Managing Economic and Virtual Economic Capital within Financial Conglomerates

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Abstract

In this paper we show how the optimal amount of economic capital can be derived such that it minimizes the economic cost of risk-bearing. The economic cost of risk-bearing takes into account the cost of the economic capital as well as the exposure to residual risk. In addition to the *absolute* problem of determining the amount of economic capital, we also consider the *relative* problem of how to establish the allocation of economic capital among subsidiaries. Because subsidiaries are juridical entities, they will also consider the absolute problem of economic capital allocation themselves. In an equilibrium situation, the relative allocation derived by the conglomerate and the absolute allocation derived by the subsidiaries coincide. We show that the diversification benefit that is typically obtained in a conglomerate construction creates a virtual economic capital for subsidiaries within the conglomerate. We show further that the approach we propose to solve the absolute problem of economic capital allocation also can be applied to the problem of optimal portfolio selection, extending the well-known Markowitz approach and providing a tool for management by economic capital.

1. Introduction: The Evolution of Beliefs for Risk Transfer

In many countries an important technique for the financing of life insurance, in particular old-age pensions, is the so-called *repartition system.* In such a system the ‘‘belief’’ in the labor force of the next generation ‘‘guarantees’’ the benefits to be paid out. Another ‘‘belief’’ governs the creation of solvency buffers and actuarial reserves by insurers, being financed out of the premiums in order to be able to fulfill the future obligations and leading to management by economic capital. In this system the ‘‘belief’’ in adequate management activity of the future generations is the ‘‘guarantee.’’ A third ‘‘belief’’ was created when investors could convince risk bearers that risk can be hedged away within financial markets, that is, can be securitized. Here the ‘‘guarantee’’ is based on the ‘‘belief’’ that the financial markets always can absorb the risk at a predetermined price.

These three types of beliefs governing repartition, funding, and financial market transfers all exhibit their own particular exposure to a default. For repartition it is clear that longevity is a threat. For funding, inflation and other financial economic factors constitute a threat, and in financial markets, for example, a breakdown of the system because of comonotonic effects—financial contagion—is a threat.

In the abstract, beliefs and preferences can be formalized as a set of axioms concerning the behavior of human beings under risk. The particular set of axioms must reflect the risk perception of the economic agents involved in the situation under consideration. The economic relevance of the axioms thus

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depends on the agents involved as well as on the specific situation under study. The axioms should be formalized such as to be representative for all the agents in the evaluation of any feasible risk defined on a particular measurable space. Markowitz (1959, chap. 10) wrote: ''We might decide that in one context one basic set of principles is appropriate, while in another context a different set of principles should be used. We might find that some patterns of preferences are consistent with a set of preferences while other patterns are not.'' Here ''principles'' mean ''axioms.'' Following this flow of ideas, we think that preference axioms and axiomatically justified risk measures constitute flexible tools to determine the risk of decisions made in a particular context. These types of risk measures have been studied extensively in actuarial science; see, for example, Bühlmann (1970), Gerber (1979), and Goovaerts, De Vylder, and Haezendonck (1984) for early accounts. Variables such as premiums, economic capital, and asset mix should then be *derived* in an optimization procedure, for example, by minimizing economic cost or by maximizing profits, given the preference patterns in the particular context.

A fundamental difference exists between financial pricing and insurance pricing. In liquid and standardized financial markets, the validity of the law-of-one-price enforces that each risk in the market has a unique price that is determined by a linear pricing functional. Moreover, when security prices exclude arbitrage, this pricing functional is positively linear. This, however, is not in general the case in insurance markets, in which often no unique price exists and hence the law-of-one-price is violated. The ''fair value'' of an insurance portfolio, which is the value of the portfolio induced by market parameters, might differ from insurer to insurer. Arbitrage opportunities in the illiquid and nonstandardized insurance markets are difficult to determine and even more difficult to exploit. In particular in nonlife (re)insurance, hedging is a fiction, and economic phenomena such as moral hazard and adverse selection play a prominent role. The number of players in this market is restricted, and different market participants well may have a different perception of the price of a portfolio. This may be due, for example, to a different risk attitude when determining the amount of economic capital (investment) needed. In the case of Value-at-Risk (VaR) capital buffers, the one might use a percentile of order 90%, while the other uses a percentile of order 95%, the degree of risk aversion being the differentiating factor.

Different risk perceptions lead to different amounts of economic capital *derived* in a tradeoff between the preferred level of risk exposure and the cost of economic capital. A similar tradeoff is encountered in statistics where there is no unique level to be used in the testing of hypotheses. Hypothesis testing is not a one-dimensional problem (with one criterion), but it is a problem where two possible errors are to be considered: the so-called *type I error* of rejecting a true null hypothesis and the *type II error* of failing to reject a false null hypothesis. Indeed, the same type of problem arises in problems of economic capital allocation and solvency measurement. To guarantee nonruin, it is clearly sufficient to have an infinite amount of economic capital. However, the cost of this amount of capital is generally infinite as well. Hence, the optimal amount of economic capital is obtained as a compromise between the cost of economic capital on the one hand and the preferred level of risk exposure on the other hand.

For the allocation of economic capital one has to consider two different problems. First, one needs to determine how large the economic capital should be for a given conglomerate. This is what we call the *absolute problem* of economic capital allocation. The second problem concerns how one should allocate a given amount of economic capital (eventually the optimal one in absolute terms) among the different subsidiaries. This is what we call the *relative problem* of economic capital allocation. Since subsidiaries are legal entities, they will also consider the absolute problem of economic capital allocation themselves. An equilibrium situation exists within the conglomerate in the case where the relative allocation of economic capital derived by the conglomerate and the absolute allocation derived by the subsidiaries coincide. The widely observed preference for a conglomerate construction in which the liabilities are pooled can be explained by the existence of a *virtual economic* capital for subsidiaries within the conglomerate.

