambda_1 \tau}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 \tau}}{\lambda_1 - \lambda_2} \right) [\delta \phi(u) - c \phi'(u)], \tag{6.4}$$

which leads to

$$\phi(u, \tau) = \phi(u) + \frac{e^{-\lambda_1 \tau} - e^{-\lambda_2 \tau}}{\lambda_2 e^{-\lambda_1 \tau} - \lambda_1 e^{-\lambda_2 \tau}} [\delta \phi(u) - c \phi'(u)]. \tag{6.5}$$

The right-hand side of (6.5) is indeed symmetric in  $\lambda_1$  and  $\lambda_2$ . If  $\lambda_1 = \lambda_2 = \lambda$ , one can take limits in (6.5) or use (3.8) and (3.7) to see that

$$\phi(u, \tau) = \phi(u) + \frac{\tau}{1 + \lambda\tau} [\delta\phi(u) - c\phi'(u)]. \quad (6.6)$$

Let us examine the limit of  $\phi(u, \tau)$  for  $\tau \rightarrow \infty$ . For simplicity, we assume that  $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct. Because  $\phi(u, \tau)$  is invariant under permutations of  $\lambda_1, \lambda_2, \dots, \lambda_n$ , we may assume that  $\lambda_n < \lambda_j$  for  $j = 1, 2, \dots, n - 1$ . Then it follows from (3.6) and (3.9) that

$$\lim_{\tau \rightarrow \infty} \frac{\Pr(A_j)}{\Pr(V > \tau)} = \begin{cases} 0 & \text{if } j = 0, 1, \dots, n - 2, \\ 1 & \text{if } j = n - 1 \end{cases}, \quad (6.7)$$

and, hence,

$$\lim_{\tau \rightarrow \infty} \phi(u, \tau) = \phi_{n-1}(u) = \left\{ \prod_{h=1}^{n-1} \left[ \left( 1 + \frac{\delta}{\lambda_h} \right) I - \frac{c}{\lambda_h} D \right] \right\} \phi(u). \quad (6.8)$$

We end this section by showing how expression (6.2) can be evaluated in the case where  $\delta = 0$  and  $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct. Observe that (6.2) is a linear combination of  $e^{-\lambda_1\tau}, e^{-\lambda_2\tau}, \dots, e^{-\lambda_n\tau}$  and that (6.2) must be symmetric in  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Hence, it suffices to determine the term containing  $e^{-\lambda_n\tau}$ , which only appears in  $\Pr(A_{n-1})$ . It follows from (3.10) with  $j = n - 1$  that the term containing  $e^{-\lambda_n\tau}$  is

$$\begin{aligned} & \left( \prod_{i=1}^{n-1} \lambda_i \right) \frac{e^{-\lambda_n\tau}}{\prod_{i=1}^{n-1} (\lambda_i - \lambda_n)} \left\{ \prod_{h=1}^{n-1} \left[ I - \frac{c}{\lambda_h} D \right] - I \right\} \phi(u) \\ &= \frac{e^{-\lambda_n\tau}}{\prod_{i=1}^{n-1} (\lambda_i - \lambda_n)} \sum_{j=1}^{n-1} (-c)^j \sigma_{n-1-j} \phi^{(j)}(u). \end{aligned} \quad (6.9)$$

Here,

$$\begin{aligned} \sigma_0 &\equiv 1, \\ \sigma_1 &= \lambda_1 + \lambda_2 + \dots + \lambda_{n-1}, \\ \sigma_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-2}\lambda_{n-1}, \\ &\vdots \\ \sigma_{n-1} &= \lambda_1\lambda_2 \dots \lambda_{n-1} \end{aligned}$$

are the *elementary symmetric functions* of  $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ . (Note that  $\sigma_{n-1}$  is not used.) Finally, the term in expression (6.2) containing  $e^{-\lambda_n\tau}$  as a factor is obtained by the exchange  $\lambda_i \leftrightarrow \lambda_n, i = 1, 2, \dots, n - 1$ . We can readily check that, with  $n = 2$ , expression (6.9) implies formula (6.4) for  $\delta = 0$ .

## 7. THE LAPLACE TRANSFORM OF $\phi$

We now solve the integro-differential equation (5.11) in terms of Laplace transforms. Let  $f(u)$  be a function with Laplace transform  $\hat{f}(\xi)$ ; the Laplace transform of the  $k$ -th derivative  $f^{(k)}(u)$  is

$$\xi^k \hat{f}(\xi) - \xi^{k-1} f(0) - \xi^{k-2} f'(0) - \dots - f^{(k-1)}(0)$$

(Spiegel 1965, p. 10). Thus, the Laplace transform of (5.11) is

$$\gamma(\xi)\hat{\phi}(\xi) + q(\xi) = \hat{\phi}(\xi)\hat{p}(\xi) + \hat{\omega}(\xi), \quad \text{Re } \xi \geq 0, \quad (7.1)$$

where  $q(\xi)$  is a polynomial of degree  $n - 1$  or less, with coefficients in terms of  $\delta, c, \lambda_1, \lambda_2, \dots, \lambda_n$ , and the values of  $\phi(u)$  and its first  $n - 1$  derivatives at  $u = 0$ . It follows from (7.1) that

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\gamma(\xi) - \hat{p}(\xi)}, \quad \text{Re } \xi \geq 0. \quad (7.2)$$

Because  $\hat{\phi}(\xi)$  is finite for  $\text{Re } \xi \geq 0$ , the numerator on the right-hand side of (7.2) must be zero whenever the denominator is zero. We have shown in Section 4 that, in the right half of the complex plane, the function in the denominator of (7.2) has  $n$  zeros  $\rho_1, \rho_2, \dots, \rho_n$ . For simplicity, we assume that they are distinct. It follows that  $q(\xi)$  is the *collocation polynomial* of the function  $\hat{\omega}(\xi)$  with respect to these  $n$  points. Then, by the *Lagrange interpolation formula*, we obtain

$$q(\xi) = \sum_{j=1}^n \hat{\omega}(\rho_j) \prod_{k=1, k \neq j}^n \frac{\xi - \rho_k}{\rho_j - \rho_k}. \tag{7.3}$$

For  $n = 2$ , this expression can be found on the right-hand side of equation (8) in Lin (2003).

The Laplace transform of the function  $\phi(u)$  is given by formulas (7.2) and (7.3). In some cases, the function  $\phi$  can be determined by identifying the right-hand side of (7.2). Also, we have the *inversion formula*:

$$\phi(u) = \frac{1}{2\pi i} \lim_{y \rightarrow \infty} \int_{b-iy}^{b+iy} \hat{\phi}(\xi) e^{\xi u} d\xi, \tag{7.4}$$

where  $i = \sqrt{-1}$ , and the path of integration is parallel to the imaginary axis in the complex plane, with the real number  $b$  being chosen so that all the singularities of the integrand lie to the left of the line of integration (Spiegel 1965, p. 201). However, the integral can be difficult to calculate.

**Example Continued**

We now revisit the example in Section 5. It follows from (5.13) that

$$\hat{\omega}(\xi) = \sum_{i=1}^m A_i \beta_i \hat{\omega}(\beta_i) \frac{1}{\xi + \beta_i}. \tag{7.5}$$

Applying (4.4) and (7.5) to (7.2), we see that

$$\hat{\phi}(-\beta_i) = -\hat{\omega}(\beta_i), \quad i = 1, 2, \dots, m. \tag{7.6}$$

If we multiply the numerator and denominator in (7.2) by the product of the  $m$  factors,  $\prod(\xi + \beta_i)$ , then  $\hat{\phi}(\xi)$  is the ratio of a polynomial of degree  $n + m - 1$  (or less) to a polynomial of degree  $n + m$ . Thus,  $\hat{\phi}(\xi)$  is *proper*. Now, the zeros of the polynomial in the denominator are  $\rho_1, \rho_2, \dots, \rho_{n+m}$ , and the first  $n$  of these are identical to the zeros of the polynomial in the numerator. After canceling these  $n$  common factors, we see that  $\hat{\phi}(\xi)$  is a proper rational function, with  $\rho_{n+1}, \rho_{n+2}, \dots, \rho_{n+m}$  being the zeros of the polynomial in the denominator. Because of (7.6), we can identify  $\hat{\phi}(\xi)$  at  $m$  distinct values of  $\xi$ . Hence, analogous to the Lagrange interpolation formula, we have

$$\begin{aligned} \hat{\phi}(\xi) &= \sum_{j=1}^m \hat{\phi}(-\beta_j) \left( \prod_{h=1, h \neq j}^m \frac{\xi + \beta_h}{-\beta_j + \beta_h} \right) \left( \prod_{k=1}^m \frac{\rho_{n+k} + \beta_j}{\rho_{n+k} - \xi} \right) \\ &= \sum_{j=1}^m \hat{\omega}(\beta_j) \left( \prod_{h=1, h \neq j}^m \frac{\xi + \beta_j}{\beta_j - \beta_h} \right) \left( \prod_{k=1}^m \frac{\beta_j + \rho_{n+k}}{\xi - \rho_{n+k}} \right). \end{aligned} \tag{7.7}$$

Now the integral (7.4) can be evaluated. The result is given by the *Heaviside expansion formula* (Spiegel 1965, p. 73; Gerber and Shiu 1998, formula 6.54). Alternatively, we can identify  $\phi$  by applying the method of partial fractions to (7.7). For simplicity, we assume that  $\rho_{n+1}, \rho_{n+2}, \dots, \rho_{n+m}$  are distinct; then we can write (7.7) as

$$\hat{\phi}(\xi) = \sum_{k=1}^m b_k \frac{1}{\xi - \rho_{n+k}}, \tag{7.8}$$

which means that  $\phi(u)$  is of the same form as (5.21). To find the coefficient  $b_i$ , we multiply each of the right-hand sides of (7.7) and (7.8) by  $(\xi - \rho_{n+i})$ , cancel, and set  $\xi = \rho_{n+i}$ . Equating the results yields

$$b_i = \sum_{j=1}^m \hat{\omega}(\beta_j) \left( \prod_{h=1, h \neq j}^m \frac{\rho_{n+i} + \beta_h}{\beta_j - \beta_h} \right) \left( \prod_{k=1, k \neq i}^m \frac{\beta_j + \rho_{n+k}}{\rho_{n+i} - \rho_{n+k}} \right) (\beta_j + \rho_{n+i}), \quad (7.9)$$

which is in agreement with (5.27) and (5.28). Incidentally, this yields an alternative approach for deriving (A2) in the Appendix.

## 8. A KEY RESULT

It follows from (2.4), (7.2), and (7.3) that

$$\begin{aligned} \phi(0) &= \lim_{\xi \rightarrow \infty} \xi \frac{\hat{\omega}(\xi) - \sum_{j=1}^n \hat{\omega}(\rho_j) \prod_{k=1, k \neq j}^n \frac{\xi - \rho_k}{\rho_j - \rho_k}}{\gamma(\xi) - \hat{\rho}(\xi)} \\ &= \frac{- \sum_{j=1}^n \hat{\omega}(\rho_j) \prod_{k=1, k \neq j}^n \frac{1}{\rho_j - \rho_k}}{\prod_{j=0}^{n-1} \frac{-c}{\lambda_{j+1}}} \\ &= \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} \sum_{j=1}^n \hat{\omega}(\rho_j) \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j}. \end{aligned} \quad (8.1)$$

For  $n = 2$ , this is formula (6) in Lin (2003).

