

PRICING OPTIONS USING LATTICE RULES

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ABSTRACT

There are many examples of option contracts in which the payoff depends on several stochastic variables. These options often can be priced by the valuation of multidimensional integrals. Quasi-Monte Carlo methods are an effective numerical tool for this task. We show that, when the dimensions of the problem are small (say, less than 10), a special type of quasi-Monte Carlo known as the *lattice rule method* is very efficient. We provide an overview of lattice rules, and we show how to implement this method and demonstrate its efficiency relative to standard Monte Carlo and classical quasi-Monte Carlo. To maximize the efficiency gains, we show how to exploit the regularity of the integrand through a periodization technique. We demonstrate the superior efficiency of the method both in the estimation of prices as well as in the estimation of partial derivatives of these prices (the so-called Greeks). In particular this approach provides good estimates of the second derivative (the gamma) of the price in contrast to traditional Monte Carlo methods, which normally yield poor estimates. Although this method is not new, it appears that the advantages of lattice rules in the context of insurance and finance applications have not been fully appreciated in the literature.

1. INTRODUCTION

The Monte Carlo method has proven to be a very useful tool for numerical analysis, particularly when the number of dimensions ranges from medium to large. Such problems occur in a broad range of applications in science, physics, and engineering. In recent years the Monte Carlo method has also become a popular computational device for problems in finance and insurance. The finance discipline has become more sophisticated and more quantitative in the last two decades. There are several applications of the Monte Carlo method in finance, and we will briefly describe two of the main ones.

The first application is to the pricing of derivative securities. Boyle, Broadie, and Glasserman (1997) provide a survey of the applications of the Monte Carlo methods in this area. The basic valuation paradigm for a derivative security states that the price of the derivative can be obtained by taking the expectation of its future cash flows under a particular probability measure known as the risk-neutral measure. Now, if there are several state variables or if the payoff is path dependent, this expectation can be expressed as a multivariate integral. The Monte Carlo method is often the most efficient way of evaluating such high-dimensional integrals. This method is also widely applied in the case of the risk analysis of portfolios. For example, it provides a practical method of obtaining the probability that the value of the portfolio falls below some threshold at the end of some period. Such calculations are needed to compute the so-called Value at Risk of the portfolio. Glasserman, Heidelberger, and Shahabuddin (2000) discuss the applications of the Monte Carlo method in computing Value at Risk.

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In the last few years new approaches have been developed that outperform standard Monte Carlo in terms of numerical efficiency for some applications. It has been found that there can be efficiency gains in using deterministic sequences rather than the random sequences that are a feature of standard Monte Carlo. These deterministic sequences are carefully selected so that they are well dispersed throughout the region of integration. Sequences with this property are known as low-discrepancy sequences. These sequences are often more efficient than standard Monte Carlo in evaluating high-dimensional integrals if the integrand is sufficiently regular, and for many finance applications this is the case. Applications of low-discrepancy sequences to finance problems have been discussed by Boyle, Broadie, and Glasserman (1997), Caflisch, Morokoff, and Owen (1997), Joy, Boyle, and Tan (1996), Ninomiya and Tezuka (1996), Tan and Boyle (2000), and Paskov and Traub (1995).

Much of the focus to date has been on high-dimensional problems since these are more challenging from a computational viewpoint. However, it is also of interest to examine small-dimension problems, and this is the objective of the present paper. There are several practical examples of such problems:

- Options whose payoff depends on the relative performance of two underlying assets. A particular version of this option is known as a spread option and is very popular in the energy industry.¹ See, for example, Pearson (1995) and Wilcox (1990).
- Basket options, where the payoff depends on the ending values of a number of assets such as different common stocks or stock market indices. The payoff could be based on the average, the maximum, or the minimum of the asset prices. See, for example, Dufresne, Keirstead, and Ross (1996) and Stultz (1982).
- Path-dependent options, where the payoff is a function of the asset price at a number of discrete monitoring points along the path. In the case of *Asian* options the payoff is based on the average of these points. In the case of *lookback* options the payoff is based on the largest (or smallest) value recorded at one of these monitoring points. The dimensions of the problem are directly related to the number of discrete points in the path. See, for example, Heynen and Kat (1995).
- Equity-indexed annuities (EIAs). These contracts typically contain embedded options related to the performance of some equity index such as the S&P 500. Common examples are the high-water-mark (or discrete lookback) options and Asian (or average) options. For the high-water-mark option, the payoff is path-dependent in the sense that it depends on the level of the index fund at each anniversary date of the contract. Since EIAs have maturities ranging from 3 to 10 years, they can be regarded as 3-to-10-dimensional problems. See, for example, Hardy (2003), Lee (2003), Lin (1999), Lin and Tan (2003), Moore and Young (2005), and Tiong (2000).

This paper shows that a particular type of low-discrepancy sequence known as lattice rules has strong advantages in the case of small-dimensional problems. We concentrate on a particular type of lattice rule known as good lattice points. The basic idea behind this method can be explained as follows. For all the functions in a certain class,² we can find an expression for the integration error. This enables us to find the *worst case function* in the class in the sense that it gives the largest integration error. The good lattice points are obtained by finding those points that minimize the integration error for this worst case function. So the method proceeds by searching for points that minimize the maximum integration error. The numerical experiments conducted by Lemieux and L'Ecuyer (1998, 1999) indicate that the lattice rules can be as efficient as other low-discrepancy point sets. In this paper we demonstrate that, for low-dimensional problems, the lattice rules not only are more efficient than the Monte Carlo method, but also significantly outperform other quasi-Monte Carlo methods.

¹ Oil refineries refine crude oil to produce a number of products including heating oil and gasoline. Their revenues depend on the difference between their costs and their output prices. The crack spread, which denotes the difference between the price of gasoline and the price of crude oil, is a measure of their profit margin. Crack spread options are based on this difference and are natural hedging tools for oil refineries. Spread options enable refineries to control their margins and stabilize their profits.

² These functions satisfy specified regularity conditions related to the existence and continuity of partial derivatives. They should be periodic, but nonperiodic functions often can be transformed into periodic functions in a sense that will be made clear later in the paper.

The layout of the rest of this paper is as follows. The next section summarizes various approaches that can be used to evaluate multidimensional integrals. These include the standard Monte Carlo method as well as approaches based on low-discrepancy sequences. We also discuss lattice rules and their special case, known as the good lattice points (g.l.p.). Although these methods already have been discussed in the literature, we feel that it is useful to describe them here for the sake of completeness. See also the survey paper by Lemieux and L'Ecuyer (2002). Section 3 discusses the pricing of European derivative securities and shows the genesis of the multivariate integrals in this context. Section 4 explores the numerical efficiency of the lattice rules against alternative approaches using various examples including the exchange options, spread options, discretely monitored lookback options, and a particular type of EIA. We find that lattice rules generally outperform alternative approaches in terms of efficiency. We also show that lattice rules dramatically outperform the standard Monte Carlo approach in the computation of the so-called Greek sensitivities. The last section contains a brief summary.

2. VALUING MULTIPLE INTEGRALS USING SAMPLING METHODS

We now summarize various methods for evaluating multiple integrals based on some form of sampling. Consider the multiple integral

$$\int_{[0,1]^s} f(\mathbf{x}) d\mathbf{x} = \int_0^1 dx^{(1)} \int_0^1 dx^{(2)} \cdots \int_0^1 dx^{(s)} f(x^{(1)}, x^{(2)}, \dots, x^{(s)}) = \theta, \quad (2.1)$$

where $\mathbf{x} = (x^{(1)}, \dots, x^{(s)}) \in [0, 1]^s$, the function f is square-integrable on $[0, 1]^s$, and $|\theta| < \infty$. For small values of s , numerical integration methods such as Simpson's rule or the trapezoidal rule (see Davis and Rabinowitz 1984) can be used to approximate the integral in equation (2.1). These methods, however, suffer from the so-called *curse of dimensionality* and become impractical as s increases beyond 3 or 4. One viable technique for higher dimensions is to use *sampling* methods. In this case an estimate of θ is given by

$$\hat{\theta} = \frac{1}{N} \sum_{n=0}^{N-1} f(\mathbf{x}_n), \quad (2.2)$$

where $P_N = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}\} \in [0, 1]^s$ is a point set. Different techniques are available for selecting these point sets, and we now examine three of them: Monte Carlo, quasi-Monte Carlo, and lattice rule approaches.

2.1 Monte Carlo Integration

First we provide a brief description of the Monte Carlo method. Let \mathbf{x} be a uniform r.v. on $[0, 1]^s$. Then the probability density function of \mathbf{x} always takes the value 1 on $[0, 1]^s$. Therefore, the multiple integral (2.1) can be interpreted as the expectation of the random variable $f(\mathbf{x})$, that is, $E[f(\mathbf{x})] = \theta$. Now, if $\mathbf{x}_0, \mathbf{x}_1, \dots$ are i.i.d. r.v.'s, then $f(\mathbf{x}_0), f(\mathbf{x}_1), \dots$ are also i.i.d. r.v.'s. If the points $\mathbf{x}_0, \mathbf{x}_1, \dots$ used in equation (2.2) are i.i.d. uniform r.v.'s on $[0, 1]^s$, then $\hat{\theta}$ is an unbiased estimator of θ . This estimate is known as the Monte Carlo estimate and is denoted by $\hat{\theta}^{\text{MC}}$. Additional properties of the Monte Carlo method are the following:

- The strong law of large numbers asserts that $\hat{\theta}^{\text{MC}}$ converges to θ almost surely.
- The Monte Carlo estimate $\hat{\theta}^{\text{MC}}$ is approximately normally distributed with mean θ and standard error σ/\sqrt{N} , where

$$\sigma^2 = \int_{[0,1]^s} (f(\mathbf{x}) - \theta)^2 d\mathbf{x}.$$

- A useful consequence of the above result is that one can easily gauge the accuracy of the Monte Carlo estimate $\hat{\theta}^{\text{MC}}$ by constructing a confidence interval. For instance, the 95% confidence interval is usually reported along with the estimate $\hat{\theta}^{\text{MC}}$:

$$\left[\hat{\theta}^{\text{MC}} - 1.96 \frac{\sigma}{\sqrt{N}}, \hat{\theta}^{\text{MC}} + 1.96 \frac{\sigma}{\sqrt{N}} \right], \quad (2.3)$$

where σ in turn can be approximated by

$$\sqrt{\frac{1}{N-1} \sum_{n=0}^{N-1} (f(x_n) - \hat{\theta}^{\text{MC}})^2}.$$

- The Monte Carlo convergence rate $O(N^{-1/2})$ is independent of the dimension s .

2.2 Quasi-Monte Carlo Integration

Whereas the points used in Monte Carlo methods are selected randomly and independently, quasi-Monte Carlo methods are based on points that are deterministic and are more evenly dispersed throughout the domain of integration. For instance, in typical quadrature methods such as the midpoint rule, the points are evenly spaced over the unit interval, and these methods yield a much higher rate of convergence than the Monte Carlo methods. Hence it appears that the *uniformity* property of a sequence plays an important role in the efficiency of the sampling methods. It has been established formally that when a sequence of points is more uniformly distributed than a random sequence, we can attain a better convergence rate, even in high dimensions. This is a consequence of the Koksma-Hlawka inequality, which states that

$$\left| \frac{\sum_{n=0}^{N-1} f(x_n)}{N} - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq V(f) D_N^*(P_N), \quad (2.4)$$

where the function f is of bounded variation in the sense of Hardy-Krause, $V(f)$ is the total variation of f , and $D_N^*(P_N)$ is the discrepancy of the point set P_N and is a measure of uniformity (see Niederreiter 1992). This inequality effectively separates the integration error into two components: these correspond to the smoothness of the function f and the uniformity of the point set used in approximating the function. Hence for a fixed function f , point sets with better uniformity (i.e., lower discrepancy) should lead to lower integration errors.

In practice, it is possible to construct an s -dimensional point set with discrepancy bound $O(N^{-1}(\log N)^{s-1})$. Such sets of points are known as low-discrepancy point sets and are more uniformly distributed than a random point set. Quasi-Monte Carlo methods rely on these low-discrepancy point sets. There are two major approaches for constructing point sets that have low discrepancy: digital nets and integration lattices. We now give some references that describe digital nets and defer the detailed discussion of integration lattices to the next subsection. The digital net construction produces (t, m, s) -nets and (t, s) -sequences with a discrepancy of $O(N^{-1}(\log N)^{s-1})$ and $O(N^{-1}(\log N)^s)$, respectively. Explicit algorithms for constructing these sequences are given by van der Corput, Halton (1960), Sobol' (1967), Faure (1982), Niederreiter (1987, 1988), Tezuka (1995), and Niederreiter and Xing (1996, 1998). For details, see Niederreiter (1992) and Tezuka (1995). It follows from the Koksma-Hlawka inequality that these nets and sequences achieve an error bound of $O(N^{-1}(\log N)^{s-1})$ and $O(N^{-1}(\log N)^s)$, respectively, and are more efficient than the Monte Carlo rate of $O(N^{-1/2})$ for large N . In recent years many researchers (see, e.g., Joy, Boyle and Tan 1996; Paskov and Traub 1995; Ninomiya and Tezuka 1996; and Tan and Boyle 2000) have considered these low-discrepancy point sets or sequences in high-dimensional finance problems and have documented their superiority over the standard Monte Carlo method.