We emphasize here that *economic* capital is different from *regulatory* capital. The latter is the bare minimum amount of capital that one is forced to hold, whereas the former is the amount of capital one "should" have in accordance with its risk preferences. Unfortunately, the literature does not always distinguish explicitly between the two concepts.

In this paper we consider the management of economic capital within financial conglomerates. First, we address allocation and diversification issues. In doing so, we advocate an explicit distinction between *risk measures* and *economic capital*. The difference between risk measures and economic capital comes from the different "levels" at which they operate: that is, there is a hierarchy between the two concepts. Economic capital is *derived* from a risk measure, by means of an optimization procedure. Previous related ideas can be found in Dhaene, Goovaerts, and Kaas (2003) and Laeven and Goovaerts (2004). The former paper presents some simple examples to support such a distinction, and the latter paper formalizes this distinction for the absolute and relative problems of economic capital allocation.

Next, we compare solutions of the relative problem to solutions of the absolute problem when the latter problem is solved from the point of view of the subsidiaries, and we demonstrate the emergence of virtual economic capital within financial conglomerates. Finally, we discuss the role of economic capital in problems of optimal portfolio selection.

The outline of the paper is as follows: in Section 2 we derive solutions to both the absolute and the relative problems of economic capital allocation. In Section 3 we compare subsidiaries when incorporated in a conglomerate construction with subsidiaries when considered as stand-alone entities and introduce the notion of virtual economic capital. Section 4 demonstrates that the absolute allocation approach also can be applied to the problem of optimal portfolio selection, taking into account the available amount of surplus capital and extending the well-known Markowitz approach.

2. ECONOMIC CAPITAL ALLOCATION

A risk is represented by a random variable X defined on a set of states of nature Ω and to be interpreted as the future net loss or deficit of a portfolio or position currently held. For simplicity and notational convenience, we henceforth restrict ourselves to random variables with continuous and strictly increasing distribution functions, although the results easily can be generalized to allow for random variables with discontinuous and nondecreasing distribution functions. Under these assumptions, the distribution function of a random variable has a true inverse.

We will first illustrate that the relative problem of economic capital allocation is similar to a top-down allocation of insurance premiums. Consider a financial conglomerate; in particular, think of an insurance company that consists of n subsidiaries. We denote by X_j the risk of subsidiary j, $j = 1, \ldots, n$. Furthermore, we denote by u the aggregate amount of economic capital and by P the aggregate premium income. Figures 1 and 2 illustrate a similar problem.

The relative problem of economic capital allocation is concerned with the question of how a given aggregate economic capital u is to be distributed among the subsidiaries, allocating u_j to subsidiary j. The aggregate amount of economic capital u depends on the aggregate risk, cost of economic capital, and risk preference of the conglomerate. We note that the relative problem of economic capital allocation is important from performance and risk evaluation perspectives. By determining u_j for $j = 1$,

Figure 1
Economic Capital Allocation among Subsidiaries

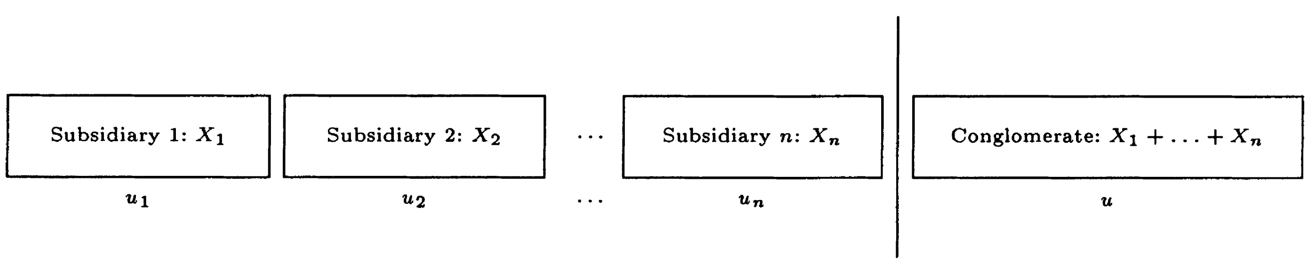
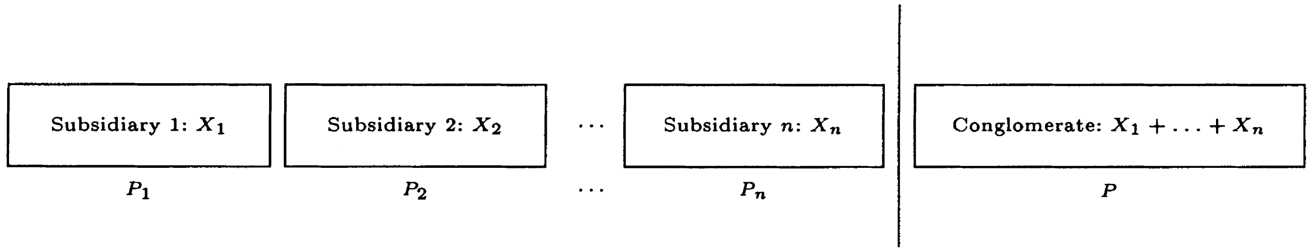


Figure 2
Premium Allocation among Subsidiaries



. . . , n, the conglomerate may evaluate the economic cost of risk-bearing and the risk-adjusted return per subsidiary and may allocate costs among subsidiaries correspondingly.