It follows from (5.10) that the Laplace transform of  $\omega$  is

$$\hat{\omega}(\rho) = \int_0^\infty \int_0^\infty e^{-\rho x} \omega(x, y) p(x + y) dx dy. \quad (8.2)$$

Applying (8.2) to (8.1) and then comparing the resulting formula with (2.3), we see that

$$f(x, y|0) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} p(x + y) \sum_{j=1}^n e^{-\rho_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j}. \quad (8.3)$$

For  $n = 1$ , this is equation (3.3) in Gerber and Shiu (1998).

In terms of divided differences, (8.1) can be rewritten as

$$\phi(0) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} (-1)^{n-1} \hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]. \quad (8.4)$$

Similarly, if we define  $\varepsilon_x(\rho) = e^{-\rho x}$ , then (8.3) can be rewritten as

$$f(x, y|0) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} p(x + y) (-1)^{n-1} \varepsilon_x[\rho_1, \rho_2, \dots, \rho_n]. \quad (8.5)$$

These formulas are meaningful even if some of the  $\rho$ 's coincide; see (3.16).

## 9. LI'S RENEWAL EQUATION

Equation (2.2), with  $f(x, y|0)$  given by (8.3) or (8.5), is a (defective) renewal equation for  $\phi$ . Li (2003, theorem 2) presented a seemingly different renewal equation for  $\phi$ ; the result can also be found in Li and Garrido (2004a). We now show that the two renewal equations are equivalent. For a number  $r$  with nonnegative real part,  $\text{Re } r \geq 0$ , and for an integrable function  $\varphi$ , let

$$(T_r \varphi)(y) = \int_y^\infty e^{-r(u-y)} \varphi(u) du = \int_0^\infty e^{-rx} \varphi(y + x) dx, \quad y \geq 0. \quad (9.1)$$

Thus,  $(T_r \varphi)(y)$  is the Laplace transform, for argument  $r$ , of the translated function  $\varphi(y + \cdot)$ . *Li's renewal equation* is

$$\phi = \phi^*(Sp) + S\omega, \tag{9.2}$$

where the operator S is defined by

$$S = \frac{\lambda_1 \lambda_2 \dots \lambda_n}{c^n} \prod_{j=1}^n T_{\rho_j}. \tag{9.3}$$

For  $n = 1$ , (9.2) is (2.34) of Gerber and Shiu (1998). For  $n = 2$ , (9.2) is (19) of Lin (2003). For  $n = 2$  and  $\varpi \equiv 1$ , (9.2) is (5.1) of Dickson and Hipp (2001).

Let E denote the *translation operator* that actuaries know from *finite differences*. Since  $E^x \varphi(y) = \varphi(y + x)$ , we have the operator equation

$$T_r = \int_0^\infty e^{-rx} E^x dx. \tag{9.4}$$

Actuarial authors such as Steffensen (1950, p. 186) and Freeman (1960, p. 127) have noted that Taylor’s formula leads to the operator identity

$$E^x = e^{xD}, \tag{9.5}$$

with which (9.4) becomes

$$T_r = \int_0^\infty e^{-x(rI-D)} dx = (rI - D)^{-1}. \tag{9.6}$$

A rigorous discussion of (9.6) can be found in Butzer and Berens (1967, sec. 1.3.3). It is shown that the operator equation is valid under the assumption that  $\text{Re } r > 0$  and that the domain of the operators is the Banach space of bounded, uniformly continuous functions on  $[0, \infty)$ . As an operator-valued function of  $r$ ,  $T_r$  is called the *resolvent* of the operator D. Because the differentiation operator D is an unbounded operator, some authors would call  $T_r$  a *pseudo-resolvent* (Hille and Phillips 1957). The beginning words in Chapter XIII of Feller (1966) are “The Laplace transforms are a powerful practical tool, but at the same time their theory is of intrinsic value and opens the door to other theories such as semi-groups. . . . Resolvents are basic for semi-group theory.”

Formula (9.6) leads to three observations. The first is that  $T_r T_s = T_s T_r$  and, hence, the product of operators  $\prod_{j=1}^n T_{\rho_j}$  in (9.3) is unambiguous. The second observation is that the partial fraction formula

$$\prod_{j=1}^n \frac{1}{\rho_j - z} = \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) \frac{1}{\rho_j - z} \tag{9.7}$$

can be translated into the operator identity

$$\prod_{j=1}^n T_{\rho_j} = \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j}. \tag{9.8}$$

When  $n = 2$ , such an equation is called the (*first*) *resolvent equation*, which “is of great importance in analysis” (Hille and Phillips 1957, p. 126). If the  $\rho$ ’s are not distinct, then we need to modify the partial fraction formula above. In this case, our third observation becomes relevant: the  $k$ -fold integral arising from applying the operator  $T_\rho$   $k$  times can be reduced to a single integral. It follows from

$$\frac{\partial^{k-1}}{\partial \rho^{k-1}} \frac{1}{\rho - z} = (-1)^{k-1} (k - 1)! \left( \frac{1}{\rho - z} \right)^k$$

that

$$T_\rho^k = \frac{(-1)^{k-1}}{(k - 1)!} \frac{\partial^{k-1}}{\partial \rho^{k-1}} T_\rho.$$

Thus,

$$(T_{\rho}^k \varphi)(y) = \frac{(-1)^{k-1}}{(k-1)!} \frac{\partial^{k-1}}{\partial \rho^{k-1}} \int_y^{\infty} e^{-\rho(u-y)} \varphi(u) du = \frac{1}{(k-1)!} \int_y^{\infty} e^{-\rho(u-y)} (u-y)^{k-1} \varphi(u) du.$$

We are now ready to show that (2.2) is the same as (9.2). As pointed out immediately after formula (2.7), the first integral on the right-hand side of (2.2) is  $\phi * g$ . Thus, we are to show that

$$Sp = g. \quad (9.9)$$

Because

$$\int_0^{\infty} e^{-\rho x} p(x+y) dx = (T_{\rho} p)(y),$$

it follows from formulas (2.7), (2.6), and (8.3) that

$$g(y) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} p \right](y), \quad (9.10)$$

which is  $(Sp)(y)$  by (9.8) and (9.3). Hence, we have proved (9.9).

It remains to check that the last integral in (2.2),

$$\int_0^{\infty} \int_0^{\infty} \varpi(x+u, s) f(x, s+u|0) dx ds, \quad (9.11)$$

is the same as  $(S\omega)(u)$ ,  $u \geq 0$ . By (8.3),

$$f(x, s+u|0) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} p(x+s+u) \sum_{j=1}^n e^{-\rho_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j}.$$

From (5.10),

$$\int_0^{\infty} \varpi(x+u, s) p(x+s+u) ds = \omega(x+u).$$

From (9.1),

$$\int_0^{\infty} \omega(x+u) e^{-\rho x} dx = (T_{\rho} \omega)(u).$$

Hence, the integral (9.11) is, indeed,  $(S\omega)(u)$ . This finishes the proof that the renewal equation (2.2) and Li's renewal equation (9.2) are the same.

#### REMARK

With  $p(x)$  satisfying (4.3), we have

$$\left( \prod_{j=1}^n T_{\rho_j} p \right)(y) = \sum_{i=1}^m A_i \beta_i e^{-\beta_i y} \prod_{j=1}^n \frac{1}{\beta_i + \rho_j};$$

hence,

$$g(y) = (Sp)(y) = \frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} \sum_{i=1}^m A_i \beta_i e^{-\beta_i y} \prod_{j=1}^n \frac{1}{\beta_i + \rho_j}. \quad (9.12)$$

Comparing the coefficient of  $e^{-\beta_k y}$  in (9.13) with the coefficient of  $\hat{\varpi}(\beta_k)$  in (5.29), we obtain the identity

$$\frac{\lambda_1 \lambda_2 \cdots \lambda_n}{c^n} A_k \beta_k \prod_{j=1}^n \frac{1}{\beta_k + \rho_j} = \frac{\prod_{1 \leq j \leq m} (\beta_k + \rho_{n+j})}{\prod_{1 \leq i \leq m, i \neq k} (\beta_k - \beta_i)}. \quad (9.13)$$

For the special case of  $\delta = 0$  and  $n = 1$ , this identity has been verified in Dufresne and Gerber (1988, sec. 5).

## 10. DIRECT DERIVATION OF LI'S RENEWAL EQUATION

We now show how the integro-differential equation (5.11) can be converted into Li's renewal equation (9.2) in  $n$  successive steps. In each step, we shall apply the following two results:

(i) It follows from (9.1) that  $(T_r\varphi)(0) = \hat{\varphi}(r)$ ; thus, by (9.8) and (3.13),

$$\left[ \left( \prod_{j=1}^m T_{r_j} \right) \varphi \right] (0) = (-1)^{m-1} \hat{\varphi}[r_1, r_2, \dots, r_m]. \quad (10.1)$$

(ii) By interchanging the order of integration, we have

$$T_r(\varphi * \eta) = \varphi * (T_r\eta) + (T_r\eta)(0) (T_r\varphi). \quad (10.2)$$

For the first step, we follow (3.12a) and write

$$\gamma(s) = \gamma(\rho_1) + (s - \rho_1)\gamma[\rho_1, s]. \quad (10.3)$$

Because  $\gamma(s)$  is a polynomial of degree  $n$ , the function  $\gamma[\rho_1, s]$  is a polynomial of degree  $n - 1$  in  $s$ . We replace the operator  $\gamma(D)$  in (5.11) according to (10.3), and apply the operator  $T_{\rho_1}$  on both sides, yielding the equation

$$\gamma(\rho_1)T_{\rho_1}\phi - \gamma[\rho_1, D]\phi = T_{\rho_1}(\phi * p) + T_{\rho_1}\omega. \quad (10.4)$$

Because  $\gamma(\rho_1) = \hat{p}(\rho_1) = (T_{\rho_1}p)(0)$ , we can apply (10.2), with  $\varphi = \phi$ ,  $\eta = p$  and  $r = \rho_1$ , to (10.4) and obtain

$$-\gamma[\rho_1, D]\phi = \phi * (T_{\rho_1}p) + T_{\rho_1}\omega. \quad (10.5)$$

For the second step, we write

$$\gamma[\rho_1, s] = \gamma[\rho_1, \rho_2] + (s - \rho_2)\gamma[\rho_1, \rho_2, s]. \quad (10.6)$$