2.3 Lattice Rules

We now discuss another approach for approximating the integral (2.1) using the point set generated from the integration lattices. These methods, which were first introduced by Sloan and Kachoyan (1987), are known as lattice rules and are the generalization of the number theoretic quadrature rules of Korobov (1959) for integration over an s -dimensional hypercube. To describe this method, we first

introduce several definitions. A comprehensive treatment of this topic can be found in Sloan and Joe (1995) and Niederreiter (1992, chap. 5).

First, we define an s -dimensional *integration lattice* \mathcal{L} . An s -dimensional integration lattice is a discrete subset of \mathbb{R}^s that is closed under addition and subtraction and contains \mathbb{Z}^s as a subset. If the integration nodes P for equation (2.2) are a set of N points that lie in the intersection of the hypercube and an integration lattice, that is, $P = \mathcal{L} \cap [0, 1)^s$, then the resulting method is known as the lattice rule of order N , or simply an N -point lattice rule.

An important classification of lattice rules was established by Sloan and Lyness (1989), who showed that for any s -dimensional lattice rule, there exists a unique integer r , $1 \leq r \leq s$ and positive integers n_1, \dots, n_r , where n_{i+1} is an integer divisor of n_i for $i = 1, \dots, r-1$, and $n_r > 1$ such that the node set P can be derived from

$$\left\{ \frac{k_1}{n_1} \mathbf{z}_1 + \dots + \frac{k_r}{n_r} \mathbf{z}_r \right\} \quad \text{with } 0 \leq k_i < n_i \text{ for } 1 \leq i \leq r. \quad (2.5)$$

Here $\mathbf{z}_1, \dots, \mathbf{z}_r \in \mathbb{Z}^s$ are some suitable integer vectors, and $\{\mathbf{z}\}$ is defined by

$$\{\mathbf{z}\} = (\{\mathbf{z}_1\}, \dots, \{\mathbf{z}_s\}) \in [0, 1)^s,$$

where $\{\mathbf{z}\}$ denotes the fractional part of \mathbf{z} , that is, $\{\mathbf{z}\} = \mathbf{z} \pmod{1}$. The points in expression (2.5) are all distinct with $N = n_1 \cdots n_r$. The smallest integer r satisfying expression (2.5) is known as the rank of the lattice rule, while n_1, n_2, \dots, n_r are known as the invariants. Consequently the N -point lattice rule estimator of θ is given by

$$\hat{\theta}^{\text{LR}} = \frac{1}{N} \sum_{j_r=0}^{n_r-1} \cdots \sum_{j_1=0}^{n_1-1} f \left(\left\{ \frac{j_1}{n_1} \mathbf{z}_1, + \dots + \frac{j_r}{n_r} \mathbf{z}_r \right\} \right). \quad (2.6)$$

In this paper we consider a special case of lattice rules with rank 1. This particular class of lattice rules also corresponds to the method of good lattice points (g.l.p.) proposed by Korobov (1959) and Hlawka (1961).

The remainder of this subsection describes various characteristics of lattice rules. Subsection 2.3.1 discusses the error bounds arising from using these point sets. Subsection 2.3.2 describes an algorithm for constructing the g.l.p. Subsection 2.3.3 discusses the periodization method, and Subsection 2.3.4 presents a practical error estimation procedure.

2.3.1 Error Bounds

We now discuss briefly the error bounds associated with the lattice rules. Let $f(\mathbf{u})$, $\mathbf{u} \in [0, 1)^s$ be a periodic function on \mathbb{R}^s with period one in each of its s variables. Let $\hat{f}(\mathbf{h})$, $\mathbf{h} \in \mathbb{Z}^s$, denote the Fourier coefficient of f so that

$$f(\mathbf{u}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \hat{f}(\mathbf{h}) \exp(2\pi i \mathbf{h} \cdot \mathbf{u}), \quad (2.7)$$

where

$$\hat{f}(\mathbf{h}) = \int_{[0,1)^s} f(\mathbf{u}) \exp(-2\pi i \mathbf{h} \cdot \mathbf{u}) d\mathbf{u}, \quad \mathbf{h} \in \mathbb{Z}^s, \quad (2.8)$$

and the inner product $\mathbf{h} \cdot \mathbf{u} = h_1 u_1 + h_2 u_2 + \dots + h_s u_s$. Then it can be shown (see Theorem 2.8 of Sloan and Joe 1995) that the error arising from using the integration lattice \mathcal{L} can be expressed in terms of Fourier coefficients as

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\mathbf{x}_k) - \int_{[0,1)^s} f(\mathbf{u}) d\mathbf{u} \right| = \sum_{\mathbf{h} \neq \mathbf{0}, \mathbf{h} \in \mathcal{L}^\perp} \hat{f}(\mathbf{h}), \quad (2.9)$$

where $\{\mathbf{x}_0, \dots, \mathbf{x}_{N-1}\} \in \mathcal{L} \cap [0, 1)^s$ and \mathcal{L}^\perp corresponds to the *dual lattice* of the s -dimensional integration lattice \mathcal{L} , that is,

$$\mathcal{L}^\perp = \{\mathbf{h} \in \mathbb{R}^s : \mathbf{h} \cdot \mathbf{x} \in \mathbb{Z} \text{ for all } \mathbf{x} \in \mathcal{L}\}. \tag{2.10}$$

We now impose additional regularity on the function f . More formally, we say f belongs to the class of continuous periodic function $\mathcal{E}_\alpha^s(C)$ for a fixed $\alpha > 1$ provided that, for all nonzero $\mathbf{h} \in \mathbb{Z}^s$, we have

$$|f(\mathbf{h})| \leq C \left[\prod_{i=1}^s \max(1, |h_i|) \right]^{-\alpha} = Cr(\mathbf{h})^{-\alpha}, \tag{2.11}$$

where $r(\mathbf{h}) = \prod_{i=1}^s \max(1, |h_i|)$ and $C > 0$ is a constant that does not depend on \mathbf{h} . The rate of decay of the Fourier coefficient of a function is related to the smoothness of the function. Hence $\mathcal{E}_\alpha^s(C)$ can be interpreted as characterizing a class of functions that has a certain smoothness property.

If $f \in \mathcal{E}_\alpha^s(C)$, it follows from expression (2.11) that the quadrature error in equation (2.9) is bounded from above by

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f(\mathbf{x}_k) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq C \cdot \sum_{\mathbf{h} \neq \mathbf{0}, \mathbf{h} \in \mathcal{L}^\perp} r(\mathbf{h})^{-\alpha}. \tag{2.12}$$

In the special case of lattice rules of rank 1, the dual lattice in equation (2.10) becomes

$$\mathcal{L}^\perp = \{\mathbf{h} \in \mathbb{Z}^s : \mathbf{h} \cdot \mathbf{x} \equiv 0 \pmod{N}\} \tag{2.13}$$

so that the error bound corresponding to expression (2.12) reduces to

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f\left(\left\{\frac{k}{N} \mathbf{z}\right\}\right) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq CP_\alpha(\mathbf{z}, N), \tag{2.14}$$

where

$$P_\alpha(\mathbf{z}, N) = \sum_{\substack{\mathbf{h} \neq \mathbf{0} \\ \mathbf{z} \cdot \mathbf{h} \equiv 0 \pmod{N}}} r(\mathbf{h})^{-\alpha}. \tag{2.15}$$

Korobov (1959) establishes that for each $s > 1$ and $\alpha > 1$ and each prime number N , there exists \mathbf{z} and $\beta(s, \alpha)$ such that

$$P_\alpha(\mathbf{z}, N) = O(N^{-\alpha}(\log N)^{\beta(s,\alpha)}). \tag{2.16}$$

Sequences of vectors $\mathbf{z} = \mathbf{z}(N)$ satisfying equation (2.16) are traditionally known as g.l.p., and the components of \mathbf{z} are referred to as the *optimal coefficients*. When N is prime, Bahvalov (1959) shows that $\beta(s, \alpha) = \alpha(s - 1)$. The work of Šargin (1963) indicates that the lower bound on $\beta(s, \alpha)$ is at least $s - 1$. Šargin's lower bound is attained only for $s = 2$. Niederreiter (1993) also generalizes the bound for composite N with $\beta(s, \alpha)$ either equal to $\alpha(s - 1)$ or $\alpha(s - 1) + 1$.

It follows from equation (2.16) that error bound expression (2.14) can be expressed as

$$\left| \frac{1}{N} \sum_{k=0}^{N-1} f\left(\left\{\frac{k}{N} \mathbf{z}\right\}\right) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| = O\left(\frac{(\log N)^{\beta(s,\alpha)}}{N^\alpha}\right). \tag{2.17}$$

2.3.2 Construction of g.l.p.

The result described in the last subsection indicates that optimal coefficients \mathbf{z} exist for which the error bound (2.17) is attained. We now discuss an approach of finding these optimal coefficients. It turns out that these g.l.p. are neither rare nor unique. There exists a simple construction for two-dimensional g.l.p. In higher dimensions, we have existence theorems only for g.l.p. In 1959 Korobov demonstrated the existence of the g.l.p. for N that is either prime or the product of two primes. It also follows from his existence proof that for prime N , half or more of all the possible vectors \mathbf{z} can be suitable choices for g.l.p. Several authors have subsequently relaxed the restriction on N in Korobov's existence proof. Keast (1973) shows the existence of g.l.p. of order N when N is a product of more than two primes, while Niederreiter (1978) shows that g.l.p. exists for all N .

A common approach to the construction of g.l.p. is to formulate it as an optimization algorithm. The key idea underlying this method is to recognize that the worst function in the class of $\mathcal{E}_\alpha^s(C)$ can be expressed explicitly. The associated vector \mathbf{z} that minimizes the error from estimating the worst function is then the required optimal coefficient. It follows that the g.l.p. that minimizes the worst function error bound tends to give a smaller error for any arbitrary function $f \in \mathcal{E}_\alpha^s(C)$.

Let us define f_α as a function with the following representation:

$$f_\alpha(\mathbf{u}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \frac{1}{r(\mathbf{h})^\alpha} \exp(2\pi i \mathbf{h} \cdot \mathbf{u}).$$

Then $\int_{[0,1]^s} f_\alpha(\mathbf{u}) d\mathbf{u} = 1$, and f_α is the worst function in $\mathcal{E}_\alpha^s(1)$ in that equality holds for the error bound (2.14), that is,

$$\left| \frac{1}{N} \sum_{k=1}^N f_\alpha\left(\frac{k}{N} \mathbf{z}\right) - 1 \right| = \hat{P}_\alpha(\mathbf{z}, N). \tag{2.18}$$

Hence for arbitrary $f \in \mathcal{E}_\alpha^s(C)$, we have

$$\left| \frac{1}{N} \sum_{k=1}^N f\left(\frac{k}{N} \mathbf{z}\right) - \int_{[0,1]^s} f(\mathbf{u}) d\mathbf{u} \right| \leq C \hat{P}_\alpha(\mathbf{z}, N) \tag{2.19}$$

with equality holding if $f = C f_\alpha$. This explicit bound replaces the asymptotic result (2.17) and holds for any given N and \mathbf{z} . More importantly, the above bound also provides a way of assessing integration lattices. Lattice points with smaller $\hat{P}_\alpha(\mathbf{z}, N)$ should therefore lead to smaller error bounds.