The relative problem of premium allocation is concerned with the question of how a given aggregate premium P is to be distributed among the subsidiaries, allocating P_j to subsidiary j. The aggregate premium P depends on the aggregate risk and the risk preference of the conglomerate; see in this respect Bühlmann (1985).

Other problems in insurance have the same structure. For instance, think of the determination of optimal reinsurance contracts, where, for example, in proportional reinsurance the relative problem consists of the determination of the proportionality levels for the different contracts such that at the aggregate a required level of stability is obtained; see, for instance, Bühlmann (1970) for a ruin probability approach to this problem.

In this section we will consider both the absolute and the relative problems of economic capital allocation. We demonstrate that optimal values $(u^*, u_1^{*s}, \ldots, u_n^{*s})$ generally exist that solve the absolute economic capital problem for the conglomerate, respectively for the subsidiaries, when considered as *stand-alone* entities. The relative problem consists of the distribution of u^* among the subsidiaries allocating u_j^* to subsidiary j, under the constraint that $u^* = u_1^* + \cdots + u_n^*$. The solutions $(u_1^{*s}, \ldots, u_n^{*s})$ and $(u_1^*, \ldots, u_n^*)$ generally differ.

2.1 The Relative Problem of Economic Capital Allocation

We distinguish between two approaches to economic capital allocation. The first approach, which is the one commonly encountered in the literature, consists of defining (or rather: axiomatizing) a risk measure $\rho[\cdot]$ that distributes the capital u^* in a *direct* way:

$$u_j^* = \rho[X_j], \quad j = 1, \ldots, n. \tag{2.1}$$

In this approach the risk measure and the economic capital coincide. In that case properties (axioms) of the risk measure are *directly* translated to properties (axioms) of the relative allocation of economic capital. Notice that in the case where full allocation of the economic capital is required, that is, $u_1^* + \cdots + u_n^* = u^*$, and the risk measure $\rho[\cdot]$ is also used to determine the economic capital at the conglomerate level (the absolute problem), then the axiom of additivity of the risk measure is a necessary axiom to describe the above direct allocation approach. Imposing additivity of a risk measure for all forms of dependence structure between the subsidiaries characterizes an expectation principle under very general conditions (see, e.g., Goovaerts, De Vylder, and Haezendonck 1984).

The second approach, which is the better one in our opinion, distinguishes between the risk measure and the economic capital. It consists of considering the *residual risk* of each of the subsidiaries. In this approach one has to relate the residual risk of the conglomerate in which the liabilities are pooled, given by

$$\max(X_1 + \cdots + X_n - u^*, 0) = (X_1 + \cdots + X_n - u^*)_+, \tag{2.2}$$

to the sum of the residual risks of the subsidiaries seen as separated juridical entities, given by

$$(X_1 - u_1)_+ + \cdots + (X_n - u_n)_+. \tag{2.3}$$

As long as $u^* \geq u_1 + \cdots + u_n$, the diversification effect, being the situation that the conglomerate has a residual risk that is smaller than that of the subsidiaries, follows because of the following stochastic dominance relation:

$$(X_1 + \cdots + X_n - u^*)_+ \leq_1 (X_1 - u_1)_+ + \cdots + (X_n - u_n)_+, \quad u^* \geq u_1 + \cdots + u_n. \tag{2.4}$$

Thus, any risk measure that preserves stochastic dominance is consistent with the diversification effect. The advantage of considering the residual risk is that we can establish a sound objective function from which the optimal allocation can be *derived,* rather than using a direct allocation rule.

Using the second approach, a distribution of the capital u^* is *derived* by minimizing the residual risk of the subsidiaries, as measured by the conglomerate. To illustrate the approach, let us take as an example the expectation to measure the residual risk. It is obvious that the expectation is a stochastic dominance-preserving risk measure and therefore preserves the so-called diversification effects; see relation (2.4). Notice that the probability measure under which the expectation is calculated can be left unspecified and is not necessarily the ''physical'' probability measure, since relation (2.4) holds even for every state of nature. Hence, the risk of the subsidiaries as measured by the conglomerate is given by

$$\mathbb{E}[(X_1 - u_1)_+ + \cdots + (X_n - u_n)_+]. \tag{2.5}$$

We then solve the problem

$$\min_{u_1,\ldots,u_n | \sum_{j=1}^n u_j = u^*} \mathbb{E}\left[\sum_{j=1}^n (X_j - u_j)_+\right], \tag{2.6}$$

which minimizes the risk measure (here the expectation) applied to the sum of the risk residuals representing the subsidiaries after the capital allocation has been performed. Solving this problem by means of Lagrange multipliers gives (see Laeven and Goovaerts 2004 for a detailed proof)

$$u_j^* = F_{X_j}^{-1}(1 - s), \quad j = 1, \ldots, n, \tag{2.7}$$

where s is determined as $F_{X_1^c+\cdots+X_n^c}(u^*) = 1 - s$, in which $F_{X_1^c+\cdots+X_n^c}$ denotes the distribution function of the comonotonic random vector $(X_1^c, \ldots, X_n^c)$ with the same marginal distribution functions as $(X_1, \ldots, X_n)$. The interested reader is referred to Dhaene et al. (2002a, 2002b) for an elaborate treatment of the concept of comonotonicity. Hence, $u_j^* = F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(u^*))$, which is the VaR of X_j at a confidence level $F_{X_1^c+\cdots+X_n^c}(u^*)$.