The function  $\gamma[\rho_1, \rho_2, s]$  is a polynomial of degree  $n - 2$  in  $s$ . We replace the operator  $\gamma[\rho_1, D]$  in (10.5) according to (10.6), and apply the operator  $T_{\rho_2}$  to both sides, yielding the equation

$$-\gamma[\rho_1, \rho_2]T_{\rho_2}\phi + \gamma[\rho_1, \rho_2, D]\phi = T_{\rho_2}[\phi * (T_{\rho_1}p)] + T_{\rho_2}T_{\rho_1}\omega. \quad (10.7)$$

Because  $\gamma[\rho_1, \rho_2] = \hat{p}[\rho_1, \rho_2] = -(T_{\rho_2}T_{\rho_1}p)(0)$  by (10.1), we can apply (10.2), with  $\varphi = \phi$ ,  $\eta = T_{\rho_1}p$  and  $r = \rho_2$ , to (10.7) and obtain

$$\gamma[\rho_1, \rho_2, D]\phi = \phi * (T_{\rho_2}T_{\rho_1}p) + T_{\rho_2}T_{\rho_1}\omega. \quad (10.8)$$

Continuing this way, we find for  $j = 1, 2, \dots, n$  the equations

$$(-1)^j \gamma[\rho_1, \rho_2, \dots, \rho_j, D]\phi = \phi * (T_{\rho_j} \cdots T_{\rho_1}p) + T_{\rho_j} \cdots T_{\rho_1}\omega. \quad (10.9)$$

As  $\gamma(s)$  is a polynomial of degree  $n$ , the  $n$ -th divided difference  $\gamma[\rho_1, \dots, \rho_n, s]$  is the coefficient of  $s^n$ ,

$$\gamma[\rho_1, \dots, \rho_n, s] = \prod_{j=1}^n \frac{-c}{\lambda_j}. \quad (10.10)$$

Thus,

$$(-1)^n \gamma[\rho_1, \rho_2, \dots, \rho_n, D] = \frac{c^n}{\lambda_1 \lambda_2 \cdots \lambda_n} I,$$

and equation (10.9), with  $j = n$ , is indeed Li's renewal equation (9.2).

## 11. SOME RESULTS BY DIVIDED DIFFERENCES

We conclude this paper with some results derived by means of divided differences. The Laplace transform of (9.2) is

$$\hat{\phi}(\xi) = \hat{\phi}(\xi)(T_{\xi}Sp)(0) + (T_{\xi}S\omega)(0). \quad (11.1)$$

By (9.3) and (10.1),

$$(T_{\xi}S\omega)(0) = \frac{\lambda_1\lambda_2 \cdots \lambda_n}{c^n} (-1)^n \hat{\omega}[\xi, \rho_1, \dots, \rho_n], \quad (11.2)$$

which cannot be further simplified without explicit knowledge of  $\omega$ . Similarly,

$$(T_{\xi}Sp)(0) = \frac{\lambda_1\lambda_2 \cdots \lambda_n}{c^n} (-1)^n \hat{p}[\xi, \rho_1, \dots, \rho_n], \quad (11.3)$$

but this can be further evaluated because  $\hat{p}$  coincides with the polynomial  $\gamma$  at the points  $\rho_1, \rho_2, \dots, \rho_n$ . It follows from the identity  $\hat{p} = \gamma + (\hat{p} - \gamma)$  that

$$\hat{p}[\xi, \rho_1, \dots, \rho_n] = \gamma[\xi, \rho_1, \dots, \rho_n] + (\hat{p} - \gamma)[\xi, \rho_1, \dots, \rho_n].$$

By (10.10),

$$\gamma[\xi, \rho_1, \dots, \rho_n] = \prod_{j=1}^n \frac{-c}{\lambda_j},$$

and by (3.13),

$$(\hat{p} - \gamma)[\xi, \rho_1, \dots, \rho_n] = [\hat{p}(\xi) - \gamma(\xi)] \prod_{k=1}^n \frac{1}{\xi - \rho_k}.$$

Combining these formulas, we have

$$\hat{g}(\xi) = (T_{\xi}Sp)(0) = 1 + \frac{\lambda_1\lambda_2 \cdots \lambda_n}{c^n(\rho_1 - \xi)(\rho_2 - \xi) \cdots (\rho_n - \xi)} [\hat{p}(\xi) - \gamma(\xi)]. \quad (11.4)$$

Two interesting results immediately follow from formula (11.4). The first result is that, for  $\delta > 0$ ,

$$\begin{aligned} \mathbb{E}[e^{-\delta T} 1_{(T < \infty)} | U(0) = 0] &= \int_0^{\infty} g(y) dy \\ &= \hat{g}(0) \\ &= 1 + \frac{\lambda_1\lambda_2 \cdots \lambda_n}{c^n \rho_1 \rho_2 \cdots \rho_n} \left[ 1 - \prod_{j=1}^n \left( 1 + \frac{\delta}{\lambda_j} \right) \right] \\ &= 1 - \frac{\prod_{k=1}^n (\lambda_k + \delta) - \lambda_1\lambda_2 \cdots \lambda_n}{c^n \rho_1 \rho_2 \cdots \rho_n}, \end{aligned} \quad (11.5)$$

which is formula (10) in Li (2003). For  $n = 1$ , this is formula (3.9) of Gerber and Shiu (1998). For  $n = 2$ , it is the last formula in Dickson and Hipp (2001). The last part of Theorem 4 in Li and Garrido (2004a) evaluates the limit  $\delta \downarrow 0$  in the Erlang( $n$ ) case.

The second result is that to solve for a negative  $\xi$  satisfying the equation  $\hat{g}(\xi) = 1$  is equivalent to solve for a negative  $\xi$ , satisfying (4.2). For  $n = 1$ , this equivalence has been pointed out in Gerber and Shiu (1998, p. 54, remark v). Such a  $\xi$ , the negative of which is the adjustment coefficient  $R$ , is needed for obtaining an *asymptotic formula* for  $\phi(u)$ ; see Gerber and Shiu (1998, sec. 4).

**REMARK**

For the example in Section 5, there are two alternative expressions for (11.5). One is the summation of (5.35), with  $\rho = 0$ , over  $i$ . The other is a translation of the right-hand side of (A14) with  $b = 0$ , namely,

$$\mathbb{E}[e^{-\delta T} 1_{(T < \infty)} | U(0) = 0] = \sum_{k=1}^m \left( 1 + \frac{\beta_k}{\rho_{n+k}} \right) \prod_{j=1, j \neq k}^m \frac{\beta_k + \rho_{n+j}}{\beta_k - \beta_j}. \tag{11.6}$$

In the limit  $\delta \rightarrow 0$ , these expressions evaluate the probability of ultimate ruin,  $\psi(0, 0)$ , defined in the introduction section of this paper. For  $n = 1$ , Gerber and Shiu (1998, p. 55) pointed out that, in the limit, the right-hand side of (11.5) becomes  $c\mathbb{E}[X]/\lambda_1$ , a well-known result. On the other hand, the two alternative expressions do not change in form in the limit  $\delta \rightarrow 0$ .

**APPENDIX**

For the convenience of the reader, this Appendix presents some known results concerning the Cauchy matrix. Let  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_m$  be  $2m$  numbers such that  $a_i + b_j \neq 0$  for all  $i$  and  $j$ . Let  $C$  be the  $m$ -by- $m$  matrix whose entry at the  $i$ - $j$  position is  $1/(a_i + b_j)$ . Cauchy published the formula for its determinant in 1841 (Pólya and Szegő 1976, p. 279, #3),

$$\det C = \frac{\prod_{i < j} (a_i - a_j)(b_i - b_j)}{\prod_{i,j} (a_i + b_j)}. \tag{A1}$$

A proof of (A1) can be found in Knuth (1973, p. 473, #38), Pólya and Szegő (1976, p. 279, #3), and Prasolov (1994, p. 3). Thus, both  $C$  and  $C^{-1}$  exist if, and only if, the  $2m$  numbers,  $a_1, a_2, \dots, a_m, -b_1, -b_2, \dots, -b_m$ , are distinct.

If a row and a column are removed from a Cauchy matrix, what remains is still a Cauchy matrix. Hence, the *cofactors* of a Cauchy matrix can be calculated by a modification of formula (A1), and there is an explicit expression for the inverse of  $C$ . The  $i$ - $j$  entry of  $C^{-1}$  is

$$\left( \prod_{h=1, h \neq i}^m \frac{b_h + a_j}{b_h - b_i} \right) (a_j + b_i) \left( \prod_{k=1, k \neq j}^m \frac{a_k + b_i}{a_k - a_j} \right); \tag{A2}$$

see also Knuth (1973, p. 36, #41).

Because of formula (5.27), we are interested in evaluating expressions of the form  $\mathbf{x}^t C^{-1} \mathbf{y}$ , where  $\mathbf{x}$  and  $\mathbf{y}$  are  $m$ -dimensional column vectors. This is a sum with  $m^2$  terms. In some cases, this sum can be simplified to one with substantially fewer terms. One case is where  $\mathbf{y}$  is a vector whose  $i$ -th entry is (proportional to)  $1/(a_i + b)$ . Then  $C^{-1} \mathbf{y}$  can be readily calculated using *Cramer's rule*. The denominator is given by (A1). For the  $i$ -th entry of  $C^{-1} \mathbf{y}$ , the numerator is also given by (A1) but with  $b_i$  replaced by  $b$ . Thus, after cancellation, the  $i$ -th entry of  $C^{-1} \mathbf{y}$  is

$$\left( \prod_{h=1, h \neq i}^m \frac{b_h - b}{b_h - b_i} \right) \left( \prod_{k=1}^m \frac{a_k + b_i}{a_k + b} \right). \tag{A3}$$

By symmetry, there is also a simplified formula for  $\mathbf{x}^t C^{-1}$  if the  $j$ -th entry of  $\mathbf{x}$  is (proportional to)  $1/(a + b_j)$ , but it has no applications in this paper.