Korobov showed that for fixed α , the g.l.p. can be constructed by calculating $\hat{P}_\alpha(\mathbf{z}, N)$ in inequality (2.19) over all possible vectors \mathbf{z} and choosing the \mathbf{z} that minimizes that quantity. In other words, searching the g.l.p. reduces to finding the vector \mathbf{z} that minimizes $\hat{P}_\alpha(\mathbf{z}, N)$. To facilitate the computation, a more convenient representation of f_α is to express it as a product of functions of a single variable (see Haber 1983):

$$f_\alpha(\mathbf{u}) = \prod_{i=1}^s F_\alpha(u_i),$$

where

$$F_\alpha(u) = \sum_{h=-\infty}^{\infty} \frac{\exp(2\pi i h u)}{\max(1, |h|)^\alpha}.$$

The regularity parameter α is typically assumed to be an even number since in these cases F_α can be expressed explicitly in terms of a Bernoulli polynomial. For instance, with $\alpha = 2$ and 4 , we have

$$F_2(u) = 1 + 2\pi^2 \left(u^2 - u + \frac{1}{6} \right),$$

$$F_4(u) = 1 + \frac{\pi^4}{45} (1 - 30u^2(1 - u)^2),$$

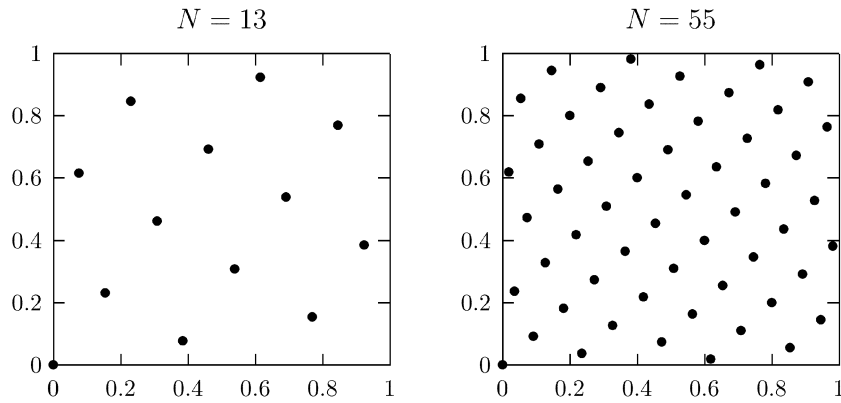
respectively, for $u \in [0, 1)$.

The direct approach of finding a g.l.p. over all possible \mathbf{z} becomes prohibitive for large values of N and s since there are N^s possible choices of \mathbf{z} to search through. For this reason, Korobov (1959) considers a more restricted but manageable structure of the lattice points that is of the form

$$\mathbf{z}(l) = (1, l, l^2 \bmod N, \dots, l^{s-1} \bmod N), \quad 1 \leq l < N.$$

Each vector \mathbf{z} thus consists of only one parameter with a total of $N - 1$ possible choices, as opposed to N^s . A further reduction in the work involved is by noting that

Figure 1
Good Lattice Points Generated from Fibonacci Numbers with $N = 13$ and 55



$$F_\alpha(u) = F_\alpha(1 - u), \quad u \in [0, 1).$$

This implies $f_\alpha(kz(l)/N) = f_\alpha((N - k)z(l)/N)$ so that the search is reduced to almost half by considering only $z(l)$ for $1 \leq l \leq \lfloor N/2 \rfloor$.

Tables of g.l.p. obtained by minimizing $\hat{P}_2(s, N)$ can be found in Haber (1983). Other tables of g.l.p. can be found in Saltykov (1963) (which are reproduced in Stroud 1970), Hua and Wang (1981), and Sloan and Joe (1995).³

To conclude this subsection, we remark that there exists an explicit construction for specific values of N in the two-dimensional case. This is the only known situation where Šargin’s lower bound is achieved. The construction algorithm for this particular case is relatively simple. It is based on the Fibonacci numbers and can briefly be described as follows: first, recall that the Fibonacci numbers are defined as

$$F_1 = 1, \quad F_2 = 1, \quad F_m = F_{m-1} + F_{m-2}, \quad m \geq 3,$$

where F_m is the m -th Fibonacci number. The g.l.p. of N elements is then constructed as

$$\mathbf{u}_k = \left(\left\{ \frac{k}{N} \right\}, \left\{ \frac{kF_{m-1}}{N} \right\} \right), \quad k = 1, \dots, N$$

with the optimal coefficient $\mathfrak{x} = (1, F_{m-1})$, $N = F_m$ and $m \geq 3$. See Bahvalov (1959) for details. Since the Fibonacci numbers are defined only on certain set of integers, the above construction yields two-dimensional g.l.p. for specific values of N . The first 15 Fibonacci numbers are $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610\}$. Figure 1 demonstrates the g.l.p. generated from the Fibonacci numbers with $N = 13$ and 55 . Note that the points are evenly distributed over the unit-square.

2.3.3 Periodization

We have mentioned that the g.l.p. method exploits the additional regularity of the function such as the periodicity (with respect to each component of \mathbf{u}). Most functions encountered in practice do not meet these criteria. Some functions may be periodic in some of the components but not all. In these situations it is necessary to carry out a preliminary transformation to convert a sufficiently regular nonperiodic integrand into an integrand that has the required periodicity. One of the most common

³Note that the g.l.p. constructed by Hua and Wang (1981) is based on ideas from algebraic number theory and rational approximation methods.

periodization procedures is through the change-of-variable technique by first defining a nonlinear transformation for each variable as

$$u = \psi(t),$$

where ψ is a smooth increasing function that maps $[0, 1)$ onto $[0, 1)$ and $\psi^{(j)}(0) = \psi^{(j)}(1) = 0$ for $1 \leq j \leq \alpha$. It follows from the standard change-of-variable techniques that the transformed periodized ϕ is given by

$$\phi(\mathbf{u}) = f(\psi(u_1), \dots, \psi(u_s))\psi'(u_1) \cdots \psi'(u_s). \quad (2.20)$$

Here we list two common functions of $\psi(t)$ that are based on polynomial and trigonometric functions. We refer to these transformations as the polynomial- m and \sin^m -transformation, respectively. The first three polynomial- m transformations are

$$\begin{aligned} \psi_2(t) &= 3t^2 - 2t^3, \\ \psi_3(t) &= 10t^3 - 15t^4 + 6t^5, \\ \psi_4(t) &= 35t^4 - 84t^5 + 70t^6 - 20t^7, \end{aligned}$$

while the first four \sin^m -transformations are

$$\begin{aligned} \psi_1(t) &= \frac{1}{2} (1 - \cos \pi t), \\ \psi_2(t) &= \frac{1}{2\pi} (2\pi t - \sin 2\pi t), \\ \psi_3(t) &= \frac{1}{16} (8 - 9 \cos \pi t + \cos 3\pi t), \\ \psi_4(t) &= \frac{1}{12\pi} (12\pi t - 8 \sin 2\pi t + \sin 4\pi t). \end{aligned}$$

For additional information on the various periodization methods, see Hua and Wang (1981), Zaremba (1972), and Beckers and Haegemans (1992).

2.3.4 Error Estimation

In this subsection we consider a practical approach of estimating the error associated with the lattice methods. This was originally proposed by Cranley and Patterson (1976) in the context of rank-1 rule and subsequently extended to lattice rules of arbitrary rank by Joe (1990). This method is known as the “random shift” approach and can be described briefly as follows: For a given lattice point set $P = \{x_0, \dots, x_{N-1}\}$ we generate m i.i.d. vectors of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ uniformly distributed numbers in $[0, 1)^s$. Let θ_i be the estimate of the function f corresponding to the set of N nodes $\{P + \mathbf{v}_i\}$, that is, P is shifted by \mathbf{v}_i modulo 1. If $\mathbf{v}_i, i = 1, \dots, m$ are chosen independently and randomly from a multivariate uniform distribution on $[0, 1)^s$, then $\theta_i, i = 1, \dots, m$ are i.i.d. random variables with expected value equal to θ . This implies that the sample mean over m independent replications

$$\hat{\theta}^{\text{g.l.p.}} = \frac{1}{m} \sum_{i=1}^m \theta_i$$

is an unbiased estimator of θ with estimated standard error $\sqrt{1/m(m-1) \sum_{i=1}^m (\theta_i - \hat{\theta}^{\text{g.l.p.}})^2}$. This facilitates the construction of the confidence intervals.

It was also pointed out in Tuffin (1996) that the random shift technique can be applied to other low-discrepancy sequences such as those by Halton (1960), Sobol' (1967), Faure (1982), and Nieder-

reiter (1987, 1988) in providing an estimate of the standard errors. In our numerical work, we will also be using Sobol' sequences. We will use the same technique to estimate the standard errors.

3. DERIVATIVE PRICING

In this section we provide a brief overview of derivative pricing and then discuss a special case that will be useful in our later examples. For a detailed analysis see Duffie (1996). In modern finance, security prices are often modeled as stochastic processes to reflect future uncertainty. If we assume that there is no arbitrage in the financial market, then loosely speaking this implies the market prices of securities suitably normalized are martingales.⁴ This martingale property means that we can write the current price of an asset as an expectation of its future cash flows under a probability measure known as the equivalent martingale measure. When the market is complete, the equivalent probability measure is unique. We shall make this assumption in the rest of the paper and call it the Q measure. We now illustrate how this approach is used to develop a valuation formula for a European option based on several assets. A European option is a security that gives its owner the right to purchase a certain asset or receive a certain payoff at some fixed future date.

We consider an economy with s risky assets with prices $S_t = (S_{1t}, \dots, S_{st})'$ at time t . We assume that each asset i has a proportional dividend yield δ_i that is constant for each asset. The volatility of asset i , $1 \leq i \leq s$, is σ_i , which is also assumed constant for each asset. The asset returns follow a multivariate lognormal distribution so that our asset price dynamics correspond to the multivariate extension of the single-asset Black-Scholes-Merton model. The risk-free rate r is assumed to be constant. We now describe the asset price distribution more fully.

Let $\vartheta_{it} = \log S_{it}$, $1 \leq i \leq s$. Then $\boldsymbol{\vartheta}_t = (\vartheta_{1t}, \dots, \vartheta_{st})'$ is normally distributed with mean

$$\begin{aligned} \boldsymbol{\mu}_t &= (\mu_{1t}, \dots, \mu_{st})' \\ &= \left(\log S_{10} + \left(r - \delta_1 - \frac{1}{2} \sigma_1^2 \right) t, \dots, \log S_{s0} + \left(r - \delta_s - \frac{1}{2} \sigma_s^2 \right) t \right)' \end{aligned} \quad (3.1)$$

and covariance matrix

$$\Sigma_t = (\sigma_{ij}) = \rho_{ij} \sigma_i \sigma_j t. \quad (3.2)$$

The current price of any derivative security in this market is equal to the (discounted) expected value of its payoff under the Q measure. For example, if the payoff on a European derivative security is $p(S_T)$, where $p(S_T) = p(S_{1T}, \dots, S_{sT})$ depends on the terminal asset prices (S_{1T}, \dots, S_{sT}) , the current price of this derivative security is

$$\begin{aligned} V &= e^{-rT} E_Q[p(S_T)] \\ &= e^{-rT} \int p(\mathbf{s}) f(\mathbf{s}) d\mathbf{s}, \end{aligned} \quad (3.3)$$

where $f(\mathbf{s})$ is an s -variate lognormal distribution with probability density function

$$f(\mathbf{s}) = \frac{1}{\sqrt{|\Sigma_T|} (2\pi)^s s_1 \cdots s_s} \exp \left\{ -\frac{1}{2} (\log \mathbf{s} - \boldsymbol{\mu}_T)' \Sigma_T^{-1} (\log \mathbf{s} - \boldsymbol{\mu}_T) \right\},$$

where $\boldsymbol{\mu}_T$ and Σ_T are defined respectively in equations (3.1) and (3.2), and $\log \mathbf{s} = (\log(s_1), \dots, \log(s_s))'$.

Thus to obtain the price of a European derivative security we need to evaluate the integral in equation (3.3). The complexity of the problem depends on the structure of the payoff function $p(S_T)$. It is only

⁴ For a more precise statement see Delbaen and Schachermayer (2004).

in rare cases that analytic solutions are available, hence numerical methods are often used to approximate equation (3.3). The next section considers some specific examples and compares the relative efficiency of the three sampling methods discussed in the last section.

4. APPLICATIONS

In this section we illustrate the numerical valuation of several exotic options using the g.l.p. approach and analyze its efficiency. We also discuss the transformation that is necessary to convert the integration domain to the required $[0, 1]^s$ domain. We demonstrate that for reasonably low dimensions, the multiple integral (3.3) can be effectively approximated to high precision using the g.l.p. method. This method also compares favorably to other numerical techniques. The following subsection considers spread options (as well as the exchange options). Subsection 4.2 examines the discretely monitored path-dependent options such as lookback options. Subsection 4.3 discusses the pricing of a particular type of equity-indexed annuity.

4.1 Spread Options

In this subsection we consider the pricing of spread options. Spread options are options whose payoff depends on the difference in the prices of two underlying assets. In the general case, the payoff of a spread call option at maturity is given by

$$p(S_T) = \max[\varpi_2 S_{2T} - \varpi_1 S_{1T} - K, 0], \quad (4.1)$$

where K is the strike price of the option, and the weights ϖ_1 and ϖ_2 are assumed to be positive. When $K = 0$, spread options reduce to exchange options. The pricing formula for this particular case with $\varpi_1 = \varpi_2 = 1$ is given in Margrabe (1978) and Stultz (1982).