The expectation of the residual risk of the conglomerate is bounded from above by the expectation of the sum over the residual risks of the subsidiaries:

$$\mathbb{E}[(X_1 + \cdots + X_n - u^*)_+] \leq \mathbb{E}\left[\sum_{j=1}^n (X_j - F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(u^*)))_+\right]. \tag{2.8}$$

Furthermore, because

$$\sum_{j=1}^n \mathbb{E}[(X_j - F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(u^*)))_+] = \mathbb{E}[(X_1^c + \cdots + X_n^c - u^*)_+],$$

we find that

$$\mathbb{E}[(X_1 + \cdots + X_n - u^*)_+] \leq \mathbb{E}[(X_1^c + \cdots + X_n^c - u^*)_+]. \tag{2.9}$$

The difference between the right-hand side and the left-hand side of inequality (2.9) can be regarded as the diversification effect within the conglomerate. That is, the *diversification benefit* (DB) of incorporating subsidiaries in a conglomerate is given by

$$\mathrm{DB} = \mathbb{E}[(X_1^c + \cdots + X_n^c - u^*)_+] - \mathbb{E}[(X_1 + \cdots + X_n - u^*)_+]. \tag{2.10}$$

Note that the diversification benefit stems from the diversification effect on the residual risk of the conglomerate (see also in this context the report of the Casualty Actuarial Society 1999). For the expectation risk measure, the diversification benefit is nonnegative. Consequently, one might consider a diversification gain to be flowing back to the subsidiaries. A possible choice for the allocation of the diversification benefit would be to distribute it proportionally to the expected residual risk of each of the subsidiaries:

$$\mathrm{DB}_j = \mathrm{DB}\,\frac{\mathbb{E}[(X_j - F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(u^*)))_+]}{\mathbb{E}[(X_1^c + \cdots + X_n^c - u^*)_+]}, \quad j = 1, \ldots, n, \tag{2.11}$$

with $\mathrm{DB} = \sum_{j=1}^n \mathrm{DB}_j$. Notice that the diversification benefit is not a physical benefit, that is, it does not appear on the balance sheet or on the profit-and-loss account. However, as will become apparent in Section 3, it is of eminent importance when assessing the desirability of a conglomerate construction.

2.2 The Absolute Problem of Economic Capital Allocation

In this section we illustrate the absolute problem of economic capital allocation. We denote by r the risk-free rate of return and denote by i the opportunity cost of capital. Both variables are assumed to be nonrandom and are compounded over the particular time horizon considered. We let $(i - r)$ represent the cost of raising economic capital, and we assume throughout, without loss of generality, that this cost is the same for the conglomerate as for the subsidiaries when considered as stand-alone entities. We consider a risk measure that preserves stochastic dominance:

$$X \leq_1 Y \Rightarrow \rho[X] \leq \rho[Y].$$

Let

$$\rho_1[X, u] = \mathbb{E}[(X - u)_+] + (i - r)u, \tag{2.12}$$

and (see Kaas et al. 2001, Section 5.2)

$$\rho_2[X, u] = \frac{u}{|\log(\varepsilon)|}\log\left(\mathbb{E}\left[\exp\left(\frac{|\log(\varepsilon)|}{u}X\right)\right]\right) + (i - r)u, \quad \varepsilon \in (0, 1], \tag{2.13}$$

for both of which one can easily prove that if $X = X_1 + X_2$ and $u = u_1 + u_2$, then

$$\rho[X, u] \leq \rho[X_1, u_1] + \rho[X_2, u_2], \tag{2.14}$$

which expresses the diversification effect on the residual risk. To further simplify expression (2.13), we consider the economic capitals to be relatively large and let $\mathrm{Var}[X]$ denote the variance of a random variable X. Then

$$\rho_2[X, u] \approx \rho_3[X, u] = \mathbb{E}[X] + \frac{|\log(\varepsilon)|}{2u}\mathrm{Var}[X] + (i - r)u, \tag{2.15}$$

using only two terms of the Taylor expansion. Suppose that $\mathbb{E}[X] + \mathbb{E}[Y] \leq 2u/|\log(\varepsilon)|$, such that $\rho_3[X, u] \leq \rho_3[Y, u]$ in the case $X \leq_1 Y$. However, note that in contrast to $\rho_2[X, u]$, the approximation $\rho_3[X, u]$ no longer satisfies expression (2.14).

To solve the absolute problem of economic capital allocation, we minimize $\rho[X, u]$ with respect to u. For the choice of $\rho_1[X, u] = \mathbb{E}[(X - u)_+] + (i - r)u$ we obtain the following solutions:

$$u^* = F_X^{-1}(1 - (i - r)), \tag{2.16}$$

$$u_j^{*s} = F_{X_j}^{-1}(1 - (i - r)), \quad j = 1, \ldots, n. \tag{2.17}$$

Hence, the optimal amount of economic capital can be calculated by the VaR measure of which the confidence level depends on the cost of economic capital.