Another case where  $\mathbf{x}^t C^{-1} \mathbf{y}$  can be simplified is when  $\mathbf{x}$  or  $\mathbf{y}$  is (proportional to) the vector  $(1, 1, \dots, 1)^t$ . Let us now evaluate  $\mathbf{x}^t C^{-1}$ . That is, we are to determine

$$(d_1, d_2, \dots, d_m) = (1, 1, \dots, 1) C^{-1}, \tag{A4}$$

which is equivalent to solving the system of linear equations

$$\sum_{i=1}^m \frac{1}{a_i + b_j} d_i = 1, \quad j = 1, 2, \dots, m. \tag{A5}$$

Now, consider the rational function

$$Q(z) = \sum_{i=1}^m \frac{d_i}{\alpha_i + z}. \quad (\text{A6})$$

From (A5) it follows that  $Q(b_j) = 1, j = 1, 2, \dots, m$ . Write (A6) as

$$Q(z) = \frac{R(z)}{\prod_{1 \leq i \leq m} (\alpha_i + z)}. \quad (\text{A7})$$

Then  $R(z)$  is the unique polynomial of degree  $m-1$  or less, with  $R(b_j) = \prod_{1 \leq i \leq m} (\alpha_i + b_j), j = 1, 2, \dots, m$ . Thus,

$$R(z) = \prod_{1 \leq i \leq m} (z + \alpha_i) - \prod_{1 \leq j \leq m} (z - b_j). \quad (\text{A8})$$

Note that the leading term of  $R(z)$  is

$$\left[ \sum_{1 \leq j \leq m} (\alpha_j + b_j) \right] z^{m-1}. \quad (\text{A9})$$

To find  $d_k$ , we multiply each of the right-hand sides of (A6) and (A7) by  $(\alpha_k + z)$ , cancel, and set  $z = -\alpha_k$ . Equating the results yields

$$d_k = \frac{R(-\alpha_k)}{\prod_{1 \leq i \leq m, i \neq k} (\alpha_i - \alpha_k)} = \frac{-\prod_{1 \leq j \leq m} (-\alpha_k - b_j)}{\prod_{1 \leq i \leq m, i \neq k} (\alpha_i - \alpha_k)} = (\alpha_k + b_k) \prod_{j=1, j \neq k}^m \frac{\alpha_k + b_j}{\alpha_k - \alpha_j}. \quad (\text{A10})$$

Formula (A10) matches with formula (15) in Dufresne and Gerber (1991). Furthermore, if we multiply each of the right-hand sides of (A6) and (A7) with  $z$  and let  $z$  tend to  $\infty$ , we have

$$\sum_{1 \leq i \leq m} d_i = \sum_{1 \leq j \leq m} (\alpha_j + b_j) \quad (\text{A11})$$

by noting (A9). From (A4), we see that  $d_i$  is the sum of the  $m$  entries in the  $i$ -th column of  $C^{-1}$ . Thus, (A11) is a formula for the sum of *all* entries of  $C^{-1}$ . To facilitate obtaining (5.32), we write this result as

$$(1, 1, \dots, 1) C^{-1} (1, 1, \dots, 1)^t = \sum_{1 \leq j \leq m} (\alpha_j + b_j). \quad (\text{A12})$$

Finally, evaluating

$$(1, 1, \dots, 1) C^{-1} (1/(\alpha_1 + b), 1/(\alpha_2 + b), \dots, 1/(\alpha_m + b))^t \quad (\text{A13})$$

in two ways—by first applying (A3) or by first applying (A10)—yields the identity

$$\sum_{i=1}^m \left( \prod_{h=1, h \neq i}^m \frac{b_h - b}{b_h - b_i} \right) \left( \prod_{k=1}^m \frac{\alpha_k + b_i}{\alpha_k + b} \right) = \sum_{k=1}^m \frac{\alpha_k + b_k}{\alpha_k + b} \prod_{j=1, j \neq k}^m \frac{\alpha_k + b_j}{\alpha_k - \alpha_j}. \quad (\text{A14})$$

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## DISCUSSIONS

### HANSPETER SCHMIDLİ\*

Professors Gerber and Shiu have again produced a very interesting paper. It is particularly surprising how many explicit formulae they were able to obtain. Of course, the results cannot be as simple as for

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\* Hanspeter Schmidli, Dr. scient., Dr. sc. math., is a Professor of Actuarial Sciences, Laboratory of Actuarial Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark, schmidli@act.ku.dk.

the classical risk model, but the proofs are very elegant. Without any doubt the paper will inspire a lot of further research.

Some of the results of the paper can be extended to the case where the common distribution of the interclaim times is phase-type,  $\mathbb{P}[V > \tau] = \boldsymbol{\alpha} \exp\{\tau \mathbf{B}\} \mathbf{e}^\top$ . This discussion uses the notation of Section 8.2 of Rolski et al. (1999) and let  $n = \ell$ . The model considered in the paper is then a special case with

$$\mathbf{B} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_n \end{pmatrix}, \quad \boldsymbol{\alpha} = (1, 0, \dots, 0).$$

We can then define the events  $A_j = \{X(\tau) = j + 1\}$ ,  $j = 0, 1, \dots, n - 1$ . This yields

$$(\mathbb{P}[A_0|V > \tau], \mathbb{P}[A_1|V > \tau], \dots, \mathbb{P}[A_{n-1}|V > \tau]) = \frac{\boldsymbol{\alpha} \exp\{\tau \mathbf{B}\}}{\boldsymbol{\alpha} \exp\{\tau \mathbf{B}\} \mathbf{e}^\top},$$

where  $\mathbf{e} = (1, 1, \dots, 1)$ . Recall that  $\mathbf{b}^\top = -\mathbf{B}\mathbf{e}^\top$  are the intensities for jumping to the absorbing state.

The Laplace-Stieltjes transform of a matrix exponential distribution is a rational function. The denominator is of degree  $n$ ; the numerator is uppermost of degree  $n - 1$ . Lundberg's fundamental equation can be written as in (4.2) of the paper with  $\gamma(\xi) = (\hat{f}_V(\delta - c\xi))^{-1}$ , where we now consider the rational function. The function  $|\gamma(\xi)|$  tends to infinity as either  $\Re(\xi)$  or  $|\Im(\xi)|$  tends to infinity. It also tends to infinity as it approaches one of the uppermost  $n - 1$  poles. It is therefore possible to find closed connected sets on the right half of the complex plane not containing the poles of  $\gamma(\xi)$  such that  $|\gamma(\xi)| > 1$  on the boundary. We can therefore, again from Rouché's theorem, conclude that Lundberg's equation has exactly  $n$  roots on the right half of the complex plane. For simplicity we again assume that the  $n$  roots are distinct.

Let us now introduce the functions

$$\phi_j(u) = \mathbb{E}[e^{-\delta\tau} \varpi(U(T-), -U(T)) \mathbb{1}_{T < \infty} | U(0) = u, X(0) = j + 1], \quad j = 0, 1, \dots, n - 1.$$

If the claim size distribution is continuous (e.g., absolutely continuous) and the function  $\varpi(x, y)$  is continuous with respect to  $x$ , then the functions  $\phi_j(u)$  are differentiable. They fulfill the equations

$$c\phi'_{i-1}(u) = \delta\phi_{i-1}(u) - \sum_{j=1}^n b_j \phi_{j-1}(u) - b_i \sum_{j=1}^n \alpha_j \left( \int_0^u \phi_{j-1}(u-x)p(x)dx + \int_u^\infty \varpi(u, x-u)p(x)dx \right),$$

which generalize the integro-differential equations in Section 5 of the paper. In matrix notation the equations are

$$c\boldsymbol{\Phi}'(u) = \delta\boldsymbol{\Phi}(u) - \mathbf{B}\boldsymbol{\Phi}(u) - \left( \int_0^u \boldsymbol{\alpha}\boldsymbol{\Phi}(u-x)p(x)dx + \int_u^\infty \varpi(u, x-u)p(x)dx \right) \mathbf{b}^\top,$$

where  $\boldsymbol{\Phi}(u) = (\phi_0(u), \phi_1(u), \dots, \phi_{n-1}(u))^\top$ . It is now possible to find the Laplace transforms of the functions  $\phi_j(u)$ . The boundary conditions  $\phi_j(0)$  are obtained by observing that  $\hat{\phi}_j(\xi)$  exists also at the  $n$  roots of Lundberg's equation with positive real part. In the special case where the claim size distribution is also phase-type and  $\varpi(x, y) = e^{-\kappa x}$  is a function of  $x$  only more explicit, results can be obtained by inversion of the Laplace transform because the Laplace transform of  $\phi_i(u)$  is a rational function. In particular, it follows that the distribution of  $U(T-)$  is phase-type. This implies that the distribution of  $-U(T)$  is phase-type also.

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**HANSJÖRG ALBRECHER\***

This interesting paper of Professors Gerber and Shiu provides an elegant approach to the study of several quantities related to the risk process in a Sparre Andersen model. I would like to comment on the fact that some of the results of the paper can be extended to a more general framework including models with a Markovian dependence structure among interclaim times and claim sizes. Let  $\{Z_n, n \geq 0\}$  denote an irreducible discrete-time Markov chain with state space  $\{1, \dots, M\}$  and transition matrix  $P = ((p_{ij}), 1 \leq i, j \leq M)$  and consider a Markov additive risk model with

$$\begin{aligned} \mathbb{P}(V_{n+1} \leq t, X_{n+1} \leq x, Z_{n+1} = j | Z_n = i, (V_r, X_r, Z_r), 0 \leq r \leq n) \\ = \mathbb{P}(V_1 \leq t, X_1 \leq x, Z_1 = j | Z_0 = i) = (1 - e^{-\lambda_j t}) p_{ij} B_j(x), \end{aligned} \tag{1}$$

where, as in the paper,  $V_n$  denotes the  $n$ -th interclaim time and  $X_n$  denotes the size of the  $n$ -th claim, respectively (with the convention  $V_0 = X_0 = 0$  almost surely). Thus, at each instant of a claim the Markov chain jumps from state  $i$  to a state  $j$ , and the distribution  $B_j$  of the claim depends on the new state  $j$ . Then the next interarrival time is exponentially distributed with parameter  $\lambda_j$ . Note that given the states  $Z_{n-1}$  and  $Z_n$ , the quantities  $V_n$  and  $X_n$  are independent, but there is autocorrelation among consecutive claim sizes and among consecutive interclaim times as well as cross-correlation between  $V_n$  and  $X_n$ .

This model is investigated in detail in Albrecher and Boxma (2004), where it is shown that the Laplace transform  $\hat{\phi}_{\delta,i}(\xi)$  of the discounted penalty function  $\phi_{\delta,i}(u)$  (with initial capital  $u$  and  $Z_0 = i$ ) is given by the system of linear equations

$$A_\delta(\xi) \vec{\hat{\phi}}_\delta(\xi) = c \vec{\phi}_\delta(0) - \Lambda P \vec{\hat{\omega}}(\xi),$$

with  $\vec{\hat{\phi}}_\delta(\xi) = (\hat{\phi}_{\delta,1}(\xi), \dots, \hat{\phi}_{\delta,M}(\xi))$ ,  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_M)$ , and  $\vec{\hat{\omega}}(\xi) = (\hat{\omega}_1(\xi), \dots, \hat{\omega}_M(\xi))$ , where  $\hat{\omega}_i(\xi) = \int_{x=0}^\infty e^{-sx} \int_x^\infty \varpi(x, y - x) dB_i(y) dx$  ( $i = 1, \dots, M$ ). The matrix  $A_\delta(\xi)$  is given by

$$A_\delta(\xi) := (c\xi - \delta) I - \Lambda + \Lambda P \hat{B}(\xi),$$

where  $I$  is the identity matrix and  $\hat{B}(\xi) = \text{diag}(\hat{b}_1(\xi), \dots, \hat{b}_M(\xi))$  with  $\hat{b}_i(\xi)$  denoting the Laplace transform of claim size distribution  $B_i$ . The equation

$$\det A_\delta(\xi) = 0 \tag{2}$$

generalizes the fundamental Lundberg equation (1.8) of the paper and can be shown to have exactly  $M$  zeroes  $\xi_1, \dots, \xi_M$  in the positive halfplane (cf. Albrecher and Boxma). These zeroes can be used to determine the quantities  $\phi_{\delta,i}(0)$  by finding a nontrivial solution  $\vec{k}_i$  of

$$A_\delta^T(\xi_i) \vec{k}_i = \vec{0} \tag{3}$$

for each  $i = 1, \dots, M$ . Since we then have

$$0 = \vec{\hat{\phi}}_\delta(\xi_i)^T A_\delta^T(\xi_i) \vec{k}_i = (c \vec{\phi}_\delta(0) - \Lambda P \vec{\hat{\omega}}(\xi_i))^T \vec{k}_i, \tag{4}$$

this gives  $M$  linear equations for  $\phi_{\delta,1}(0), \dots, \phi_{\delta,M}(0)$ .