The spread option with nonzero strike price is of particular interest because there is no closed-form solution available. Several numerical approximation methods have been proposed. One simple approach suggested by Wilcox (1990) is to assume that the spread is normally distributed. Although this assumption leads to a very simple pricing formula, its drawback is that it is inconsistent with the underlying lognormality assumption of the asset prices. A slightly more accurate, but also more involved, approach is that due to Pearson (1995). Other approaches to pricing the spread options include the bivariate binomial and trinomial approximations proposed by Boyle (1988) and Boyle, Evnine, and Gibbs (1989). Simulation is also an alternative approach, although the drawbacks of this method, according to Pearson, are "somewhat limited accuracy and the computational effort involved." In this subsection we show that the number-theoretic methods using g.l.p. can be a very effective tool for pricing these options. The numerical examples also indicate that this method yields much higher accuracy than the approximation algorithm proposed by Pearson (1995).

We now describe how spread options can be priced using sampling methods. By substituting equation (4.1) into equation (3.3), we obtain the current price of the spread call option, V , as

$$\begin{aligned} V &= e^{-rT} E_Q [\max[\varpi_2 S_{2T} - \varpi_1 S_{1T} - K, 0]] \\ &= e^{-rT} \int_0^\infty \int_0^\infty \max[\varpi_2 s_2 - \varpi_1 s_1 - K, 0] f(\mathbf{s}) ds. \end{aligned} \quad (4.2)$$

It is shown in Appendix A that equation (4.2) reduces to the following integrals with integration domain $[0, 1]^2$ (see expression [A.4]):

$$V = e^{-rT} \int_{[0,1]^2} h^*(\mathbf{u}) d\mathbf{u}, \quad (4.3)$$

where

$$\begin{aligned}
h^*(\mathbf{u}) &= h^*(u_1, u_2) = (1 - d_2(\mathbf{u}))h(\mathbf{u}) \\
h(\mathbf{u}) &= h_2(\mathbf{u}) - h_1(\mathbf{u}) - K \\
h_1(\mathbf{u}) &= \varpi_1 e^{h_3(\mathbf{u})} \\
h_2(\mathbf{u}) &= \varpi_2 e^{h_4(\mathbf{u})} \\
h_3(\mathbf{u}) &= c_{11}\Phi^{-1}(u_1) + \mu_{1T} \\
h_4(\mathbf{u}) &= c_{21}\Phi^{-1}(u_1) + c_{22}h_5(\mathbf{u}) + \mu_{2T} \\
h_5(\mathbf{u}) &= \Phi^{-1}(d_2(\mathbf{u}) + u_2(1 - d_2(\mathbf{u}))) \\
d_2(\mathbf{u}) &= \Phi(g(\mathbf{u})) \\
g(\mathbf{u}) &= \frac{\log(h_1(\mathbf{u}) + K) - \log(\varpi_2) - \mu_{2T} - c_{21}\Phi^{-1}(u_1)}{c_{22}},
\end{aligned}$$

and c_{ij} is the (i, j) entry of the Cholesky decomposition lower triangular matrix C , that is, $CC' = \Sigma_T$.

The double integral (4.3) is now readily approximated using either Monte Carlo or quasi-Monte Carlo sampling methods. Suppose $\mathbf{u}_n = (u_{n1}, u_{n2}) \in [0, 1]^2$ denotes the n -th term of a random or low-discrepancy sequence; the unbiased Monte Carlo or quasi-Monte Carlo estimator is then obtained as

$$\hat{V} = \frac{e^{-rT}}{N} \sum_{n=1}^N h^*(u_{n1}, u_{n2}). \quad (4.4)$$

For the lattice rules approach, we need one more step. The function $h^*(u_1, u_2)$ is usually not periodic, and hence an additional transformation is required to ensure the periodicity. Accordingly, estimate (4.4) is revised as

$$\bar{V} = \frac{e^{-rT}}{N} \sum_{n=1}^N h^*(\psi(u_{n1}), \psi(u_{n2}))\psi'(u_{n1})\psi'(u_{n2}), \quad (4.5)$$

where (u_{n1}, u_{n2}) is the g.l.p. and ψ is an appropriate periodization transformation discussed in Subsection 2.3.3.

We now demonstrate the relative merits of g.l.p. in using both equations (4.4) and (4.5). This is carried out in the following three parts.

Part A

In the first part of the analysis, we consider a special case of a spread option with $K = 0$ and $\varpi_1 = \varpi_2 = 1$. This becomes the exchange options, and a closed-form solution exists (see Margrabe 1978). The analytic value is then used to gauge the accuracy of the lattice methods. Rather than drawing conclusions based on a few predetermined sets of parameter values, we assess the efficiency of the proposed method by generating 50 option contracts randomly. The parameter values are obtained based on the following procedure: For the first asset, we set $S_{10} = 100$, $\sigma_1 = 30\%$, and $\delta_1 = 5\%$. The parameter values for the second asset are generated randomly so that they are uniformly distributed in the intervals $[50, 130]$, $[10\%, 50\%]$, and $[1\%, 10\%]$ for S_{20} , σ_2 , and δ_2 , respectively. Fixing S_{10} while randomizing S_{20} allows us to have a better control on whether the option is in-the-money, out-of-the-money, or at-the-money. The remaining parameters r , ρ_{12} , and T are also randomly selected so that they are uniformly distributed in the range $[1\%, 15\%]$, $[-0.8, 0.8]$, and $[6 \text{ months}, 1 \text{ year}]$, respectively. Based on the randomly generated option contracts, we report the root-mean-square relative error (RMSE) calculated as

$$RMSE = \sqrt{\frac{1}{50} \sum_{i=1}^{50} \left(\frac{\hat{V}_i - V_i}{V_i} \right)^2},$$

Table 1
Comparisons of the RMSE for the 50 Randomly Generated Exchange Option Contracts

N	Periodization Transformation				No Periodization
	Polynomial-3	Polynomial-4	\sin^2	\sin^3	
13	2.76679%	3.96266%	1.78029%	5.42768%	25.65%
21	0.70174	0.61241	0.39638	0.83536	16.77
34	0.29790	0.14219	0.16876	0.12909	13.00
55	0.05446	0.01800	0.02765	0.01666	8.27
89	0.02395	0.00505	0.01411	0.00458	6.24
144	0.00509	0.00109	0.00243	0.00067	3.90
233	0.00215	0.00010	0.00125	0.00006	2.91
377	0.00046	0.00004	0.00021	0.00003	1.80
610	0.00019	0.00001	0.00011	0.00001	1.33
987	0.00004	0.00000	0.00002	0.00000	0.81

where \hat{V}_i is the estimated price of the i -th option contract and V_i is the corresponding theoretical true value. The option price for the randomly selected option contract is enforced to be at least 0.5, as very low option values may lead to a less reliable estimate of RMSE.

Table 1 reports the RMSE as a percentage using the two-dimensional g.l.p. generated from Fibonacci numbers and for point sets in the range $10 < N < 1000$. The polynomial transformation with $m = 3, 4$ and sine transformation with $m = 2, 3$ are considered. To examine the impact of exploiting additional smoothness of the periodicity, the same set of g.l.p. is also applied to equation (4.4), that is, without any periodization. Two conclusions can be drawn from these results. First, the g.l.p. method yields extremely high precision, even for a small point set. For instance, with merely 55 points, the largest RMSE across different periodization techniques is only 0.05% and goes as low as 0.017%. Second, it is desirable to periodize the function. Without periodization, the resulting RMSE based on $N = 55$ is $8.27/0.01666 \approx 496$ times larger than the corresponding periodized estimate using \sin^3 -transformation.

Part B

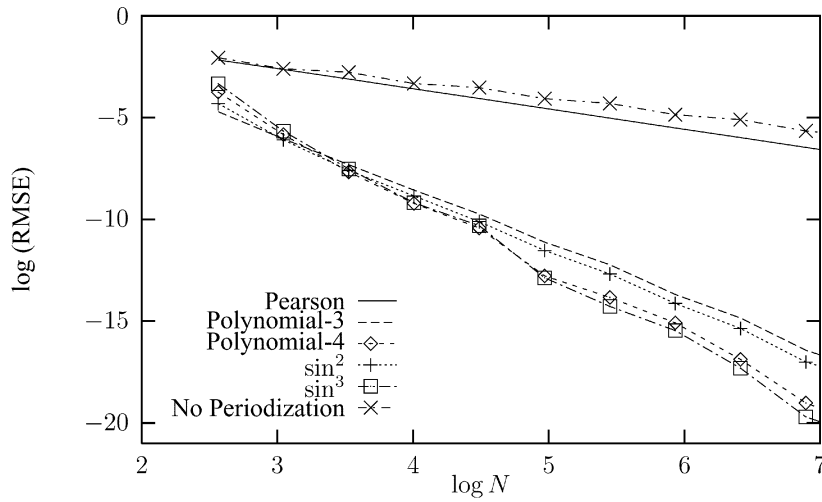
We now proceed to the more relevant problem by relaxing the constraint that $K = 0$. This is the situation in which the closed-form solution is not available. For our comparison, the option estimate obtained from the g.l.p. method using $N = 121,393$ is taken to be the “true” option value. This is reasonable given the success we observed for the exchange options in Part A. We also use the same set of parameter values as Pearson (1995), who considers 144 spread-call option contracts with the parameter values comprising the following: $S_{10} \in \{92, 96, 100, 104\}$, $S_{20} = 100$, $\sigma_1 \in \{10\%, 20\%, 30\%\}$, $\sigma_2 = 20\%$, $T = 1$ week, 1 month, 1 year, 5 years, $\rho_{12} \in \{-0.5, 0, 0.5\}$, $\delta_1 = \delta_2 = 5\%$, $r = 10\%$. The RMSEs are computed over these option contracts using both the g.l.p. and Pearson’s approximation method. Figure 2 plots $\log N$ against the computed $\log(\text{RMSE})$. The graph indicates that Pearson’s method is more efficient than the g.l.p. that does not exploit periodicity. Once we periodize the integrand, the g.l.p. method yields a faster convergence rate. Similar to the results of Part A, the transformations based on polynomial-4 and \sin^3 are the most efficient.

Part C

In this part of the analysis, we discuss the estimation of the option sensitivity parameters known as the Greeks. Accurate estimation of the Greeks (e.g., delta or gamma) is of particular importance in the context of hedging. It is well known that the Monte Carlo approach of estimating these parameters is inefficient. Here we will demonstrate that the g.l.p. method can be a very efficient tool in estimating these Greeks. Formally, the delta (Δ_i) and the gamma (γ_i) of the option with respect to first and second assets are defined as

Figure 2

Comparisons of the RMSEs Using g.l.p. and Pearson's Method Based on Pearson (1995) Examples



$$\Delta_i = \frac{\partial V}{\partial S_i},$$

$$\Gamma_i = \frac{\partial \Delta_i}{\partial S_i} = \frac{\partial^2 V}{\partial S_i^2}, \quad i = 1, 2,$$

where V is the value of the derivative security.

For the Monte Carlo methods, these sensitivity parameters are typically estimated using the finite difference resimulation technique.⁵ For example, a crude estimate of the delta can be approximated as

$$\Delta = \frac{\hat{V}(S_o + h) - \hat{V}(S_o)}{h}$$

for a small perturbation h , and $\hat{V}(S_o)$ is the simulated value of an option based on N simulation runs and a given initial level of S_o . The above expression is the standard forward difference approximation of the first derivative. A better estimate of the delta, which is proposed by Shamma (1995), is based on an extended central difference formula:

$$\Delta = \frac{\hat{V}(S_o - 2h) - 8\hat{V}(S_o - h) + 8\hat{V}(S_o + h) - \hat{V}(S_o + 2h)}{12h}. \tag{4.6}$$

Note that this formula requires twice the simulation effort as compared to the forward difference approximation. Similarly, the gamma or the second derivative can be estimated via

$$\Gamma = \frac{-\hat{V}(S_o - 2h) + 16\hat{V}(S_o - h) - 30\hat{V}(S_o) + 16\hat{V}(S_o + h) - \hat{V}(S_o + 2h)}{12h^2}. \tag{4.7}$$

In contrast to the resimulation method, this paper considers a direct approach of calculating the Greeks by recognizing that from equation (4.3), the delta and the gamma can be calculated via

⁵ See Boyle, Broadie, and Glasserman (1997) for a comprehensive treatment of this subject along with other more efficient approaches.

$$\begin{aligned} \Delta_i &= \frac{\partial V}{\partial S_i} = e^{-rT} \frac{\partial}{\partial S_i} \left\{ \int_{[0,1]^2} (1 - d_2) h(u) du \right\} \\ &= e^{-rT} \int_{[0,1]^2} \frac{\partial}{\partial S_i} \{(1 - d_2) h(u)\} du \\ &= e^{-rT} \int_{[0,1]^2} \left[(1 - d_2) \frac{\partial h}{\partial S_i} - \frac{\partial d_2}{\partial S_i} h(u) \right] du, \end{aligned} \tag{4.8}$$

$$\Gamma_i = \frac{\partial^2 V}{\partial S_i^2} = e^{-rT} \int_{[0,1]^2} \left[(1 - d_2) \frac{\partial^2 h}{\partial S_i^2} - 2 \frac{\partial h}{\partial S_i} \frac{\partial d_2}{\partial S_i} - \frac{\partial^2 d_2}{\partial S_i^2} h(u) \right] du, \tag{4.9}$$

for $i = 1, 2$, and the partial derivatives are defined in Appendix B.