Now we compare equation (2.17) with equation (2.7). We find that to enforce that the solution to the absolute problem of economic capital allocation for the subsidiaries corresponds to the preferred value of the conglomerate, which is the solution to the relative allocation problem, the conglomerate simply can charge a single *shadow cost of capital* λ to its subsidiaries (notice that the uniqueness of λ is not a trivial result; it is valid only because of the particular form of the derived solutions). Then the solution to the absolute allocation problem for subsidiary j is given by

$$u_j^{*s}(\lambda) = F_{X_j}^{-1}(1 - (i - r) - \lambda), \quad j = 1, \ldots, n, \tag{2.18}$$

where λ is to be determined as

$$F_X^{-1}(1 - (i - r)) = \sum_{j=1}^{n} F_{X_j}^{-1}(1 - (i - r) - \lambda), \tag{2.19}$$

or equivalently

$$\lambda = 1 - (i - r) - F_{X_1^c+\cdots+X_n^c}(F_{X_1+\cdots+X_n}^{-1}(1 - (i - r))). \tag{2.20}$$

Verify that the absolute solution $u_j^{*s}(\lambda)$, corresponding to a cost of economic capital $(i - r) + \lambda$, indeed equals the relative allocation u_j^*. It is not difficult to see that typically (not always) $\lambda \geq 0$ and hence $u_j^{*s}(\lambda) \leq u_j^{*s}$; see Embrechts, Höing, and Puccetti (2004). Thus, both the conglomerate and the subsidiaries incorporated in the conglomerate can use the VaR measure to determine their economic capital, which is highly attractive from a corporate governance point of view. We remark that VaR allocation is consistent with the capital adequacy requirements established by the Basel Capital Accord (see Basel Committee 1988, 1996, 2004), under which banks currently operate. The fact that the order of the percentile is not the same in both equations (2.16) and (2.18) is rather natural. It is not sound to compare a confidence level for the conglomerate directly with a confidence level for the subsidiaries.

For this particular value of λ, the conglomerate enforces not just a *Pareto optimal* allocation, for which no subsidiary can be better off (in terms of *amount* of economic capital) without making another subsidiary worse off, but even a ''full'' optimal allocation (that is, optimal for all subsidiaries). This is attractive, since in problems of risk and resource distribution, in general not even a Pareto optimal allocation is reached; see, for example, Borch (1962). Clearly, we rely strongly on the possibility of the conglomerate's charging a shadow cost of capital. We defer until Section 3 an analysis of whether or not a subsidiary will be willing to pay such a shadow cost.

Notice that when the cost of raising economic capital differs between the juridical entities, a vector $(\lambda_1, \ldots, \lambda_n)$ may be introduced that is such that

$$i - r + \lambda = i_1 - r + \lambda_1 = \cdots = i_n - r + \lambda_n,$$

with $i - r$ and $i_j - r, j = 1, \ldots, n$, denoting the cost of raising economic capital for the conglomerate and for the subsidiaries, respectively.

Next, for $\rho_3[X, u] = \mathbb{E}[X] + (|\log(\varepsilon)|/2u)\,\mathrm{Var}[X] + (i - r)u$ we obtain

$$u^* = \sqrt{\frac{|\log(\varepsilon)|}{2(i - r)}\,\mathrm{Var}[X]}, \tag{2.21}$$

$$u_j^{*s} = \sqrt{\frac{|\log(\varepsilon)|}{2(i - r)}\,\mathrm{Var}[X_j]}, \quad j = 1, \ldots, n, \tag{2.22}$$

$$u_j^{*s}(\lambda) = \sqrt{\frac{|\log(\varepsilon)|}{2(i - r + \lambda)}\,\mathrm{Var}[X_j]}, \quad j = 1, \ldots, n, \tag{2.23}$$

In this situation, because

$$\sqrt{\frac{|\log(\varepsilon)|}{2(i-r)}\operatorname{Var}[X]} \leq \sum_{j=1}^{n} \sqrt{\frac{|\log(\varepsilon)|}{2(i-r)}\operatorname{Var}[X_j]}, \tag{2.24}$$

it holds true that $\lambda \geq 0$ if it is determined such that $\Sigma_{j=1}^{n} u_j^{*s}(\lambda) = u^*$, and hence a shadow cost is due.

Another interpretation is possible by transforming ε into ε', representing a larger probability of ruin:

$$\frac{|\log(\varepsilon)|}{2(i-r+\lambda)} = \frac{|\log(\varepsilon')|}{2(i-r)} \Rightarrow \varepsilon \leq \varepsilon'. \tag{2.25}$$

This finding is also natural since within a conglomerate one can intuitively accept a probability of ruin at the subsidiary level that is higher than the probability of ruin at the conglomerate level, which brings out the diversification benefit in another way.

The values of $u_j^{*s}(\lambda)$ can be obtained by solving the following system of equations numerically:

$$\frac{u_j^{*s}(\lambda)}{u^*} = \frac{\sqrt{\operatorname{Var}[X_j]}}{\Sigma_{k=1}^{n}\sqrt{\operatorname{Var}[X_k]}}, \quad j = 1, \ldots, n. \tag{2.26}$$

3. THE EMERGENCE OF VIRTUAL ECONOMIC CAPITAL

Since subsidiaries are juridical entities, one should solve the absolute problem of economic capital allocation not only for the conglomerate, but also for the subsidiaries, when considered as stand-alone entities. The latter solutions should then be compared with the solutions to the relative allocation problem of the conglomerate. Using $\rho_1[X, u]$ defined in equation (2.12) in the absolute allocation problem, we derived in Section 2.2 that

$$u_j^{*s} = F_{X_j}^{-1}(1-(i-r)), \tag{3.1}$$

which is to be compared with

$$u_j^* = F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(u)) = F_{X_j}^{-1}(F_{X_1^c+\cdots+X_n^c}(F_{X_1+\cdots+X_n}^{-1}(1-(i-r)))), \tag{3.2}$$

using the expectation in the relative allocation problem (2.6). Clearly, instead of u_j^* we can also write $u_j^{*s}(\lambda)$ as defined in equation (2.18), corresponding to a capital cost $(i-r)+\lambda$. We have seen in Section 2.2 that in general (not always) $u_j^*(\lambda) \leq u_j^{*s}$ and hence $\lambda \geq 0$. In the case where $\lambda < 0$, the conglomerate may regard the allocation enforcing mechanism as undesirable. Obviously a subsidiary is willing to pay a positive shadow cost λ only if it is compensated by the conglomerate in some sense. Hence, we should consider the diversification benefit for the subsidiaries of being incorporated in the conglomerate.