Let  $f_i(x, y|0)$  denote the discounted joint probability density of the surplus prior to ruin and the deficit at ruin, given that  $Z_0 = i$  and zero initial capital. Furthermore let  $K := (\vec{k}_1, \dots, \vec{k}_M)^T$ . Then, from

\* Hansjörg Albrecher, PhD, is an Assistant Professor at the Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria, and currently a postdoctoral research fellow at K.U. Leuven (Fellowship F/04/009), albrecher@tugraz.at.

$$\phi_{\delta,i}(0) = \int_{x=0}^{\infty} \int_{y=0}^{\infty} \varpi(x, y) f_i(x, y|0) dy dx$$

together with (4) and, for instance, applying Cramér's rule, one can derive a generalized version of the key result (8.3) of the paper:

$$f_i(x, y|0) = \sum_{j_1=1}^M \sum_{j_2=1}^M C_{j_1, j_2}^{(i)}(\xi_1, \dots, \xi_M, \delta) e^{-s_{j_2} x} b_{j_1}(x + y), \quad (5)$$

where

$$C_{j_1, j_2}^{(i)} = \frac{(-1)^{i+j_2} \cdot \det K_{j_2, i} \cdot \sum_{l=1}^M \lambda_l p_{l, j_1} k_{j_2, l}}{c \det K}.$$

Here  $\det K_{j_2, i}$  denotes the minor of  $K$  with respect to row  $j_2$  and column  $i$ , and  $k_{j_2, l}$  denotes the  $l$ -th component of vector  $\vec{k}_{j_2}$ .

Apart from various dependence models, the Markov additive model (1) also contains renewal models with phase-type claim arrival distributions as a special case (just choose appropriate transition probabilities and let  $B_j$  be degenerate at 0 for all but one state  $j = 1$ ). In particular, the Sparre Andersen model with generalized Erlang( $n$ ) interclaim times investigated in the paper is retained for  $M = n$  by starting in state 1 (i.e.,  $Z_0 = 1$ ) and choosing

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad \hat{B}(\xi) = \begin{pmatrix} \hat{p}(\xi) & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \dots & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

In this case  $A_\delta(\xi)$  has the simple form

$$A_\delta(\xi) = \begin{pmatrix} c\xi - \delta - \lambda_1 & \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & c\xi - \delta - \lambda_2 & \lambda_2 & \dots & 0 & 0 \\ \vdots & 0 & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & c\xi - \delta - \lambda_{n-2} & \lambda_{n-2} & 0 \\ 0 & 0 & \dots & 0 & c\xi - \delta - \lambda_{n-1} & \lambda_{n-1} \\ \lambda_n \hat{p}(\xi) & 0 & \dots & \dots & 0 & c\xi - \delta - \lambda_n \end{pmatrix}$$

so that its determinant is easily calculated, yielding

$$\det A_\delta(\xi) = (-1)^n \left( \prod_{j=1}^n (\lambda_j + \delta - c\xi) - \hat{p}(\xi) \prod_{j=1}^n \lambda_j \right).$$

In that way (2) reduces to equation (4.2) of the paper, so that the corresponding zeroes  $\xi_1, \dots, \xi_n$  coincide. Since we start in state  $i = 1$  and only in state 1 there is now a nondegenerate claim size distribution, (5) simplifies to

$$f(x, y|0) = \sum_{j_2=1}^n C_{1, j_2}^{(1)}(\xi_1, \dots, \xi_n, \delta) e^{-\xi_{j_2} x} p(x + y).$$

It remains to determine the constants

$$C_{1,j_2}^{(1)}(\xi_1, \dots, \xi_n, \delta) = \frac{(-1)^{1+j_2} \lambda_n k_{j_2,n} \det K_{j_2,1}}{c \det K}.$$

From (3) one obtains

$$K = \begin{pmatrix} \prod_{j=2}^n \frac{\lambda_j + \delta - c\xi_1}{\lambda_{j-1}} & \prod_{j=3}^n \frac{\lambda_j + \delta - c\xi_1}{\lambda_{j-1}} & \dots & \frac{\lambda_n + \delta - c\xi_1}{\lambda_{n-1}} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \prod_{j=2}^n \frac{\lambda_j + \delta - c\xi_n}{\lambda_{j-1}} & \prod_{j=3}^n \frac{\lambda_j + \delta - c\xi_n}{\lambda_{j-1}} & \dots & \frac{\lambda_n + \delta - c\xi_n}{\lambda_{n-1}} & 1 \end{pmatrix}$$

and subsequently

$$\det K = \frac{1}{\lambda_1 \lambda_2^2 \dots \lambda_{n-1}^{n-1}} \begin{vmatrix} \prod_{j=2}^n (\lambda_j + \delta - c\xi_1) & \prod_{j=3}^n (\lambda_j + \delta - c\xi_1) & \dots & \lambda_n + \delta - c\xi_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \prod_{j=2}^n (\lambda_j + \delta - c\xi_n) & \prod_{j=3}^n (\lambda_j + \delta - c\xi_n) & \dots & \lambda_n + \delta - c\xi_n & 1 \end{vmatrix}.$$

Standard algebraic manipulations lead to

$$\det K = \frac{c^{\frac{n(n-1)}{2}}}{\lambda_1 \lambda_2^2 \dots \lambda_{n-1}^{n-1}} \begin{vmatrix} 1 & \xi_1 & \dots & \xi_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_n & \dots & \xi_n^{n-1} \end{vmatrix},$$

which is the well-known Vandermonde determinant given by

$$\det K = \frac{c^{\frac{n(n-1)}{2}}}{\lambda_1 \lambda_2^2 \dots \lambda_{n-1}^{n-1}} \prod_{\substack{j,k=1 \\ k>j}}^n (\xi_k - \xi_j).$$

Analogously,  $\det K_{j_2,1}$  can be expressed as

$$\det K_{j_2,1} = \frac{c^{\frac{(n-1)(n-2)}{2}}}{\lambda_2 \lambda_3^2 \dots \lambda_{n-1}^{n-2}} \prod_{\substack{j,k=1 \\ k>j, k \neq j_2, j \neq j_2}}^n (\xi_k - \xi_j),$$

from which we obtain

$$C_{1,j_2}^{(1)}(\xi_1, \dots, \xi_n, \delta) = \frac{(-1)^{1+j_2} \lambda_1 \dots \lambda_n \prod_{\substack{j,k=1 \\ k>j, k \neq j_2, j \neq j_2}}^n (\xi_k - \xi_j)}{c^n \prod_{\substack{j,k=1 \\ k>j}}^n (\xi_k - \xi_j)} = \frac{\lambda_1 \dots \lambda_n}{c^n} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{\xi_k - \xi_{j_2}}.$$

Thus we finally arrive at

$$f(x, y|0) = \sum_{j_2=1}^n \frac{\lambda_1 \dots \lambda_n}{c^n} \prod_{\substack{k=1 \\ k \neq j_2}}^n \frac{1}{\xi_k - \xi_{j_2}} e^{-\xi_{j_2} x} p(x + y),$$

which completes a somewhat alternative proof of formula (8.3) of the paper.

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## CARY CHI-LIANG TSAI\*

Professors Gerber and Shiu use an innovative approach based on a Sparre Andersen model (more general than the ones used in the study of ruin theory before) for the interclaim time assumption to derive the renewal equation for the expected discounted function  $\phi$ , which is usually used to study a variety of interesting and important quantities in the problems of classical ruin theory by appropriate choices of the penalty function  $\varpi$ . The purpose of this discussion is to show that the discounted joint distribution function of  $U(T-)$  and  $|U(T)|$  can be obtained from the expected discounted function.

Gerber and Shiu express  $g(y)$  by (eq. 9.3)

$$g(y) = \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \prod_{j=1}^n T_{\rho_j} \right] p(y),$$

and further simplify it to a very elegant and simple form (eq. 9.10) as

$$g(y) = \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} p \right] (y),$$

a linear function of  $T_{\rho_j} p(y)$ 's.

Define the distribution function  $\Gamma_j(y)$  by (see also Lin and Willmot 1999)

$$\Gamma_j(y) = \frac{\int_0^y T_{\rho_j} p(t) dt}{\int_0^\infty T_{\rho_j} p(t) dt} = \frac{\int_0^y \int_t^\infty e^{-\rho_j(x-t)} dP(x) dt}{\int_0^\infty \int_t^\infty e^{-\rho_j(x-t)} dP(x) dt},$$

$j = 1, 2, \dots, n$ . Since

$$\int_y^\infty \int_t^\infty e^{-\rho_j(x-t)} dP(x) dt = \int_y^\infty e^{-\rho_j(x-y)} \bar{P}(x) dx,$$

we have

$$\bar{\Gamma}_j(y) = 1 - \Gamma_j(y) = \frac{\int_y^\infty T_{\rho_j} p(t) dt}{\int_0^\infty T_{\rho_j} p(t) dt} = \frac{\int_y^\infty e^{-\rho_j(x-y)} \bar{P}(x) dx}{E_j} \quad (1)$$

where

$$E_j = \int_0^\infty T_{\rho_j} p(t) dt = \int_0^\infty \int_t^\infty e^{-\rho_j(x-t)} dP(x) dt = \int_0^\infty e^{-\rho_j x} \bar{P}(x) dx = [T_{\rho_j} \bar{P}](0), j = 1, 2, \dots, n.$$

Then

$$E = \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) E_j \right]$$

where  $E = \int_0^\infty g(x) dx$ . Let  $G(y) = \int_0^y g(x) dx / E$ ; then the associated "claim size" distribution  $G$  can be

\* Cary Chi-Liang Tsai, ASA, PhD, is an Assistant Professor of the Department of Statistics and Actuarial Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada, clltsai@sfu.ca.

expressed as a weighted average of  $\Gamma_j$ 's (each  $\Gamma_j$  plays the same role as  $G$  if the interclaim time random variables are independent with a common exponential distribution of mean  $1/\lambda_j$  for fixed  $j$ ), that is,

$$\bar{G}(y) = \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) E_j \bar{\Gamma}_j(y) \right] / E = \sum_{j=1}^n \varpi_j \bar{\Gamma}_j(y),$$

where

$$\varpi_j = \frac{\left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) E_j}{\sum_{i=1}^n \left( \prod_{h=1, h \neq i}^n \frac{1}{\rho_h - \rho_i} \right) E_i} = \frac{\prod_{i=1}^n \lambda_i \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) E_j}{c^n E} \tag{2}$$

with  $\sum_{j=1}^n \varpi_j = 1$ , and  $G(y) = 1 - \bar{G}(y) = \sum_{j=1}^n \varpi_j \Gamma_j(y)$ .