For our numerical illustration, we consider the parameter values: $S_1 \in \{96, 100, 104\}$, $\sigma_1 = 30\%$, $\delta_1 = 5\%$, $S_2 = 100$, $\sigma_2 = 20\%$, $\delta_2 = 5\%$, $\rho = 50\%$, $r = 5\%$, $T = 5$ years, and $K = 4$. For each example and for each N , we compute the option value, the delta, and the gamma with respect to the first and second stock using equations (4.8) and (4.9). Furthermore, we also use the random shift approach (as discussed in Subsection 2.3.4) to generate 10 independent estimates to estimate its standard error. The results are summarized in Table 2 for $N = 55$ and 233. The column labeled ‘‘MC’’ corresponds to

Table 2
Estimates of Option Values and Their Greeks

S_1	MC	Std. Error	Polynomial-3	\sin^2	No Periodization
$N = 55$					
96: V	18.40710	0.86420	17.50118(659)	17.50141(821)	17.75481(3)
Δ_1	-0.30220	0.00935	-0.29834(408)	-0.29833(485)	-0.29889(2)
Γ_1	0.00508	0.00009	0.00520(107)	0.00520(112)	0.00511(1)
Δ_2	0.49288	0.01745	0.47938(497)	0.47938(597)	0.48262(2)
Γ_2	0.00509	0.00009	0.00521(109)	0.00521(113)	0.00513(1)
100: V	17.23814	0.83425	16.34858(676)	16.34884(847)	16.59932(3)
Δ_1	-0.28249	0.00912	-0.27818(391)	-0.27817(462)	-0.27906(2)
Γ_1	0.00477	0.00010	0.00488(156)	0.00488(167)	0.00481(2)
Δ_2	0.47275	0.01730	0.45879(488)	0.45879(583)	0.46236(2)
Γ_2	0.00519	0.00011	0.00530(169)	0.00530(179)	0.00523(2)
104: V	16.14555	0.80505	15.27407(695)	15.27435(877)	15.52075(3)
Δ_1	-0.26400	0.00887	-0.25928(377)	-0.25928(443)	-0.26043(2)
Γ_1	0.00447	0.00010	0.00457(248)	0.00457(267)	0.00451(2)
Δ_2	0.45310	0.01713	0.43871(482)	0.43871(573)	0.44256(2)
Γ_2	0.00526	0.00012	0.00537(281)	0.00537(296)	0.00530(2)
$N = 233$					
96: V	17.60447	0.45778	17.50288(19242)	17.50288(23467)	17.45274(6)
Δ_1	-0.29574	0.00449	-0.29835(11235)	-0.29835(13228)	-0.29676(7)
Γ_1	0.00507	0.00007	0.00520(5929)	0.00520(7561)	0.00516(5)
Δ_2	0.47803	0.00904	0.47942(15296)	0.47942(18321)	0.47737(7)
Γ_2	0.00509	0.00007	0.00521(5626)	0.00521(6997)	0.00518(5)
100: V	16.46125	0.44137	16.35022(19468)	16.35022(23768)	16.30614(6)
Δ_1	-0.27608	0.00434	-0.27820(10809)	-0.27820(12899)	-0.27675(7)
Γ_1	0.00476	0.00006	0.00488(5430)	0.00488(6580)	0.00484(5)
Δ_2	0.45794	0.00893	0.45883(14868)	0.45883(17939)	0.45692(7)
Γ_2	0.00518	0.00007	0.00530(5037)	0.00530(6000)	0.00527(5)
104: V	15.39421	0.42535	15.27564(19757)	15.27564(24142)	15.23706(6)
Δ_1	-0.25764	0.00420	-0.25930(10477)	-0.25930(12697)	-0.25799(7)
Γ_1	0.00446	0.00006	0.00457(5084)	0.00457(6039)	0.00454(5)
Δ_2	0.43835	0.00882	0.43875(14522)	0.43875(17652)	0.43698(7)
Γ_2	0.00524	0.00007	0.00537(4665)	0.00537(5523)	0.00533(5)

Notes: The parameter values are $S_1 \in \{96, 100, 104\}$, $\sigma_1 = 30\%$, $\delta_1 = 5\%$, $S_2 = 100$, $\sigma_2 = 20\%$, $\delta_2 = 5\%$, $\rho = 50\%$, $r = 5\%$, $T = 5$ years, and $K = 4$. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The Monte Carlo standard errors are reported in the fourth column. The values in parentheses are the efficiency ratios as measured by the ratio of the standard error of the basic Monte Carlo method to the standard error of the g.l.p.

the Monte Carlo sampling method, while the other reported results are based on g.l.p. (with and without periodization). Note that with such a small set of sampling points, the standard errors of the Monte Carlo estimates are fairly large. On the other hand, the standard errors arising from the g.l.p. method with $N = 233$ are several thousands smaller than the respective Monte Carlo-based standard errors, indicating that the g.l.p. is not only efficient at estimating the option value but also extremely efficient for the Greeks. The g.l.p. without periodization, however, is marginally more efficient than the Monte Carlo methods.

4.2 Lookback Options

In this subsection we examine another type of exotic option known as lookback options. In particular, we consider a discrete fixed-strike lookback call option with the following payoff structure:

$$\max[\max(S_1, S_2, \dots, S_s) - K, 0],$$

where S_i denotes the price of the underlying asset at time t_i , and $0 < t_1 < \dots < t_s = T$ are the s discrete monitoring time points. When the sampling frequency of the assets is continuous, the value of the corresponding option converges to a simple Black-Scholes-type expression as shown in Goldman, Sosin, and Gatto (1979) and Conze and Viswanathan (1991). The discrete sampling case has been considered by Heynen and Kat (1995) and Dufresne, Keirstead, and Ross (1996). The analytic expression for the value of the option in this case can be expressed in terms of the cumulative distribution function (CDF) of a standard multivariate normal r.v. as follows:

$$V_L(S_o, K, r, \delta, \sigma, t_1, \dots, t_s) = S_o \sum_{i=1}^s H_i I_{s-i} e^{(r-\delta)t_i - rT} - Ke^{-rT} (1 - L), \quad (4.10)$$

where

$$\begin{aligned} H_i &= N_i \left(\frac{\alpha_1}{\sigma} \sqrt{t_i - t_1}, \dots, \frac{\alpha_1}{\sigma} \sqrt{t_i - t_{i-1}}, d_1(K, t_i), R_i \right), \\ I_{s-i} &= N_{s-i} \left(-\frac{\alpha_2}{\sigma} \sqrt{t_{i+1} - t_i}, \dots, -\frac{\alpha_2}{\sigma} \sqrt{t_s - t_i}, \Psi_{s-i} \right), \quad \text{with } I_0 = 1, \\ L &= N_s(-d_2(K, t_1), \dots, -d_2(K, t_s), \Sigma), \\ d_1(K, t_i) &= \frac{\log(S_o/K) + (r - \delta + \sigma^2/2)t_i}{\sigma\sqrt{t_i}}, \\ d_2(K, t_i) &= d_1(K, t_i) - \sigma\sqrt{t_i}, \\ \alpha_1 &= r - \delta + \frac{\sigma^2}{2}, \\ \alpha_2 &= \alpha_1 - \sigma^2, \\ \Sigma &= (\gamma_{jk})_{s \times s} \quad \text{with } \gamma_{jk} = \sqrt{\frac{t_{j \wedge k}}{t_{j \vee k}}}, \quad 1 \leq j, k \leq s, \\ \Psi_{s-i} &= (\rho_{jk}^{(i)})_{(s-i) \times (s-i)} \quad \text{with } \rho_{jk}^{(i)} = \sqrt{\frac{t_{i+j \wedge k} - t_i}{t_{i+j \vee k} - t_i}}, \quad 1 \leq j, k \leq s - i, \\ R_i &= (r_{jk}^{(i)})_{i \times i} \quad \text{with } r_{jk}^{(i)} = \sqrt{\frac{t_i - t_{j \vee k}}{t_i - t_{j \wedge k}}}, \quad 1 \leq j, k \leq i - 1, \\ r_{ij}^{(i)} &= r_{ji}^{(i)} = \sqrt{\frac{t_i - t_j}{t_i}}, \quad 1 \leq j \leq i - 1, \end{aligned}$$

and N_m is the CDF of a standard m -variate normal r.v., $j \wedge k = \min(j, k)$ and $j \vee k = \max(j, k)$. Hence pricing a lookback option using equation (4.10) boils down to evaluating the multivariate normal CDF. There are very efficient algorithms for computing s -variate normal CDF for small values of s (such as less than three). For larger value of s , this can be a challenging numerical problem. See, for example, Genz (1992) for various approaches to this problem.

In this paper we consider the relative efficiency of evaluating equation (4.10) using random points, low-discrepancy Sobol' points, and g.l.p.⁶ As demonstrated in the spread option example, a similar preliminary transformation is necessary to convert the region of integration of the multivariate normal probability density function to the s -dimensional hypercube. Furthermore, an additional transformation is essential for the method of g.l.p. to exploit the periodicity. Table 3 reports the results. In this comparison, we consider five-year lookback call options with annual monitoring and with parameter values $S_0 = 100$, $r = 10\%$, $\delta = 0$, $\sigma \in \{20\%, 30\%, 40\%\}$, and $K \in \{100, 110, 120\}$. Note that we consider only at-the-money and out-of-the-money cases, because the in-the-money option can be formulated as an at-the-money case as follows: for $S_0 > K$,

$$\max[\max(S_0, S_1, \dots, S_s) - K, 0] = S_0 - K + \max[\max(S_1, \dots, S_s) - S_0, 0].$$

To use g.l.p. points for a given dimension and a fixed N , we can always use the searching algorithm as outlined in Subsection 2.3.2 to obtain the optimal coefficient α . Alternatively, we can simply rely on published tables (e.g., Saltykov 1963, Haber 1983, Hua and Wang 1981, and Sloan and Joe 1995), although the optimal coefficients are available only for specific dimension and N .

Table 3

Estimates of Prices of Lookback Options Based on Monte Carlo, Sobol' Points, and g.l.p.

σ	K	MC	Std. Error	Sobol'	Good Lattice Points	
					Periodized	Not Periodized
Random and g.l.p. Use $N = 562$, Sobol' Uses $N = 512$						
0.2	100	47.027	0.421	47.373(5)	47.317(1001)	47.333(18)
	110	41.543	0.469	41.952(5)	41.884(290)	41.897(17)
	120	36.421	0.540	36.931(5)	36.843(107)	36.852(16)
0.3	100	56.861	0.496	57.336(6)	57.261(163)	57.276(17)
	110	51.864	0.535	52.397(6)	52.312(124)	52.323(17)
	120	47.230	0.582	47.838(6)	47.744(115)	47.748(16)
0.4	100	68.234	0.529	68.756(6)	68.680(193)	68.694(17)
	110	63.627	0.560	64.196(6)	64.114(205)	64.125(16)
	120	59.350	0.594	59.975(6)	59.887(221)	59.894(16)
Random and g.l.p. Use $N = 1142$, Sobol' Uses $N = 1024$						
0.2	100	47.087	0.312	47.328(17)	47.318(2864)	47.314(18)
	110	41.644	0.348	41.900(17)	41.887(1953)	41.885(18)
	120	36.579	0.393	36.868(16)	36.850(1435)	36.849(18)
0.3	100	56.964	0.365	57.282(18)	57.265(1986)	57.259(20)
	110	51.996	0.394	52.334(18)	52.315(1568)	52.309(20)
	120	47.398	0.426	47.765(17)	47.744(1270)	47.738(21)
0.4	100	68.341	0.391	68.699(18)	68.681(1663)	68.673(21)
	110	63.755	0.414	64.131(18)	64.114(1302)	64.104(22)
	120	59.502	0.439	59.902(17)	59.884(1117)	59.874(22)

Notes: The options evaluated are five-year lookback call with annual monitoring, $S_0 = 100$, $r = 10\%$, $\delta = 0$, $\sigma \in \{20\%, 30\%, 40\%\}$, and $K \in \{100, 110, 120\}$. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The Monte Carlo standard errors are reported in the fourth column. The values in parentheses are the efficiency ratios as measured by the ratio of the standard error of the basic Monte Carlo method to the standard error of the respective method.

⁶ We have also repeated our numerical examples on other types of low-discrepancy sequences such as Halton (1960) and Faure (1982) and found that these two sequences are less efficient compared to the Sobol' sequence. This is also consistent with the finding in Acworth, Broadie, and Glasserman (1997) that the Sobol'-based low-discrepancy method in general outperforms Halton and Faure. Hence we report only the results based on the Sobol' sequence.