The *cost of risk-bearing* of subsidiary *j* when considered as a stand-alone entity is defined by

$$c_j^{*s} = (i-r)u_j^{*s} + \mathbb{E}[(X_j - u_j^{*s})_+], \quad j = 1, \ldots, n. \tag{3.3}$$

Furthermore, the *cost of risk-bearing* attributed to subsidiary *j* when incorporated in the conglomerate construction, but without taking into account the diversification benefit, is defined by

$$c_j^{*s}(\lambda) = (i-r+\lambda)u_j^{*s}(\lambda) + \mathbb{E}[(X_j - u_j^{*s}(\lambda))_+], \quad j = 1, \ldots, n. \tag{3.4}$$

The conglomerate situation becomes more favorable in the case where diversification benefits are taken into account. We have seen in equation (2.10) that the diversification benefit of the conglomerate construction is given by

$$\mathrm{DB} = \sum_{j=1}^{n} \mathbb{E}[(X_j - u_j^{*s}(\lambda))_+] - \mathbb{E}[(X-u)_+] = (1-\alpha)\sum_{j=1}^{n} \mathbb{E}[(X_j - u_j^{*s}(\lambda))_+], \quad \text{for some } \alpha \in [0,1].$$

Notice that in this notation

$$\mathrm{DB}_j = (1 - \alpha)\mathbb{E}[(X_j - u_j^{*s}(\lambda))_+], \quad j = 1, \ldots, n.$$

Taking into account the diversification benefits, according to a proportional distribution among the subsidiaries, the cost of risk-bearing attributed to subsidiary j is given by

$$\begin{aligned} c_j^{*\mathrm{DB}}(\lambda) &= (i - r + \lambda)u_j^{*s}(\lambda) + \alpha\mathbb{E}[(X_j - u_j^{*s}(\lambda))_+] \\ &= (i - r + \lambda)u_j^{*s}(\lambda) + \mathbb{E}[(X_j - u_j^{*\mathrm{DB}}(\lambda))_+], \quad j = 1, \ldots, n. \end{aligned} \tag{3.5}$$

Clearly, $c_j^{*\mathrm{DB}}(\lambda) \leq c_j^{*s}(\lambda)$ and $u_j^{*\mathrm{DB}}(\lambda) \geq u_j^{*s}(\lambda)$. The difference between $u_j^{*\mathrm{DB}}(\lambda)$ and $u_j^{*s}(\lambda)$ is what we call the *virtual economic capital* of subsidiary j:

$$u_j^{*v}(\lambda) = u_j^{*\mathrm{DB}}(\lambda) - u_j^{*s}(\lambda) \geq 0, \quad j = 1, \ldots, n. \tag{3.6}$$

The existence of the diversification benefit and hence of a virtual economic capital should not be ignored by the regulatory authority when assessing the solvency position of each subsidiary.

To determine whether or not the conglomerate construction is beneficial when compared with the stand-alone situation, one should consider the difference

$$c_j^{*\mathrm{DB}}(\lambda) - c_j^{*s} = (i - r + \lambda)u_j^{*s}(\lambda) + \mathbb{E}[(X_j - u_j^{*\mathrm{DB}}(\lambda))_+] - (i - r)u_j^{*s} - \mathbb{E}[(X_j - u_j^{*s})_+],$$

which for subsidiary j may or may not be nonpositive (nonpositivity would argue in favor of the conglomerate construction), depending on the relative dependence structure within the conglomerate and the cost of economic capital.

Notice that the shadow cost can be regarded as a full gain for the conglomerate construction. From the viewpoint of the subsidiaries, the conglomerate construction becomes more favorable when this gain is redistributed among the subsidiaries, ex post.

3.1 Generalization with a Comonotonic Additive Risk Measure

We now will generalize the previous results by replacing the expectation operator by a comonotonic additive risk measure $\pi[\cdot]$. In particular, we specify

$$\begin{aligned} \rho_4[X, u] &= \pi[(X - u)_+ + (i - r)u] = \int_0^1 F^{-1}_{(X-u)+}(y)d(1 - g(1 - y)) + (i - r)u \\ &= \int_u^\infty g(1 - F_X(x))dx + (i - r)u, \end{aligned} \tag{3.7}$$

for some strictly increasing and continuous function $g : [0, 1] \to [0, 1]$, satisfying $g(0) = 0$ and $g(1) = 1$, assuming the integrals are finite. The function $g(\cdot)$ is known as a *distortion* or *probability weighting* function. We refer to Greco (1982), Schmeidler (1989), and Yaari (1987) and in an insurance context to Wang, Young, and Panjer (1997) for an axiomatic characterization of distortion risk measures. It is not difficult to verify that if $u = u_1 + \cdots + u_n$, then

$$\rho_4[X_1^c + \cdots + X_n^c, u] = \rho_4[X_1, u_1] + \cdots + \rho_4[X_n, u_n], \tag{3.8}$$

that is, $\rho_4[X, u]$ is comonotonic additive. Minimizing $\rho_4[X, u]$ with respect to u yields the following solutions:

$$u^* = F_X^{-1}(1 - g^{-1}(i - r)), \tag{3.9}$$

$$u_j^{*s} = F_{X_j}^{-1}(1 - g^{-1}(i - r)), \quad j = 1, \ldots, n, \tag{3.10}$$

$$u_j^{*s}(\lambda) = F_{X_j}^{-1}(1 - g^{-1}(i - r + \lambda)), \quad j = 1, \ldots, n, \tag{3.11}$$

where λ is determined such that

$$\sum_{j=1}^{n} F_{X_j}^{-1}(1 - g^{-1}(i - r + \lambda)) = F_X^{-1}(1 - g^{-1}(i - r)). \tag{3.12}$$