Since  $E = \int_0^\infty g(y)dy < 1$  (see eq. 11.5), let  $E = 1/(1 + \beta)$ , where  $\beta = 1/E - 1 > 0$ . Then the defective renewal equation (9.2) for  $\phi(u)$  can be rewritten as

$$\phi(u) = \frac{1}{1 + \beta} \int_0^u \phi(u - y) dG(y) + \frac{1}{1 + \beta} B(u) \tag{3}$$

where

$$B(u) = (1 + \beta) \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \prod_{j=1}^n T_{\rho_j} \right] \omega(u) = (1 + \beta) \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} \omega \right] (u). \tag{4}$$

By Lin and Willmot (1999),

$$\phi(u) = -\frac{1}{\beta} \int_0^u \bar{K}(u - x) dB(x) + \frac{1}{\beta} B(u) - \frac{1}{\beta} B(0) \bar{K}(u), \tag{5}$$

where

$$\bar{K}(u) = 1 - K(u) = \sum_{n=1}^\infty \frac{\beta}{1 + \beta} \left( \frac{1}{1 + \beta} \right)^n \bar{G}^{*n}(u), \quad u \geq 0,$$

is a compound geometric distribution function with  $\bar{K}(0) = 1/(1 + \beta)$ , and  $\bar{K}(u)$  satisfies the defective renewal equation

$$\bar{K}(u) = \frac{1}{1 + \beta} \int_0^u \bar{K}(u - x) dG(x) + \frac{1}{1 + \beta} \bar{G}(u), \quad u \geq 0. \tag{6}$$

When  $\varpi(x, y) = 1$ ,  $\phi(u)$  reduces to  $\mathbb{E}[e^{-\delta T} I(T < \infty) | U(0) = u]$ . Then  $B(u)$  in (4) becomes

$$(1 + \beta) \frac{\prod_{i=1}^n \lambda_i}{c^n} \left[ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) \int_u^\infty e^{-\rho_j(x-u)} \bar{P}(x) dx \right] = \sum_{j=1}^n \varpi_j \bar{\Gamma}_j(u) = \bar{G}(u).$$

Comparing (6) with (3), we then conclude that  $\bar{K}(u) = \phi(u)|_{\varpi=1} = \mathbb{E}[e^{-\delta T} I(T < \infty) | U(0) = u]$ .

Moreover, the discounted joint distribution function of  $U(T^-)$  and  $|U(T)|$  can be produced from  $\phi(u)$  by an appropriate choice of the penalty function  $\varpi(x, y)$ . Then an explicit expression for the discounted joint distribution function of  $U(T^-)$  and  $|U(T)|$  can be obtained from (5). To see this, for any fixed  $x$  and  $y$ , let

$$\varpi(x_1, x_2) = \begin{cases} 1, & \text{if } x_1 \leq x, x_2 \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

By (2.1),  $\phi(u)$  becomes  $\int_0^x \int_0^y \int_0^\infty e^{-\delta t} f(x_1, x_2, t|u) dt dx_1 dx_2 = F(x, y|u)$ , the discounted joint distribution function of  $U(T^-)$  and  $|U(T)|$ .

For  $j = 1, 2, \dots, n$ , if  $0 < x \leq u$ ,  $T_{\rho_j} \omega(u) = 0$ , and if  $0 \leq u < x$ ,

$$\begin{aligned} T_{\rho_j} \omega(u) &= \int_u^x e^{-\rho_j(x_1-u)} \int_{x_1}^{x_1+y} dP(x_2) dx_1 \\ &= \int_u^\infty e^{-\rho_j(x_1-u)} \bar{P}(x_1) dx_1 - e^{-\rho_j(x-u)} \int_x^\infty e^{-\rho_j(x_1-x)} \bar{P}(x_1) dx_1 \\ &\quad - \int_{u+y}^\infty e^{-\rho_j(x_1-u-y)} \bar{P}(x_1) dx_1 + e^{-\rho_j(x-u)} \int_{x+y}^\infty e^{-\rho_j(x_1-x-y)} \bar{P}(x_1) dx_1 \\ &= E_j[\bar{\Gamma}_j(u) - \bar{\Gamma}_j(u+y)] - E_j e^{-\rho_j(x-u)} [\bar{\Gamma}_j(x) - \bar{\Gamma}_j(x+y)] \end{aligned}$$

by (1). Therefore,

$$B(u) = [\bar{G}(u) - \bar{G}(u+y)] - \sum_{j=1}^n \omega_j e^{-\rho_j(x-u)} [\bar{\Gamma}_j(x) - \bar{\Gamma}_j(x+y)], \quad (7)$$

and for  $u > 0$

$$dB(u) = dG(u+y) - dG(u) - \sum_{j=1}^n \omega_j \rho_j e^{-\rho_j(x-u)} [\bar{\Gamma}_j(x) - \bar{\Gamma}_j(x+y)] du. \quad (8)$$

Equation (5) becomes

$$F(x, y|u) = -(1/\beta) \int_0^u \bar{K}(u-t) dB(t) + (1/\beta) B(u) - (1/\beta) B(0) \bar{K}(u)$$

for  $0 \leq u < x$ , and

$$F(x, y|u) = -(1/\beta) \int_0^x \bar{K}(u-t) dB(t) - (1/\beta) B(0) \bar{K}(u)$$

for  $0 < x \leq u$ . Replacing  $B(u)$  and  $dB(t)$  with (7) and (8), respectively, and using (6) gives the discounted joint distribution function of  $U(T-)$  and  $|U(T)|$  as follows (setting  $u = 0$  yields  $F(x, y|0)$  from  $\bar{K}(0) = 1/(1 + \beta)$  and (6)):

$$F(x, y|u) = \begin{cases} \frac{1 + \beta}{\beta} [\bar{K}(u) - \bar{K}(u+y)] - \frac{1}{\beta} G(y) \bar{K}(u) + \frac{1}{\beta} \int_0^y \bar{K}(u+y-t) dG(t) \\ \quad + \frac{1}{\beta} \sum_{j=1}^n \omega_j e^{-\rho_j x} [\bar{\Gamma}_j(x) - \bar{\Gamma}_j(x+y)] \left[ \rho_j \int_0^u e^{\rho_j t} \bar{K}(u-t) dt + \bar{K}(u) - e^{\rho_j u} \right], & \text{if } 0 \leq u < x, \\ \frac{1}{\beta} \int_0^x \bar{K}(u-t) [dG(t) - dG(y+t)] - \frac{1}{\beta} G(y) \bar{K}(u) \\ \quad + \frac{1}{\beta} \sum_{j=1}^n \omega_j e^{-\rho_j x} [\bar{\Gamma}_j(x) - \bar{\Gamma}_j(x+y)] \left[ \rho_j \int_0^x e^{\rho_j t} \bar{K}(u-t) dt + \bar{K}(u) \right], & \text{if } 0 < x \leq u, \end{cases}$$

with

$$F(x, y|0) = \frac{1}{1 + \beta} \left\{ G(y) + \sum_{j=1}^n \omega_j e^{-\rho_j x} [\Gamma_j(x) - \Gamma_j(x+y)] \right\}.$$

The discounted joint probability density function of  $U(T-)$  and  $|U(T)|$  can be derived from

$$f(x, y|u) = \frac{\partial^2 F(x, y|u)}{\partial x \partial y}$$

by (2),

$$\bar{K}(0) = 1/(1 + \beta), \rho_i \bar{\Gamma}_i(x) + \Gamma'_i(x) = -e^{\rho_i x} [e^{-\rho_i x} \bar{\Gamma}_i(x)]' = \bar{P}(x)/E_i,$$

and

$$\frac{\partial}{\partial x} [e^{-\rho_i x} \bar{\Gamma}'_i(x + y)] = e^{-\rho_i x} \frac{p(x + y)}{E_i}.$$

$$f(x, y|u) = \begin{cases} \frac{\prod_{i=1}^n \lambda_i}{c^n} \frac{p(x + y)}{1 - \bar{K}(0)} \\ \times \left\{ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) e^{-\rho_j x} \left[ e^{\rho_j u} - \bar{K}(u) - \rho_j \int_0^u e^{\rho_j t} \bar{K}(u - t) dt \right] \right\}, & \text{if } 0 \leq u < x, \\ \frac{\prod_{i=1}^n \lambda_i}{c^n} \frac{p(x + y)}{1 - \bar{K}(0)} \\ \times \left\{ \sum_{j=1}^n \left( \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right) e^{-\rho_j x} \left[ e^{\rho_j x} - \bar{K}(u - x) - \bar{K}(u) - \rho_j \int_0^x e^{\rho_j t} \bar{K}(u - t) dt \right] \right\}, & \text{if } 0 < x \leq u, \end{cases}$$

with

$$f(x, y|0) = \frac{\prod_{i=1}^n \lambda_i}{c^n} p(x + y) \sum_{j=1}^n \left[ e^{-\rho_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j} \right],$$

which agrees with equation (8.3). Note that  $f(x, y|u)$  is a weighted average of  $n$  functions with the sum of the weights,

$$\sum_{j=1}^n \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j},$$

equal to zero for  $n \geq 2$ .

For  $n = 1$ ,  $F(x, y|u)$  and  $f(x, y|u)$  simplify to equations (2.9) and (2.12) of Tsai (2001), respectively; for  $n = 2$  and  $\lambda_1 = \lambda_2$ ,  $F(x, y|u)$  and  $f(x, y|u)$  reduce to equations (2.3) and (2.4) of Tsai and Sun (2004), respectively. With the expression for  $F(x, y|u)$ , the discounted marginal distribution functions and probability density functions of  $U(T-)$ ,  $|U(T)|$ , and  $[U(T-) + |U(T)|]$  can be easily obtained.

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## SHUANMING LI\*

In this paper Professors Gerber and Shiu show how to extend the results in Gerber and Shiu (1998) to a Sparre Andersen risk model with generalized Erlang interclaim times in a very elegant way. I would like to comment that the Laplace transform of the function  $\phi(u)$  can be obtained in an alternative way.