For simplicity, our g.l.p.-based results rely on the optimal coefficients as constructed by Hua and Wang (1981). We consider g.l.p. with $N = 562$ and $1,142$ together with the \sin^2 periodization transformation. For the Monte Carlo method, we also use the same number of random point sets as the g.l.p. For Sobol', we deliberately consider $N = 512$ and $1,024$. This is motivated by the property of the Sobol' sequence, which has greater uniformity when N is a power of 2.

For each set of parameter values, we replicate the simulation independently 10 times using the random shift method discussed earlier to obtain an estimate of the standard error. Based on our numerical results we draw the following conclusions:

- Although the Sobol'-based quasi-Monte Carlo method is an improvement over the classical Monte Carlo method, the g.l.p. method is the most efficient.
- Similar to the result in the spread option case, the gain in the efficiency of the g.l.p. with periodization is very striking, particularly when we use more points.
- A feature of the periodized g.l.p.-based approach is that its efficiency deteriorates as we increase the volatilities and/or the strikes. To explain this, recall that the examples considered here are the lookback call options. Hence when the underlying asset price is more volatile, it is harder to estimate its maximum value. Also, when we increase the strike price, the integration domain that leads to a positive value of the option decreases. Consequently the function of interest becomes less regular with higher volatilities and/or strikes, and the periodization works best when the function is most regular.

We now consider the computation of option sensitivity parameters. As illustrated in the last subsection, a similar technique can be applied to equation (4.10) to obtain the desired Greeks. Table 4 provides the comparison using at-the-money discrete lookback options in Table 3 with $\sigma = 20\%$ and 30% . For random points and g.l.p. (with periodization), we use $N = 1,142$ and $5,003$, while for Sobol' points, we consider $1,024$ and $4,096$ points. The standard errors are estimated based on 10 independent replications. The conclusions are similar to the previous example. The g.l.p. method is again the most efficient, with the efficiency ratio in some cases over 1,000 relative to the Monte Carlo method.

Table 4
Estimates of Greeks: Delta (Δ), Gamma (Γ), Vega (\mathcal{V})

	σ	MC	Std. Error	Sobol'	g.l.p.	Resimulation	
						MC	Sobol'
MC and g.l.p. Use $N = 1142$, Sobol' Uses $N = 1024$							
Δ	0.2	1.0317	0.00086	1.0333(3)	1.0334(71)	1.0335(7)	1.0334(103)
Δ	0.3	1.0843	0.00117	1.0859(8)	1.0859(106)	1.0861(6)	1.0859(71)
Γ	0.2	0.0023	0.00015	0.0031(6)	0.0031(99)	0.0030(4)	0.0031(5)
Γ	0.3	0.0035	0.00007	0.0036(11)	0.0036(64)	0.0035(2)	0.0036(2)
\mathcal{V}	0.2	87.34	0.88	88.41(8)	88.32(167)	88.34(15)	88.32(147)
\mathcal{V}	0.3	107.73	0.37	108.26(11)	108.24(120)	108.24(6)	108.24(34)
MC and g.l.p. Use $N = 5003$, Sobol' Uses $N = 4096$							
Δ	0.2	1.0334	0.00073	1.0335(5)	1.0334(1211)	1.0335(6)	1.0334(87)
Δ	0.3	1.0857	0.00087	1.0860(9)	1.0859(312)	1.0861(4)	1.0859(53)
Γ	0.2	0.0030	0.00010	0.0031(9)	0.0031(1306)	0.0030(3)	0.0031(3)
Γ	0.3	0.0035	0.00005	0.0036(20)	0.0036(210)	0.0035(1)	0.0036(1)
\mathcal{V}	0.2	87.96	0.48	88.29(16)	88.37(825)	88.34(8)	88.32(79)
\mathcal{V}	0.3	108.20	0.26	108.24(39)	108.24(391)	108.24(4)	108.24(24)

Notes: The parameter values for the options are five-year lookback call with annual monitoring, $S_0 = 100$, $r = 10\%$, $\delta = 0$, $K = 100$, $\sigma \in \{20\%, 30\%\}$. The estimates of Greeks in columns 3, 5, and 6 are based on direct evaluation of integrals using Monte Carlo, Sobol', and g.l.p. points. The last two columns are based on resimulation using random and Sobol' points with $N = 1, 048, 576$. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The Monte Carlo standard errors are reported in the fourth column. The values in parentheses are the efficiency ratios as measured by the ratio of the standard error of the basic Monte Carlo method (fourth column) to the standard error of the respective method.

Table 5
**Estimates of the Prices of Lookback Options Based on Unrandomized g.l.p.
 and Unrandomized Sobol' Points**

N	g.l.p.	Error	Sobol'	Error
2129	47.317169	$-3.475e-5$	47.421309	$1.041e-1$
3001	47.317154	$-5.024e-5$	47.312888	$-4.316e-3$
5003	47.317287	$8.315e-5$	47.297708	$-1.950e-2$
8191	47.317200	$-3.987e-6$	47.331035	$-1.383e-2$

Notes: The parameter values are $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.2$, $t = 5$ years.

The results in the last two columns of Table 4 are based on the finite difference resimulation approximations (4.6) and (4.7). We consider both the random points and Sobol' points, and in each case we generate 10 independent batches with each replication consisting of $N = 2^{20} = 1,048,576$ resimulations. The values in parentheses are the ratios of the standard errors from Monte Carlo-based direct evaluation of multiple integrals (fourth column of Table 4) to those from the resimulation approach (based on random or Sobol' points). One immediate conclusion that can be drawn from these results is that the Sobol'-based resimulation approach is substantially better than that using random points. The results also indicate that the resimulation method is particularly inefficient in estimating the second derivative (i.e., γ). This is in contrast to the direct evaluation of multiple integrals using g.l.p. The g.l.p. method is extremely efficient in estimating both delta and γ . Also, despite the enormous simulation runs for the resimulation approach, the g.l.p. method with a mere point set of 5,003 significantly outperforms the resimulation method. The huge number of resimulations also imply a much higher computational effort. For instance, to obtain the delta estimates, the resimulation is approximately 270 times more time consuming than that using the direct evaluation of multiple integrals with $N = 1,142$.

Our comparison so far is based on the randomized version of g.l.p. and Sobol-based low-discrepancy sequence. We conclude this subsection by briefly considering the effectiveness of the unrandomized version of these methods. Using $S_0 = K = 100$, $r = 0.1$, $\sigma = 0.2$, $t = 5$ years, Table 5 compares the option estimates for the lookback option with $N \in \{2,129, 3,001, 5,003, 8,191\}$. These values of N allow us to use the optimal coefficients from Hua and Wang (1981). As a benchmark, we assume that the result used a much larger g.l.p. point set of $N = 39,029$ as the "true" value. For $N = 39,029$, the estimated option value is 47.317204. The "errors" reported in the third and fifth columns of Table 5 are computed relative to this value. Similar to findings based on the randomized methods, the unrandomized g.l.p. consistently yields a smaller error relative to the Sobol'-based approach.

4.3 Application to Equity-Indexed Annuities

Our next example concerns the valuation of a particular option embedded in equity-indexed annuities (EIAs). These contracts have been very popular in the United States and Canada in recent years.⁷ A typical EIA allows the contract holder to benefit from the growth of a stock index such as the S&P 500 in addition to providing a basic guaranteed return. Different methods are used to tie the benefits under the contract to the performance of the index, and these are known as indexing methods. One popular indexing method is known as the high-water-mark method. In this case the contract holder owns a fixed-strike lookback option on the underlying index. Typically high-water-mark contracts have discrete monitoring, normally on a yearly basis. Because the maturity of an EIA generally is in the range of 3 to 10 years, this implies that the price of the discrete lookback option can be efficiently

⁷ According to the survey conducted by Marrion (see www.indexannuity.org), the sales of the indexed annuity products in the United States for the first quarter of 2004 were \$4.186 billion.

calculated using the g.l.p. method as we saw in the previous subsection. See Hardy (2003), Lee (2003), Tiong (2000), and Lin and Tan (2003) for additional details on EIAs.

We now consider in greater detail an EIA with high-water-mark indexing. We use the following notation:

- P : Amount invested initially, that is, the pure premium
- T : Time to maturity of the contract from inception at time zero
- S_0, S_1, \dots, S_T : Index levels at times 0, 1, 2, . . . , T
- g : Minimum guarantee rate compounded continuously
- β : Proportion of the pure premium to be accumulated at the minimum interest rate
- α : Participation rate in the growth of the index.

Using this notation, the maturity value of this EIA can be represented as

$$\max \left[\beta P e^{gT}, P + \alpha P \max_{j=0,1,\dots,T} \left\{ \frac{S_j}{S_0} - 1 \right\} \right]. \tag{4.11}$$

The first component gives the required minimum guarantee at time T , while the second component captures the contribution of the high-water-mark feature. Rearranging the above expression yields

$$P \beta e^{gT} + \frac{P}{S_0} \alpha \max \left[\max_{j=0,1,\dots,s} S_j - \frac{S_0 \beta e^{gT} - (1 - \alpha) S_0}{\alpha}, 0 \right]. \tag{4.12}$$

The first term arises from the minimum guarantee, while the second term provides an explicit link to the lookback options considered in the previous subsection. From the above formulation, it is also obvious that issuing a high-water-mark EIA is equivalent to taking the following two positions simultaneously:

- Invest in a T -year zero-coupon bond with initial amount $P \beta e^{(g-r)T}$ that yields a risk-free return r . At maturity, this bond grows to $P \beta e^{gT}$, which coincides with the minimum guarantee.
- Go long $P \alpha / S$ units of a T -year lookback call option with yearly discrete sampling and strike price $[S_0 \beta e^{gT} - (1 - \alpha) S_0] / \alpha$.

From no-arbitrage pricing theory, the total cost of investing an amount P in this EIA with initial index level S_0 must be equal to the sum of the above two positions.

We use the same parameter values as Lin (1999), who proposed an approximation algorithm for pricing an EIA with high-water-mark features. For the first case, the parameter values are $S_0 = P = 100$, $r = 6\%$, $\delta = 0$, $\sigma = 20\%$, $\beta = 90\%$, and $T = 5$ years. The second set of examples is similar to the first case except that the interest rate, volatility, and maturity of the EIA are increased to 10%, 30%, and 7 years, respectively.

Table 6
Five-Year EIAs with $S_0 = P = 100$, $r = 6\%$, $\delta = 0$, $\sigma = 20\%$, $\beta = 90\%$

α	Random	Std. Error	Sobol'	g.l.p.
1.00	35.910	0.5451	36.493 (6)	36.413 (133)
0.95	33.975	0.5194	34.537 (6)	34.455 (133)
0.90	32.041	0.4937	32.576 (6)	32.498 (130)
0.85	30.107	0.4680	30.615 (6)	30.541 (130)
0.80	28.173	0.4424	28.655 (6)	28.584 (130)
0.75	26.240	0.4167	26.695 (6)	26.629 (126)
0.70	24.308	0.3911	24.736 (6)	24.674 (126)
0.65	22.377	0.3655	22.779 (6)	22.720 (122)

Notes: Both random and g.l.p. use 562 points; Sobol' is based on 512 points. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The third column shows the standard errors corresponding to the Monte Carlo methods. For the Sobol' and g.l.p. point sets, we report the estimated prices as well as the efficiency ratios (the value in parentheses) as measured by the ratios of the standard errors of the basic Monte Carlo method to the standard errors of the method.

Table 7
Seven-Year EIAs with $S_0 = P = 100$, $r = 10\%$, $\delta = 0$, $\sigma = 30\%$, $\beta = 90\%$

α	Random	Std. Error	Sobol'	g.l.p.
1.00	65.417	0.5209	65.756 (19)	65.775 (124)
0.95	61.918	0.4959	62.241 (19)	62.259 (124)
0.90	58.420	0.4709	58.727 (19)	58.744 (121)
0.85	54.922	0.4459	55.213 (19)	55.231 (121)
0.80	51.426	0.4209	51.702 (19)	51.718 (120)
0.75	47.931	0.3959	48.191 (19)	48.207 (116)
0.70	44.438	0.3709	44.682 (19)	44.697 (116)
0.65	40.948	0.3460	41.176 (19)	41.190 (115)

Notes: Both random and g.l.p. use 2,129 points; Sobol' is based on 2,048 points. Each method is replicated independently 10 times to obtain an estimate of the standard errors. The third column shows the standard errors corresponding to the Monte Carlo methods. For the Sobol' and g.l.p. point sets, we report the estimated prices as well as the efficiency ratios (the value in parentheses) as measured by the ratios of the standard errors of the basic Monte Carlo method to the standard errors of the method.