As before, λ typically satisfies $\lambda \geq 0$. We remark that $u_j^{*s}(\lambda)$ as defined in equation (3.11) is not in general equal to the relative allocation u_j^* obtained by solving

$$\min_{u_1,\ldots,u_n | \sum_{j=1}^n u_j = u^*} \pi\left[\sum_{j=1}^{n} (X_j - u_j)_+\right],$$

Clearly, one may introduce a vector $(\lambda_1, \ldots, \lambda_n)$ to establish the equality $u_j^{*s}(\lambda_j) = u_j^*$, $j = 1, \ldots, n$, though such a distinction may be questionable from the viewpoint of corporate governance. Henceforth, we restrict ourselves to a single shadow cost λ. The diversification benefit of incorporating subsidiaries in a conglomerate construction is then given by

$$\begin{aligned} \text{DB} &= \sum_{j=1}^{n} \pi[(X_j - u_j^{*s}(\lambda))_+] - \pi[(X_1 + \cdots + X_n - u^*)_+] \\ &= \pi[(X_1^c + \cdots + X_n^c - u^*)_+] - \pi[(X_1 + \cdots + X_n - u^*)_+]. \end{aligned} \tag{3.13}$$

It can be verified that for a concave distortion function $g(\cdot)$, the diversification benefit is nonnegative in general. Clearly, if a negative diversification benefit occurs, the subsidiaries typically will not be willing to pay a positive shadow cost λ, and a conglomerate construction will appear to be undesirable. In the case of a nonnegative diversification benefit, we have that

$$\text{DB} = (1 - \alpha)\pi[(X_1^c + \cdots + X_n^c - u^*)_+], \quad \text{for some } \alpha \in [0, 1], \tag{3.14}$$

which may be distributed among the subsidiaries. Hence, the cost of risk-bearing attributed to subsidiary j, taking into account the shadow cost λ and the diversification benefit according to a proportional distribution, is given by

$$\begin{aligned} c_j^{*DB}(\lambda) &= (i - r + \lambda)u_j^{*s}(\lambda) + \alpha\pi[(X_j - u_j^{*s}(\lambda))_+] \\ &= (i - r + \lambda)u_j^{*s}(\lambda) + \pi[(X_j - u_j^{*DB}(\lambda))_+], \quad j = 1, \ldots, n. \end{aligned} \tag{3.15}$$

Again a virtual economic capital $u_j^{*v}(\lambda) = u_j^{*DB}(\lambda) - u_j^{*s}(\lambda)$ emerges.

4. THE ROLE OF ECONOMIC CAPITAL IN OPTIMAL PORTFOLIO SELECTION

In practice, people working in the framework of optimal portfolio selection generally agree that in order to benefit from ‘‘supplementary returns’’ some surplus capital (call it *economic capital*) is desirable. Optimal portfolio selection often is performed on the basis of the well-known Markowitz portfolio selection approach (Markowitz 1952, 1959). An alternative procedure employing a shortfall constraint has been developed in Basak and Shapiro (2001). In this section we present an approach that explicitly takes into account the amount of economic capital available and the economic cost of raising it.

We denote by V the future random gain obtained by an investment θ_0 in risk-free assets generating a nonrandom return r, and investments θ_j in risky assets j, $j = 1, \ldots, m$ with random return X_j:

$$V = \theta_0 r + \theta_1 X_1 + \cdots + \theta_m X_m. \tag{4.1}$$

Furthermore, we let $\theta = \theta_0 + \cdots + \theta_m$ denote the total amount invested, which is assumed to be fixed and given. The Markowitz portfolio theory considers a utility function

$$U = V - \iota(V - \mathbb{E}[V])^2, \tag{4.2}$$

where ι denotes a tolerance level, that is, a degree of risk aversion. The optimal investment portfolio then is obtained by maximizing

$$\mathbb{E}[U] = \mathbb{E}[V] - \iota\mathbb{E}[(V - \mathbb{E}[V])^2]. \tag{4.3}$$

Since the dimension of $\mathbb{E}[V]$ is the monetary unit and the dimension of $\mathbb{E}[(V - \mathbb{E}[V])^2]$ is the "squared monetary unit," it is reasonable to think of ι as a tolerance level expressed in "1 over monetary units." This natural interpretation is often overlooked, typically leading to surprising results. One could choose, for instance, $\iota = \alpha/u$, where α is a dimensionless constant and u denotes the amount of economic capital available.

An alternative approach following the Markowitz general concept could be to consider the expected gain $\mathbb{E}[(V - \theta r)_+]$ on the risky portfolio and deduct from it the cost of the downside risk $\beta\mathbb{E}[(\theta r - V)_+]$, in which $\beta > 1$ is due to the safety margin required in the actuarial framework, representing *loss aversion*. Hence, we introduce the following generalized utility function:

$$U = (V - \theta r)_+ - \beta(\theta r - V)_+, \quad \beta > 1. \tag{4.4}$$

Then

$$\mathbb{E}[U] = \mathbb{E}[(V - \theta r)_+] - \beta\mathbb{E}[(\theta r - V)_+], \quad \beta > 1 \tag{4.5}$$

represents the expected generalized utility of the investment strategy $(\theta_0, \theta_1, \ldots, \theta_m)$ that should be maximized. However, the available economic capital does not play a role in this approach, which is against common sense.