Conditioning on the time and the amount of the first claim  $X_1$ , one has

$$\phi(u) = \int_0^\infty e^{-\delta t} f_V(t) E[\phi(u + ct - X)] dt.$$

Setting  $y = u + ct$  yields

$$c\phi(u) = \int_u^\infty e^{-\frac{\delta(y-u)}{c}} f_V\left(\frac{y-u}{c}\right) E[\phi(y - X)] dy. \quad (1)$$

Taking Laplace transforms gives

$$\begin{aligned} c\hat{\phi}(s) &= \int_0^\infty e^{-su} \int_u^\infty e^{-\frac{\delta(y-u)}{c}} f_V\left(\frac{y-u}{c}\right) E[\phi(y - X)] dy du \\ &= \int_0^\infty e^{-\frac{\delta y}{c}} E[\phi(y - X)] \int_0^y e^{-\left(\frac{cs-\delta}{c}\right)u} f_V\left(\frac{y-u}{c}\right) du dy. \end{aligned} \quad (2)$$

Now, if  $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct, the probability density function of the interclaim times can be expressed as

$$f_V(t) = \sum_{i=1}^n a_i e^{-\lambda_i t}, \quad t > 0, \quad (3)$$

where

$$a_i = \lambda_i \prod_{j=1, j \neq i}^n \frac{\lambda_j}{\lambda_j - \lambda_i}, \quad i = 1, 2, \dots, n.$$

Substituting (3) into (2) yields

$$\begin{aligned} c\hat{\phi}(s) &= \int_0^\infty e^{-\frac{\delta y}{c}} E[\phi(y - X)] \sum_{i=1}^n a_i \int_0^y e^{-\left(\frac{cs-\delta}{c}\right)u} e^{-\lambda_i \left(\frac{y-u}{c}\right)} du dy \\ &= \sum_{i=1}^n a_i \int_0^\infty e^{-\left(\frac{\delta+\lambda_i}{c}\right)y} E[\phi(y - X)] \int_0^y e^{-\left(\frac{cs-\delta-\lambda_i}{c}\right)u} du dy \\ &= \sum_{i=1}^n \frac{ca_i}{cs - \delta - \lambda_i} \left\{ \int_0^\infty e^{-\left(\frac{\delta+\lambda_i}{c}\right)y} E[\phi(y - X)] dy - \int_0^\infty e^{-sy} E[\phi(y - X)] dy \right\}. \end{aligned}$$

Then

$$\hat{\phi}(s) = \sum_{i=1}^n \frac{a_i e_i}{cs - \delta - \lambda_i} + \hat{f}_V(\delta - cs) \int_0^\infty e^{-sy} E[\phi(y - X)] dy, \quad (4)$$

where

\* Shuanming Li, PhD, is a Senior Lecturer at the Centre for Actuarial Studies, the University of Melbourne, Victoria 3031, Australia, shli@unimelb.edu.au.

$$e_i = \int_0^\infty e^{-\left(\frac{\delta+\lambda_i}{c}\right)y} E[\phi(y - X)] dy, \quad i = 1, 2, \dots, n.$$

Since

$$E[\phi(y - X)] = \int_0^y \phi(y - x)p(x) dx + \omega(y),$$

where  $\omega(y) = \int_y^\infty \varpi(y, x - y)p(x)dx$ , then equation (4) reduces to

$$\hat{\phi}(s) = \frac{\sum_{i=1}^n \frac{a_i e_i}{cs - \delta - \lambda_i} + \hat{f}_V(\delta - cs)\hat{\omega}(s)}{1 - \hat{f}_V(\delta - cs)\hat{p}(s)}. \tag{5}$$

Multiplying both the denominator and numerator by  $\gamma(s) = 1/\hat{f}_V(\delta - cs)$  yields

$$\hat{\phi}(s) = \frac{\hat{\omega}(s) - q(s)}{\gamma(s) - \hat{p}(s)}, \tag{6}$$

with

$$q(s) = \frac{1}{\prod_{i=1}^n \lambda_i} \left[ \prod_{i=1}^n (\delta + \lambda_i - cs) \right] \left[ \sum_{i=1}^n \frac{a_i e_i}{\delta + \lambda_i - cs} \right]$$

being a polynomial of degree  $n - 1$  or less. Since  $\hat{\phi}(s)$  is finite for all  $s$  with  $\Re(s) > 0$ , and we note that  $\rho_j$  (with  $\Re(\rho_j) > 0, j = 1, 2, \dots, n$ ) are zeros of the denominator of (6), then they must also be zeros of the numerator. It follows that  $q(s)$  is determined by the condition.

$$q(\rho_j) = \hat{\omega}(\rho_j), \quad j = 1, 2, \dots, n. \tag{7}$$

Further, if  $\rho_1, \rho_2, \dots, \rho_n$  are distinct, then by the *Lagrange interpolation formula*, we have

$$q(s) = \sum_{j=1}^n \left\{ \hat{\omega}(\rho_j) \prod_{k=1, k \neq j}^n \frac{(s - \rho_k)}{(\rho_j - \rho_k)} \right\}. \tag{8}$$

If instead some  $\lambda_i$ 's coincide, then

$$\hat{f}_V(t) = \sum_{i=1}^k \sum_{j=1}^{n_i} a_{i,j} \frac{t^{j-1} e^{-\lambda_i t}}{(j - 1)!}, \quad t \geq 0,$$

where

$$a_{i,j} = \frac{\prod_{l=1}^k \lambda_l^{n_l}}{(n_i - j)!} \frac{d^{n_i-j}}{ds^{n_i-j}} \prod_{m=1, m \neq i}^k \frac{1}{(s + \lambda_m)^{n_m}} \Big|_{s=-\lambda_i}.$$

By a similar argument

$$\hat{\phi}(s) = \frac{-\sum_{i=1}^k \sum_{j=1}^{n_i} \sum_{m=0}^{j-1} \frac{a_{i,j} c_{i,m}}{c^m m! (\delta + \lambda_i - cs)^{j-m}} + \hat{f}_V(\delta - cs)\hat{\omega}(s)}{1 - \hat{f}_V(\delta - cs)\hat{p}(s)}, \tag{9}$$

where

$$e_{i,m} = \int_0^\infty y^m e^{-\left(\frac{\delta+\lambda_i}{c}\right)y} E[\phi(y - X)] dy, \quad i = 1, 2, \dots, k < n.$$

It is easy to show that  $\hat{\phi}(s)$  can also be expressed in the form of (6), but this time with

$$q(s) = \frac{1}{\prod_{i=1}^k \lambda_i^{n_i}} \left[ \prod_{i=1}^k (\delta + \lambda_i - cs)^{n_i} \right] \left[ \sum_{i=1}^k \sum_{j=1}^{n_i} \sum_{m=0}^{j-1} \frac{\alpha_{i,j} e_{i,m}}{c^m m! (\delta + \lambda_i - cs)^{j-m}} \right]$$

being a polynomial of degree  $n - 1$  or less, which can also be determined by conditions (7) and obtained explicitly by (8) if  $\rho_1, \dots, \rho_n$  are distinct. Note that the Laplace transform given in (6) is the same as (7.2) of the paper.

Finally, this technique can also be used to derive the Laplace transform of the expected discounted penalty function  $\phi(u)$  in a Sparre Andersen risk model with more general interclaim times distributions, such as phase-type and  $K_n$  distributions. Detailed discussions of the evaluation of the function  $\phi(u)$  in a Sparre Andersen risk model with  $K_n$  interclaim times can be found in Li and Garrido (2005).

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## AUTHORS' REPLY

We are delighted to have four such insightful discussions. Professor Schmidli's discussion shows how the common distribution of the interclaim time random variables,  $\{V_j\}$ , can be generalized to phase-type. It is interesting to note that phase-type distributions appear in two papers (Drekic and Willmot 2005; Ng and Yang 2005) in this issue, 9(2), of the *North American Actuarial Journal* and in one paper (Avram and Usabel 2004) in the latest issue of the *ASTIN Bulletin*. It may be useful for some readers that we expand on some of the points made by Professor Schmidli. Rolski et al. (1999, Theorem 8.2.3) have given the formula

$$\Pr(V > \tau) = \alpha e^{\tau \mathbf{B}} \mathbf{e}^T. \quad (\text{R1})$$

Thus, the probability density function of  $V$  is

$$f_V(\tau) = -\frac{d}{d\tau} \alpha e^{\tau \mathbf{B}} \mathbf{e}^T = -\alpha e^{\tau \mathbf{B}} \mathbf{B} \mathbf{e}^T = \alpha e^{\tau \mathbf{B}} \mathbf{b}^T, \quad (\text{R2})$$

with  $\mathbf{b} = -\mathbf{e} \mathbf{B}^T$ , the negative of the row sums of  $\mathbf{B}$ . Multiplying (R2) with  $e^{-\xi \tau}$  and integrating from  $\tau = 0$  to  $\tau = \infty$  yields the Laplace transform

$$\hat{f}_V(\xi) = \alpha (\xi \mathbf{I} - \mathbf{B})^{-1} \mathbf{b}^T. \quad (\text{R3})$$

Thus, the zeros of the function  $\gamma(\xi) = 1/\hat{f}_V(\xi) = 1/(\alpha (\xi \mathbf{I} - \mathbf{B})^{-1} \mathbf{b}^T)$  occur at  $\xi = (\delta - \zeta)/c$ , where  $\zeta$  ranges over all  $n$  eigenvalues of  $\mathbf{B}$ . Formulas (R2) and (R3) can be found in Rolski et al. (1999, Theorem 8.2.5).

With Professor Schmidli's notation, we have a very elegant formulation of equation (3.5) of the paper, namely,

$$\phi(u, \tau) = \frac{\alpha e^{\tau \mathbf{B}} \phi(u)}{\alpha e^{\tau \mathbf{B}} \mathbf{e}^T}. \quad (\text{R4})$$

To get a better feeling for this formula, we now illustrate the relationship,

$$\alpha e^{\tau \mathbf{B}} = (\Pr(A_0), \Pr(A_1), \dots, \Pr(A_{n-1})), \quad (\text{R5})$$

in two cases where we have explicit formulas; we shall evaluate  $e^{\tau \mathbf{B}}$  to obtain formulas (3.8) and (3.10) of the paper. Because  $\alpha = (1, 0, \dots, 0)$ , formula (R5) is simply that the first row of  $e^{\tau \mathbf{B}}$  is

$$(\Pr(A_0), \Pr(A_1), \dots, \Pr(A_{n-1})).$$

For simplicity, we restrict ourselves to  $n = 4$ . Also, it seems a bit clearer if we slightly generalize  $\mathbf{B}$  as

$$\mathbf{B} = \begin{pmatrix} -\lambda_1 & \mu_1 & 0 & 0 \\ 0 & -\lambda_2 & \mu_2 & 0 \\ 0 & 0 & -\lambda_3 & \mu_3 \\ 0 & 0 & 0 & -\lambda_4 \end{pmatrix}. \tag{R6}$$

Let

$$\mathbf{D} = \text{diag}[-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4] \tag{R7}$$

and

$$\mathbf{J} = \begin{pmatrix} 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{R8}$$

Then  $\mathbf{B} = \mathbf{D} + \mathbf{J}$ . We remark that the matrix  $e^{\tau\mathbf{B}}$  plays a key role in Ramsay (1989).