The results are reported in Tables 6 and 7 for a various levels of participation rates $\alpha \in \{65\%, 70\%, \dots, 100\%\}$. These results show that the g.l.p. method again outperforms both standard Monte Carlo and Sobol'-based quasi-Monte Carlo methods. Relative to the basic Monte Carlo method, the gains in efficiencies for the Sobol' points are around 6 and 19 for the 5- and 7-year EIAs, respectively. In contrast, the efficiency ratios based on g.l.p. are in the range 115–33.

5. CONCLUSION

In this paper we have described the lattice rules method and shown its effectiveness in the computation of low-dimensional integrals. Such problems are of interest in a variety of applications in finance and insurance. We examined several practical examples from option pricing and showed that the method of good lattice points outperforms not only standard Monte Carlo but also other quasi-Monte Carlo methods (based on typical low-discrepancy sequences such as those of Halton, Faure, and Sobol').

The method of good lattice points is based on the construction of a deterministic point set that is chosen in a special way. We assume that the function to be integrated has a high degree of regularity. For a certain class of functions we can identify the function that gives the largest integration error, and the g.l.p. are picked so as to minimize the size of the integration error. Then by construction the integration error associated with any arbitrary function from this class will be less than the worst case error. We described how these points were constructed and drew attention to the results available in the literature. The power of the g.l.p. approach is fully realized only if we periodize the integrand. There are different ways of doing this, and we illustrate the method that works best for our applications.

Initially we applied this method to spread options. We demonstrated the computational power of the g.l.p. method in this case. It outperforms the competing methodologies and is especially effective in computing the delta and gamma of these options. Delta and gamma are important for hedging and risk management purposes, and standard Monte Carlo methods are notoriously poor in computing gamma. We also applied the method to compute the price and sensitivities of lookback options. In this case the g.l.p. method was vastly superior to both standard Monte Carlo and quasi-Monte Carlo when we incorporate the periodization scheme. The efficiency gains were realized not only in the case of the price but also in the case of the price sensitivities. We also performed a similar analysis for discrete lookback options that are sometimes embedded in equity indexed annuity contracts and reached the same conclusion.

To summarize, the lattice rules method is a simple but powerful computational tool for certain applications in insurance and finance.

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APPENDIX A: DERIVATION OF EXPRESSION (4.3)

In this appendix we derive the expression (4.3). It follows from equation (4.2) that we have

$$\begin{aligned}
 e^{rT}V &= E_Q [\max[\tau w_2 S_{2T} - \tau w_1 S_{1T} - K, 0]] \\
 &= \int_0^\infty \int_0^\infty \max[\tau w_2 S_{2T} - \tau w_1 S_{1T} - K, 0] f(S_T) dS_T \\
 &= \int_0^\infty \int_{(\tau w_1 S_{1T} + K)/\tau w_2}^\infty (\tau w_2 S_{2T} - \tau w_1 S_{1T} - K) f(S_T) dS_{2T} dS_{1T} \\
 &= \int_{-\infty}^\infty \int_{\log(\tau w_1 e^{\varphi_1 + \mu_{1T}} + K) - \log(\tau w_2) - \mu_{2T}}^\infty (\tau w_2 e^{\varphi_2 + \mu_{2T}} - \tau w_1 e^{\varphi_1 + \mu_{1T}} - K) f^*(\boldsymbol{\varphi}) d\boldsymbol{\varphi}, \tag{A.1}
 \end{aligned}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)'$ and $f^*(\boldsymbol{\varphi})$ is a bivariate normal distribution with mean 0 and covariance matrix $\boldsymbol{\Sigma}_T$. To obtain expression (A.1), we have used the substitution $S_{iT} = \exp(\varphi_i + \mu_{iT})$, where $\mu_{iT} = \log S_{i0} + (r - \delta_i - \frac{1}{2}\sigma_i^2)T$ for $i = 1, 2$, as defined in equation (3.1).

Note that the integration domain of the integral (A.1) is not in $[0, 1]^2$. For either the g.l.p. method or Monte Carlo integration, we always assume that the integration domain lies in $[0, 1]^s$. This can be accomplished by the following sequence of transformations: first, we let $\boldsymbol{\varphi} = C\mathbf{y}$, where C is the Cholesky decomposition of the covariance matrix $\boldsymbol{\Sigma}_T$, that is, $CC' = \boldsymbol{\Sigma}_T$, where

$$C = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1 \sqrt{T} & 0 \\ \rho_{12} \sigma_2 \sqrt{T} & \sqrt{1 - \rho_{12}^2} \sigma_2 \sqrt{T} \end{bmatrix}.$$

Applying this transformation to integral (A.1), we obtain the expression

$$\frac{1}{2\pi} \int_{a'_1}^{b'_1} \int_{a'_2}^{b'_2} (e^{c_{21}y_1 + c_{22}y_2 + \mu_{2T}} - e^{c_{11}y_1 + \mu_{1T}} - K) e^{-\frac{1}{2}(y_1^2 + y_2^2)} dy_2 dy_1, \tag{A.2}$$

where $a'_1 = -\infty$, $b'_1 = b'_2 = \infty$ and

$$a'_2 = \frac{\log(\tau w_1 e^{c_{11}y_1 + \mu_{1T}} + K) - \log(\tau w_2) - \mu_{2T} - c_{21}y_1}{c_{22}}.$$

Now consider the second phase of the transformation by letting $y_i = \Phi^{-1}(z_i)$. This results in

$$\int_{d_1}^{e_1} \int_{d_2}^{e_2} (e^{c_{21}\Phi^{-1}(z_1) + c_{22}\Phi^{-1}(z_2) + \mu_{2T}} - e^{c_{11}\Phi^{-1}(z_1) + \mu_{1T}} - K) dz, \tag{A.3}$$

where $d_1 = \Phi(-\infty) = 0$, $e_1 = e_2 = \Phi(\infty) = 1$, and

$$d_2 = \Phi \left(\frac{\log(\tau w_1 e^{c_{11}\Phi^{-1}(z_1) + \mu_{1T}} + K) - \log(\tau w_2) - \mu_{2T} - c_{21}\Phi^{-1}(z_1)}{c_{22}} \right).$$

The final change-of-variable $z_i = d_i + u_i(e_i - d_i)$ leads to

$$\int_0^1 \int_0^1 (1 - d_2) (\varpi \omega_2 e^{c_{21}\Phi^{-1}(u_1) + c_{22}\Phi^{-1}(d_2 + u_2(1-d_2)) + \mu_2 T} - \varpi \omega_1 e^{c_{11}\Phi^{-1}(u_1) + \mu_1 T} - K) du, \quad (\text{A.4})$$

which has the required integration domain $[0, 1]^2$.

APPENDIX B: DELTA AND GAMMA OF SPREAD OPTIONS

This appendix collects the formulas for computing the option sensitivity parameters of spread options.

To derive the first partial derivatives (4.8) with respect to S_1 , first note that

$$\begin{aligned} \frac{\partial g}{\partial S_1} &= \frac{h_1(\mathbf{u})}{c_{22}[h_1(\mathbf{u}) + K]S_1}, \\ \frac{\partial d_2}{\partial S_1} &= \frac{1}{\sqrt{2\pi}} e^{-g^2(\mathbf{u})/2} \frac{\partial g}{\partial S_1} \\ &= \frac{1}{\sqrt{2\pi}} \frac{h_1(\mathbf{u})}{c_{22}[h_1(\mathbf{u}) + K]S_1} e^{-g^2(\mathbf{u})/2}, \\ \frac{\partial h_5}{\partial S_1} &= \sqrt{2\pi}(1 - u_2) e^{h_5^2(\mathbf{u})/2} \frac{\partial d_2}{\partial S_1} \\ &= \frac{h_1(\mathbf{u})}{c_{22}[h_1(\mathbf{u}) + K]S_1} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial h}{\partial S_1} &= \frac{\partial h_2}{\partial S_1} - \frac{\partial h_1}{\partial S_1} \\ &= h_2(\mathbf{u}) c_{22} \frac{\partial h_5}{\partial S_1} - h_1(\mathbf{u}) \frac{\partial h_3}{\partial S_1} \\ &= \frac{h_1(\mathbf{u})}{S_1} \left(\frac{h_2(\mathbf{u})}{h_1(\mathbf{u}) + K} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} - 1 \right). \end{aligned}$$

The second partial derivatives (4.9) with respect to S_1 are derived similarly as follows:

$$\begin{aligned} \frac{\partial^2 g}{\partial S_1^2} &= \frac{1}{c_{22}[h_1(\mathbf{u}) + K]S_1} \left[\frac{\partial h_1}{\partial S_1} - c_{22} \frac{\partial g}{\partial S_1} \left(\frac{\partial h_1}{\partial S_1} S_1 + h_1(\mathbf{u}) + K \right) \right] \\ &= -\frac{1}{c_{22}} \left(\frac{h_1(\mathbf{u})}{[h_1(\mathbf{u}) + K]S_1} \right)^2, \\ \frac{\partial^2 d_2}{\partial S_1^2} &= \frac{\partial d_2}{\partial S_1} \left[\frac{\partial^2 g}{\partial S_1^2} \left(\frac{\partial g}{\partial S_1} \right)^{-1} - \frac{\partial g}{\partial S_1} g(\mathbf{u}) \right] \\ &= -\frac{1}{c_{22}\sqrt{2\pi}} \left(\frac{h_1(\mathbf{u})}{[h_1(\mathbf{u}) + K]S_1} \right)^2 e^{-g^2(\mathbf{u})/2} \left(\frac{g(\mathbf{u})}{c_{22}} + 1 \right), \\ \frac{\partial^2 h_5}{\partial S_1^2} &= \frac{\partial h_5}{\partial S_1} \left[\frac{\partial h_5}{\partial S_1} h_5(\mathbf{u}) + \frac{\partial^2 d_2}{\partial S_1^2} \left(\frac{\partial d_2}{\partial S_1} \right)^{-1} \right] \\ &= \frac{1}{c_{22}} \left(\frac{h_1(\mathbf{u})}{[h_1(\mathbf{u}) + K]S_1} \right)^2 (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left[(1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \frac{h_5(\mathbf{u})}{c_{22}} - \frac{g(\mathbf{u})}{c_{22}} - 1 \right], \end{aligned}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial S_1^2} &= \frac{\partial^2 h_2}{\partial S_1^2} - \frac{\partial^2 h_1}{\partial S_1^2} \\ &= h_2(\mathbf{u}) \left(\frac{h_1(\mathbf{u})}{[h_1(\mathbf{u}) + K]S_1} \right)^2 (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \times \left[(1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left(\frac{h_5(\mathbf{u})}{c_{22}} + 1 \right) - \frac{g(\mathbf{u})}{c_{22}} - 1 \right],\end{aligned}$$

since

$$\begin{aligned}\frac{\partial^2 h_1}{\partial S_1^2} &= \frac{1}{S_1} \frac{\partial h_1}{\partial S_1} - \frac{h_1(\mathbf{u})}{S_1^2} = 0, \\ \frac{\partial^2 h_2}{\partial S_1^2} &= c_{22} \left[\frac{\partial h_2}{\partial S_1} \frac{\partial h_5}{\partial S_1} + h_2(\mathbf{u}) \frac{\partial^2 h_5}{\partial S_1^2} \right] \\ &= h_2(\mathbf{u}) c_{22} \left[c_{22} \left(\frac{\partial h_5}{\partial S_1} \right)^2 + \frac{\partial^2 h_5}{\partial S_1^2} \right] \\ &= h_2(\mathbf{u}) \left(\frac{h_1(\mathbf{u})}{[h_1(\mathbf{u}) + K]S_1} \right)^2 (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \times \left[(1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left(\frac{h_5(\mathbf{u})}{c_{22}} + 1 \right) - \frac{g(\mathbf{u})}{c_{22}} - 1 \right],\end{aligned}$$

For Δ_2 , we have

$$\begin{aligned}\frac{\partial h}{\partial S_2} &= \frac{\partial h_2}{\partial S_2} = h_2(\mathbf{u}) \left(c_{22} \frac{\partial h_5}{\partial S_2} + \frac{1}{S_2} \right) \\ &= \frac{h_2(\mathbf{u})}{S_2} (1 - (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]}),\end{aligned}$$

since

$$\begin{aligned}\frac{\partial d_2}{\partial S_2} &= \frac{1}{\sqrt{2\pi}} e^{-g^2(\mathbf{u})/2} \frac{\partial g}{\partial S_2} \\ &= -\frac{1}{c_{22} \sqrt{2\pi} S_2} e^{-g^2(\mathbf{u})/2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial h_5}{\partial S_2} &= \sqrt{2\pi} e^{h_5^2(\mathbf{u})/2} (1 - u_2) \frac{\partial d_2}{\partial S_2} \\ &= -\frac{1}{c_{22} S_2} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]}.\end{aligned}$$