To capture the effect of economic capital in the selection procedure, we employ a specific risk measure for the downside risk. In particular, we use a measure expressed in terms of the economic capital, namely,

$$\frac{u}{|\log(\varepsilon)|} \log\left(\mathbb{E}\left[\exp\left(\frac{|\log(\varepsilon)|}{u}(\theta r - V)_+\right)\right]\right), \quad \varepsilon \in (0, 1], \tag{4.6}$$

based on a ruin criterion (see, e.g., Kaas et al. 2001). Introducing expression (4.6) into the expected generalized utility expression yields

$$\mathbb{E}[U] = \mathbb{E}[(V - \theta r)_+] - \frac{u}{|\log(\varepsilon)|} \log\left(\mathbb{E}\left[\exp\left(\frac{|\log(\varepsilon)|}{u}(\theta r - V)_+\right)\right]\right). \tag{4.7}$$

Suppose that u is relatively large. Then the above expression can be approximated by

$$\mathbb{E}[(V - \theta r)_+] - \mathbb{E}[(\theta r - V)_+] - \frac{|\log(\varepsilon)|}{2u}\text{Var}[(\theta r - V)_+] = \mathbb{E}[V] - \theta r - \frac{|\log(\varepsilon)|}{2u}\text{Var}[(\theta r - V)_+], \tag{4.8}$$

using only the first two terms of the Taylor expansion. When maximizing the expected generalized utility of the investment strategy, we naturally restrict ourselves to solutions $(\theta_0^*, \theta_1^*, \ldots, \theta_m^*)$ for which the value of the objective function (4.8) is nonnegative. It follows that the economic capital u should be large enough to construct a "feasible" portfolio. For the case of only one risky asset, the optimal portfolio with total amount invested equal to $\theta = \theta_0 + \theta_1$ is given by

$$\theta_1^* = \theta - \theta_0^* = \frac{u}{|\log(\varepsilon)|} \frac{\mathbb{E}[X_1] - r}{\text{Var}[(r - X_1)_+]}. \tag{4.9}$$

Observe that the amount invested in the risky asset is naturally increasing in the amount of economic capital. We remark that the expected generalized utility in expression (4.8) does not take into account the cost of economic capital $(i - r)u$. Clearly, for a given level of economic capital the solution $(\theta_0^*, \theta_1^*, \ldots, \theta_m^*)$ is independent of whether or not the economic cost of capital is taken into account. Indeed, adding a constant $-(i - r)u$ to the objective function (4.8) does not affect the solution.

Let us now consider the case of a variable level of economic capital. In that case the expected generalized utility maximization problem is given by

$$\max_{(\theta_0,\theta_1,\ldots,\theta_m),u\,|\,\Sigma_{j=0}^m\ \theta_j=\theta} \mathbb{E}[V] - \theta r - \frac{|\log(\varepsilon)|}{2u}\mathrm{Var}[(\theta r - V)_+] - (i - r)u. \tag{4.10}$$

For the case of only one risky asset and $\theta = \theta_0 + \theta_1$, we obtain the following optimal investment strategy:

$$\theta_1^* = \theta - \theta_0^* = \begin{cases} \theta, & \mathbb{E}[X_1] - r - \sqrt{2|\log(\varepsilon)|(i - r)\mathrm{Var}[(r - X_1)_+]} > 0, \\ \alpha\theta,\ \alpha \in [0, 1], & \mathbb{E}[X_1] - r - \sqrt{2|\log(\varepsilon)|(i - r)\mathrm{Var}[(r - X_1)_+]} = 0, \\ 0, & \mathbb{E}[X_1] - r - \sqrt{2|\log(\varepsilon)|(i - r)\mathrm{Var}[(r - X_1)_+]} < 0, \end{cases}$$

not allowing short-selling and hence restricting to $\theta_j \geq 0$, $j = 1, 2$. Furthermore, the optimal amount of surplus capital is given by

$$u^* = \theta_1^* \sqrt{\frac{|\log(\varepsilon)|}{2(i - r)}\mathrm{Var}[(r - X_1)_+]}. \tag{4.11}$$

Hence, we find that in the case of only one risky asset the optimal investment strategy is a boundary solution. Indeed, if the risk premium $\mathbb{E}[X_1] - r$ is sufficiently large, the total amount θ should be invested in the risky asset. Otherwise, it is optimal to invest only in the risk-free asset and to hold no surplus capital. Whether or not the risk premium is sufficiently large naturally depends on the economic cost of capital.

The above can be extended readily to the case of multiple risky assets, for which a boundary solution will not be obtained in general.

5. CONCLUSIONS

We propose a solution to both the absolute and the relative problems of economic capital allocation based on a residual risk consideration. The solutions take into account the cost of economic capital. We compare the relative allocation among subsidiaries with the absolute allocation for subsidiaries when considered as stand-alone entities. We demonstrate the existence of a virtual economic capital for subsidiaries in a conglomerate. Furthermore, we consider the role of economic capital in optimal portfolio selection. We find that for a variable level of economic capital and the case of only one risky asset a boundary solution is obtained: it is optimal to invest either in the risky asset or in the risk-free asset.

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