We first consider the case  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ . Thus,  $\mathbf{D} = -\lambda\mathbf{I}$ . Because the matrices  $\mathbf{D}$  and  $\mathbf{J}$  commute, we have

$$e^{\tau\mathbf{B}} = e^{\tau(\mathbf{D}+\mathbf{J})} = e^{\tau\mathbf{D}}e^{\tau\mathbf{J}} = e^{-\tau\lambda} e^{\tau\mathbf{J}}. \tag{R9}$$

To evaluate  $e^{\tau\mathbf{J}}$ , we expand it by the exponential series formula,

$$e^{\tau\mathbf{J}} = \mathbf{I} + \frac{\tau}{1!} \mathbf{J} + \frac{\tau^2}{2!} \mathbf{J}^2 + \frac{\tau^3}{3!} \mathbf{J}^3. \tag{R10}$$

The series has only four terms because

$$\mathbf{J}^2 = \begin{pmatrix} 0 & 0 & \mu_1\mu_2 & 0 \\ 0 & 0 & 0 & \mu_2\mu_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{J}^3 = \begin{pmatrix} 0 & 0 & 0 & \mu_1\mu_2\mu_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{J}^m = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } m \geq 4.$$

Thus, with  $\mu_1 = \mu_2 = \mu_3 = \lambda$ , the first row of the matrix  $e^{\tau\mathbf{B}}$  is

$$e^{-\tau\lambda} \left( 1, \tau\lambda, \frac{\tau^2\lambda^2}{2!}, \frac{\tau^3\lambda^3}{3!} \right), \tag{R11}$$

which matches the right-hand side of formula (3.8) of the paper.

Next, we consider the case where the  $\lambda_i$ 's are distinct. Let

$$\mathbf{U} = \begin{pmatrix} 1 & \frac{\mu_1}{\lambda_1 - \lambda_2} & \frac{\mu_1\mu_2}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} & \frac{\mu_1\mu_2\mu_3}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \\ 0 & 1 & \frac{\mu_2}{\lambda_2 - \lambda_3} & \frac{\mu_2\mu_3}{(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \\ 0 & 0 & 1 & \frac{\mu_3}{\lambda_3 - \lambda_4} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{R12}$$

Then, it can readily be checked that

$$\mathbf{BU} = \mathbf{UD}, \tag{R13}$$

or

$$\mathbf{B} = \mathbf{UDU}^{-1}. \tag{R14}$$

Consequently,

$$e^{\tau \mathbf{B}} = \mathbf{U} e^{\tau \mathbf{D}} \mathbf{U}^{-1} = \mathbf{U} \operatorname{diag}[e^{-\tau \lambda_1}, e^{-\tau \lambda_2}, e^{-\tau \lambda_3}, e^{-\tau \lambda_4}] \mathbf{U}^{-1}. \quad (\text{R15})$$

Thus, the first row of  $e^{\tau \mathbf{B}}$  is

$$\left( e^{-\tau \lambda_1}, \frac{\mu_1 e^{-\tau \lambda_2}}{\lambda_1 - \lambda_2}, \frac{\mu_1 \mu_2 e^{-\tau \lambda_3}}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}, \frac{\mu_1 \mu_2 \mu_3 e^{-\tau \lambda_4}}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)} \right) \mathbf{U}^{-1}. \quad (\text{R16})$$

We obtained  $\mathbf{U}$  by noting that its column vectors are eigenvectors of  $\mathbf{B}$ . From

$$\mathbf{U}^{-1} \mathbf{B} = \mathbf{D} \mathbf{U}^{-1},$$

we observe that the row vectors of  $\mathbf{U}^{-1}$  are eigenvectors of  $\mathbf{B}^T$ . This observation and also that each entry in the main diagonal of  $\mathbf{U}^{-1}$  is 1 help one find the formula

$$\mathbf{U}^{-1} = \begin{pmatrix} 1 & \frac{\mu_1}{\lambda_2 - \lambda_1} & \frac{\mu_1 \mu_2}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)} & \frac{\mu_1 \mu_2 \mu_3}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_4 - \lambda_1)} \\ 0 & 1 & \frac{\mu_2}{\lambda_3 - \lambda_2} & \frac{\mu_2 \mu_3}{(\lambda_3 - \lambda_2)(\lambda_4 - \lambda_2)} \\ 0 & 0 & 1 & \frac{\mu_3}{\lambda_4 - \lambda_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{R17})$$

Now, we can see that formula (3.10) in the paper is verified.

Professor Albrecher points out that some results in the paper can be extended to a more general framework including models with a Markovian dependence structure among interclaim times and claim sizes. His Markov additive model contains renewal models with phase-type interclaim times as special cases. He also derives a generalized version of formula (8.3) and shows how it reduces to the form given in the paper.

Professor Tsai's discussion also contains formula (8.3). In fact, by an appropriate choice of the penalty function  $\varpi$ , he provides an elegant derivation for the two expressions, which are the formula for the "discounted" joint density function  $f(x, y|u)$ . Formula (8.3) is the special case when  $u = 0$ .

We take this opportunity to present a somewhat more direct derivation of formula (8.4), which is the divided difference form of (8.1), from which (8.3) follows. From (3.12) of the paper we see that, if a function  $h(s)$  vanishes at  $s = r_1, r_2, \dots, r_n$ , then

$$h(s) = h[s, r_1, r_2, \dots, r_n] \prod_{k=1}^n (s - r_k). \quad (\text{R18})$$

We apply (R18) to the numerator and to the denominator of (6.2), with  $s = \xi$  and  $r_k = \rho_k$ . After canceling  $\prod_{k=1}^n (\xi - \rho_k)$ , we obtain

$$\hat{\phi}(\xi) = \frac{\hat{\omega}[\xi, \rho_1, \dots, \rho_n] - q[\xi, \rho_1, \dots, \rho_n]}{\gamma[\xi, \rho_1, \dots, \rho_n] - \hat{p}[\xi, \rho_1, \dots, \rho_n]}, \quad \operatorname{Re} \xi \geq 0. \quad (\text{R19})$$

Because  $q(\xi)$  is a polynomial of degree  $n - 1$  or less, we have  $q[\xi, \rho_1, \rho_2, \dots, \rho_n] = 0$ . Because  $\gamma(\xi)$  is a polynomial of degree  $n$ , the  $n$ -th divided difference  $\gamma[\xi, \rho_1, \rho_2, \dots, \rho_n]$  is the coefficient of  $\xi^n$  in  $\gamma(\xi)$ ; this leading coefficient is given by (10.10), and we shall denote it as  $\gamma_n$ . Thus, equation (R19) simplifies as

$$\hat{\phi}(\xi) = \frac{\hat{\omega}[\xi, \rho_1, \dots, \rho_n]}{\gamma_n - \hat{p}[\xi, \rho_1, \dots, \rho_n]}, \quad \operatorname{Re} \xi \geq 0. \quad (\text{R20})$$

Applying (R20) to (2.4) of the paper yields

$$\phi(0) = \lim_{\xi \rightarrow \infty} \xi \frac{\hat{\omega}[\xi, \rho_1, \dots, \rho_n]}{\gamma_n - \hat{p}[\xi, \rho_1, \dots, \rho_n]}. \tag{R21}$$

Now,

$$\lim_{\xi \rightarrow \infty} \hat{p}[\xi, \rho_1, \rho_2, \dots, \rho_n] = 0$$

because  $\hat{p}(\xi) \rightarrow 0$ , and

$$\begin{aligned} \lim_{\xi \rightarrow \infty} \xi \hat{\omega}[\xi, \rho_1, \rho_2, \dots, \rho_n] &= \lim_{\xi \rightarrow \infty} \xi \frac{\hat{\omega}[\xi, \rho_2, \dots, \rho_n] - \hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]}{\xi - \rho_1} \\ &= 0 - \hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]. \end{aligned}$$

Thus, (R21) becomes

$$\phi(0) = \frac{-\hat{\omega}[\rho_1, \rho_2, \dots, \rho_n]}{\gamma_n} = \frac{\lambda_1 \lambda_2 \dots \lambda_n}{c_n} (-1)^{n-1} \hat{\omega}[\rho_1, \rho_2, \dots, \rho_n],$$

which is (8.4) of the paper. The divided difference form of (8.3) is

$$f(x, y|0) = \frac{\lambda_1 \lambda_2 \dots \lambda_n}{c^n} p(x + y) (-1)^{n-1} \epsilon_x[\rho_1, \rho_2, \dots, \rho_n]. \tag{R22}$$

Let us indicate an approach, somewhat different from Professor Tsai’s, for determining an expression for the “discounted” joint density function  $f(x, y|u)$ . Recall the first half of Section 2 of the paper. By distinguishing whether or not ruin occurs at the first instance when the surplus falls below the initial value  $u$ , we have

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z)dz + f(x - u, y + u|0), \quad 0 \leq u < x, \tag{R23}$$

and

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z)dz, \quad 0 < x \leq u. \tag{R24}$$

By (R22),

$$f(x - u, y + u|0) = \frac{\lambda_1 \lambda_2 \dots \lambda_n}{c^n} p(x + y) (-1)^{n-1} \epsilon_{x-u}[\rho_1, \rho_2, \dots, \rho_n].$$

Thus, (R23) and (R24) become

$$\begin{aligned} f(x, y|u) &= \int_0^u f(x, y|u - z)g(z)dz \\ &\quad + \frac{\lambda_1 \lambda_2 \dots \lambda_n}{c^n} p(x + y) (-1)^{n-1} \epsilon_{x-u}[\rho_1, \rho_2, \dots, \rho_n]_{(u < x)}, \quad u \geq 0. \end{aligned} \tag{R25}$$

Because  $\epsilon_{x-u}[\rho_1, \rho_2, \dots, \rho_n]$  is the  $(n - 1)$ -th divided difference of the function  $e^{-(x-u)\rho}$ , it is 0 for  $n \geq 2$  and  $x = u$ . Hence, for  $n \geq 2$ , the last term in (R25) is a continuous function.

In the literature of integral equations, the renewal equation

$$\phi = \phi * g + h \tag{R26}$$

is classified as a *Volterra equation of the second kind*. The solution of (R26) can be expressed as an infinite series of functions, sometimes called a *Neumann series*,

$$\phi = h + g*h + g*g*h + g*g*g*h + g*g*g*g*h + \dots \quad (\text{R27})$$

One may obtain (R27) from (R26) by the method of *successive substitution*. Applying (R27) to (R25) yields an expression for  $f(x, y|u)$ . We invite interested readers to verify that our formula is equivalent to Professor Tsai's.

Dr. Li has given a direct and elegant approach for deriving formula (7.2) of the paper. The method of Laplace transforms is indeed powerful. Although the function  $\phi(u)$  is not defined for negative values of  $u$ , there is a natural interpretation for  $E[\phi(y - X)]$ ,  $y \geq 0$ , and hence the simplicity in Dr. Li's formulas. For example, the first equation in the discussion is equation (1.5) of the paper with  $\tau = 0$ .

Beginning from Section 3 of the paper and in Dr. Li's discussion, the density function of  $V$  is

$$f_V(t) = (-1)^{n-1} \lambda_1 \lambda_2 \dots \lambda_n \varepsilon_t[\lambda_1, \lambda_2, \dots, \lambda_n], \quad t > 0. \quad (\text{R28})$$

It is an interesting exercise to use (3.16) in the paper to verify the two formulas between (8) and (9) of the discussion.

We thank the discussants again for their thought-provoking contributions.

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*Additional discussions on this paper can be submitted until October 1, 2005. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.*