The second partial derivatives with respect to S_2 are derived similarly as

$$\begin{aligned}\frac{\partial^2 h_5}{\partial S_2^2} &= h_5(\mathbf{u}) \left(\frac{\partial h_5}{\partial S_2} \right)^2 + \frac{\partial h_5}{\partial S_2} \left(\frac{\partial d_2}{\partial S_2} \right)^{-1} \frac{\partial^2 d_2}{\partial S_2^2} \\ &= \frac{1}{c_{22} S_2^2} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left[\frac{h_5(\mathbf{u})}{c_{22}} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} + 1 - \frac{g(\mathbf{u})}{c_{22}} \right], \\ \frac{\partial^2 d_2}{\partial S_2^2} &= \frac{1}{c_{22} \sqrt{2\pi} S_2^2} e^{-g^2(\mathbf{u})/2} \left(1 - \frac{g(\mathbf{u})}{c_{22}} \right),\end{aligned}$$

$$\begin{aligned} \frac{\partial^2 h}{\partial S_2^2} &= \frac{\partial^2 h_2}{\partial S_2^2} = h_2(\mathbf{u}) \left(c_{22} \frac{\partial^2 h_5}{\partial S_2^2} - \frac{1}{S_2^2} \right) + \left(c_{22} \frac{\partial h_5}{\partial S_2} + \frac{1}{S_2} \right) \frac{\partial h_2}{\partial S_2} \\ &= \frac{h_2(\mathbf{u})}{S_2^2} (1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left[(1 - u_2) e^{\frac{1}{2}[h_5^2(\mathbf{u}) - g^2(\mathbf{u})]} \left(\frac{h_5(\mathbf{u})}{c_{22}} + 1 \right) - \frac{g(\mathbf{u})}{c_{22}} - 1 \right]. \end{aligned}$$

REFERENCES

- ACWORTH, PETER, MARK BROADIE, AND PAUL GLASSERMAN. 1998. A Comparison of Some Monte Carlo and Quasi-Monte Carlo Methods for Option Pricing. In *Monte Carlo and Quasi-Monte Carlo Methods 1996*, ed. Harald Niederreiter et al., pp. 1–18. New York: Springer.
- BAIVALOV, N. S. 1959. On Approximation Calculation of Multiple Integrals. *Vestnik Moskovskogo Universiteta, Seriya Matematiki, Mehaniki, Astronomii, Fiziki, Himii* 4: 3–18 (in Russian).
- BECKERS, MARC, AND ANN HAEGEMANS. 1992. Transformations of Integrand for Lattice Rules. In *Numerical Integration: Recent Development, Software and Applications*, ed. T. O. Espelid and A. Genz, pp. 329–40. NATO ASI Series. Boston: Kluwer.
- BOYLE, PHELIM P. 1988. A Lattice Framework for Options Pricing with Two State Variables. *Journal of Financial and Quantitative Analysis* 23: 1–12.
- BOYLE, PHELIM P., MARK BROADIE, AND PAUL GLASSERMAN. 1997a. Monte Carlo Methods for Security Pricing. *Journal of Economic Dynamics and Control* 21: 1267–1321.
- . 1997b. Estimating Security Price Derivatives by Simulation. *Management Science* 42: 269–85.
- BOYLE, PHELIM P., JEREMY EVNINE, AND STEPHEN GIBBS. 1989. Numerical Evaluation of Multivariate Contingent Claims. *Review of Financial Studies* 2(2): 241–50.
- CAPLISCH, RUSSEL E., WILLIAM MOROKOFF, AND ART B. OWEN. 1997. Valuation of Mortgage-Backed Securities Using Brownian Bridges to Reduce Effective Dimension. *Journal of Computational Finance* 1(1): 27–46.
- CONZE, ANTOINE, AND VISWANATHAN. 1991. Path-Dependent Options: The Case of Lookback Options. *Journal of Finance* 46: 1893–1907.
- CRANLEY, R., AND T. N. L. PATTERSON. 1976. Randomization of Number Theoretic Methods for Multiple Integration. *S.I.A.M. Journal of Numerical Analysis* 23: 904–14.
- DAVIS, PHILIP J., AND PHILIP RABINOWITZ. 1984. *Methods of Numerical Integration*. 2nd ed. London: Academic Press.
- DELBAEN, FREDDY, AND WALTER SCHACHERMAYER. 2004. What Is a Free Lunch? *Notices of the AMS* 51(5): 526–28.
- DUFFIE, DARRELL. 1996. *Dynamic Asset Pricing Theory*. 2nd ed. Princeton: Princeton University Press.
- DUFRESNE, PIERRE COLLIN, WILLIAM KEIRSTEAD, AND MICHAEL P. ROSS. 1996. Pricing Derivatives the Martingale Way. Working Paper.
- FAURE, HENRI. 1982. Discrépance de suites associées à un système de numération (en dimension s). *Acta Arithmetica* 41: 337–51.
- GENZ, ALAN. 1992. Numerical Computation of Multivariate Normal Probabilities. *Journal of Computational and Graphical Statistics* 1: 141–49.
- GLASSERMAN, PAUL, PHILIP HEIDELBERGER, AND PERVEZ SHAHABUDDIN. 2000. Variance Reduction Techniques for Estimating Value-at-Risk. *Management Science* 46: 1349–64.
- GOLDMAN, BARRY, HOWARD SOSIN, AND MARY ANN GATTO. 1979. Path-Dependent Options: Buy at the Low, Sell at the High. *Journal of Finance* 34: 1111–27.
- HABER, SEYMOUR. 1983. Parameters for Integrating Periodic Functions of Several Variables. *Mathematics of Computation* 41(163): 115–29.
- HALTON, JOHN H. 1960. On the Efficiency of Certain Quasi-Random Sequences of Points in Evaluating Multi-dimensional Integrals. *Numerische Mathematik* 2: 84–90.
- HARDY, MARY R. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. Hoboken, N.J.: John Wiley.
- HEYNEEN, RONALD C., AND HARRY M. KAT. 1995. Lookback Options with Discrete and Partial Monitoring of the Underlying Price. *Applied Mathematical Finance* 2: 273–84.
- HLAWKA, EDMUND. 1961. Funktionen von beschränkter Variation in der Theorie der Gleichverteilung. *Annali di Matematica Pura Applicata* 54: 324–34.
- HUA, LUOGENG, AND YUAN WANG. 1981. *Applications of Number Theory to Numerical Analysis*. Berlin: Springer.
- JOE, STEPHEN. 1990. Randomization of Lattice Rules for Numerical Multiple Integration. *Journal of Computational and Applied Mathematics* 31: 299–304.
- JOY, CORWIN, PHELIM P. BOYLE, AND KEN SENG TAN. 1996. Quasi-Monte Carlo Methods in Numerical Finance. *Management Science* 42(6): 926–38.
- KEAST, PATRICK. 1973. Optimal Parameters for Multidimensional Integration. *S.I.A.M. Journal of Numerical Analysis* 10(5): 831–38.

- KOROBOV, N. M. 1959. Properties and Calculation of Optimal Coefficients. *Doklady Akademii Nauk SSSR* 132: 1009–12 (in Russian). English translation: *Soviet Mathematics Doklady* 1(1960): 696–700.
- LEE, HANGSUK. 2003. Pricing Equity-Indexed Annuities with Path-Dependent Options. *Insurance: Mathematics and Economics* 33(3): 677–90.
- LEMIEUX, CHRISTIANE, AND PIERRE L'ECUYER. 1998. Efficiency Improvement by Lattice Rules for Pricing Asian Options. In *Proceedings of the 1998 Winter Conference*, ed. D. J. Medeiros, pp. 579–86. New York: IEEE Press.
- . 1999. A Comparison of Monte Carlo, Lattice Rules and Other Low-Discrepancy Point Sets. *Monte and Quasi-Monte-Carlo Methods 1998*, ed. H. Niederreiter and J. Spanier. Lecture Notes in Computational Science and Engineering. New York: Springer.
- . 2002. Recent Advances in Randomized Quasi-Monte Carlo Methods. In *Modeling Uncertainty: An Examination of Stochastic Theory, Methods, and Applications*, ed. M. Dror et al., pp. 419–74. New York: Kluwer.
- LIN, X. SHELDON. 1999. Valuation of Options on the Maximum/Minimum of Multiple Assets, Discrete Lookback Options and Equity-Indexed Annuities. *Finance* 20: 95–114.
- LIN, X. SHELDON, AND KEN SENG TAN. 2003. Valuation of Equity-Indexed Annuities under Stochastic Interest Rates. *North American Actuarial Journal* 7(4): 72–91.
- MARGRABE, WILLIAM. 1978. The Value of an Option to Exchange One Asset for Another. *Journal of Finance* 33: 177–86.
- MOORE, KRISTEN S., AND VIRGINIA R. YOUNG. 2005. Optimal Design of a Perpetual Equity-Indexed Annuity. *North American Actuarial Journal* 9(1): 57–72.
- NIEDERREITER, HARALD. 1978. Existence of Good Lattice Points in the Sense of Hlawka. *Monatshefte für Mathematik* 86: 203–19.
- . 1987. Point Sets and Sequences with Small Discrepancy. *Monatshefte für Mathematik* 104: 273–337.
- . 1988. Low-Discrepancy and Low-Dispersion Sequences. *Journal of Number Theory* 30: 51–70.
- . 1992. *Random Number Generation and Quasi-Monte Carlo Methods*. Philadelphia: S.I.A.M.
- . 1993. Improved Error Bounds for Lattice Rules. *Journal of Complexity* 9: 60–75.
- NIEDERREITER, HARALD, AND CHAOPING XING. 1996. *Explicit Global Function Fields over the Binary Field with Many Rational Points*. *Acta Arithmetica* 75: 383–96.
- . 1998. Global Function Fields with Many Rational Points over the Ternary Field. *Acta Arithmetica* 83: 65–86.
- NINOMIYA, SYOITI, AND SHU TEZUKA. 1996. Toward Real-Time Pricing of Complex Financial Derivatives. *Applied Mathematical Finance* 3: 1–20.
- PASKOV, SASSIMIR H., AND JOSEPH F. TRAUB. 1995. Faster Valuation of Financial Derivatives. *Journal of Portfolio Management* 22(1): 113–20.
- PEARSON, NEIL D. 1995. An Efficient Approach for Pricing Spread Options. *Journal of Derivatives* 3(1): 76–91.
- SALTYKOV, A. I. 1963. Tables for Computing Multiple Integrals by the Method of Optimal Coefficients. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki* 3: 181–86 (in Russian). English translation: *U.S.S.R. Computational Mathematics and Mathematical Physics* 3: 235–42.
- ŠARGIN, I. F., 1963. A Lower Estimate for the Error of Quadrature Formulas for Certain Class of Functions. *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki* 3: 370–76 (in Russian). English translation: *U.S.S.R. Computational Mathematics and Mathematical Physics* 3: 489–97.
- SHAMMAS, NAMIR C. 1995. *C/C++ Mathematical Algorithms for Scientists and Engineers*. New York: McGraw-Hill.
- SLOAN, IAN H., AND STEPHEN JOE. 1995. *Lattice Methods for Multiple Integration*. New York: Oxford University Press.
- SLOAN, IAN H. AND PHILIP J. KACHOYAN. 1987. Lattice Methods for Multiple Integration: Theory, Error Analysis and Examples. *S.I.A.M. Journal on Numerical Analysis* 14: 117–28.
- SLOAN, IAN H. AND JAMES N. LYNES. 1989. The representation of Lattice Quadrature Rules as Multiple Sums. *Mathematics of Computation* 52: 81–94.
- SOBOL', ILYA M. 1967. The Distribution of Points in a Cube and the Approximate Evaluation of Integrals. *U.S.S.R. Computational Mathematics and Mathematical Physics* 7(4): 86–112.
- STROUD, A. H. 1970. *Approximate Calculation of Multiple Integrals*. Englewood Cliffs, N.J.: Prentice-Hall.
- STULZ, RENÉ M. 1982. Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications. *Journal of Financial Economics* 10: 161–85.
- TAN, KEN SENG, AND PHELM P. BOYLE. 2000. Applications of Randomized Low Discrepancy Sequences to the Valuation of Complex Securities. *Journal of Economic Dynamics and Control* 24: 1747–82.
- TEZUKA, SHU. 1995. *Uniform Random Numbers: Theory and Practice*. Norwell, Mass.: Kluwer.
- TIONG, SERENA. 2000. Valuing Equity-Indexed Annuities. *North American Actuarial Journal* (4): 149–70.
- TUFFIN, BRUNO. 1996. On the Use of Low Discrepancy Sequences in Monte Carlo Methods. *Monte Carlo Methods and Applications* 2: 295–320.
- WILCOX, DARREN. 1990. *Energy Futures and Options: Spread Options in Energy Markets*. New York: Goldman Sachs & Co.

ZAREMBA, S. K. 1972. La méthode des “Bons Treillis” pour le calcul des intégrales multiples. In *Applications of Number Theory to Numerical Analysis*, ed. S. K. Zaremba, pp. 39–119. New York: Academic Press.

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