Some Ruin Problems for a Risk Process with Stochastic Interest

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Abstract

As investment plays an increasingly important role in the insurance business, ruin analysis in the presence of stochastic interest (or stochastic return on investments) has become a key issue in modern risk theory, and the related results should be of interest to actuaries. Although the study of insurance risk models with stochastic interest has attracted a fair amount of attention in recent years, many significant ruin problems associated with these models remain to be investigated. In this paper we consider a risk process with stochastic interest in which the basic risk process is the classical risk process and the stochastic interest process (or the stochastic return-on-investment-generating process) is a compound Poisson process with positive drift. Within this framework, we first derive an integro-differential equation for the Gerber-Shiu expected discounted penalty function, and then obtain an exact solution to the equation. We also obtain closed-form expressions for the expected discounted penalty function in some special cases. Finally, we examine a lower bound for the ruin probability of the risk process.

1. Introduction

In practice, insurance companies often have a huge amount of assets to manage. For various reasons, an insurer may invest part of its assets in a portfolio with investment risk. However, poor performance of risky investments together with an unexpected large amount of claims could put a company into a very adverse financial situation, if not insolvency. In view of this, the impact of risky investments on the insurance business needs to be examined carefully. In fact, a fair amount of attention has been paid to risk models with stochastic interest (or stochastic return on investments) in recent years. Since most of the existing results for these models mainly focus on ruin probability, much research on other important and informative quantities of ruin such as the surplus immediately before ruin and the deficit at ruin remains to be carried out.

Paulsen (1993) proposed a general risk process with stochastic return on investments, taking account of three factors, namely, insurance risk, investment risk, and inflation. He used semimartingales to model the three factors and obtained an integro-differential equation and an analytical expression for ruin probability under certain conditions. Since then, various models ignoring the factor of inflation have been established within the framework of his model. These models usually assume that a basic risk process describes the underlying insurance risk and that the basic risk process is invested in a stochastic interest process (or a stochastic return-on-investment-generating process) with investment risk.

Paulsen and Gjessing (1997) derived integro-differential equations for the probability of ultimate ruin and the Laplace transform of the time of ruin when the basic risk process and the stochastic return-on-investment-generating process are compound Poisson processes perturbed by diffusion. Wang

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and Wu (2001) discussed the probability of ruin, surplus distribution at the time of ruin, and supremum distribution of the surplus before ruin for the basic risk process being the classical one and the stochastic return on investments following a Brownian motion with positive drift. In the case that the total claim and the stochastic return on investments are both Lévy processes, Kalashikov and Norberg (2002) constructed lower and upper bounds for the probability of ruin, while Paulsen (2002) studied the asymptotic ruin probability. Assuming that the basic risk process is the classical risk process and the return-on-investment-generating process is a compound Poisson process plus a Brownian motion with positive drift, Yuen, Wang, and Ng (2004) used the martingale approach and the Markov property to derive an integral equation and a lower bound for the ruin probability. For a compound Poisson risk process, Cai (2004) derived lower and upper bounds for the ultimate ruin probability with a stochastic interest process being a Lévy process and studied the Gerber-Shiu expected discounted penalty function with the stochastic interest process being a Brownian motion with drift.

Let $(\Omega, \mathcal{T}, P)$ be a complete probability space containing all the variables defined in this paper. Here the basic risk process is a compound Poisson process given by

$$S_t = u + ct - \sum_{k=1}^{N_{S,t}} Y_{S,k}, \quad t \geq 0, \tag{1.1}$$

where u is the initial surplus, $c > 0$ is the rate of premium, $N_{S,t}$ is the number of claims occurring in $(0, t]$ following a Poisson process with intensity λ_S, and $Y_{S,k}$ are independent and identically distributed (i.i.d.) claim amounts having common distribution F_S with $F_S(0) = 0$.

Suppose that the insurer is allowed to invest in an asset or investment portfolio. The stochastic return-on-investment-generating process is assumed to be another compound Poisson process

$$I_t = rt + \sum_{k=1}^{N_{I,t}} Y_{I,k}, \quad t \geq 0, \tag{1.2}$$

where $r > 0$ is the fixed interest rate, $N_{I,t}$ is a Poisson process with intensity λ_I, and $Y_{I,k}$ are i.i.d. random variables with common distribution F_I. The compound Poisson process, $\sum_{k=1}^{N_{I,t}} Y_{I,k}$, may be interpreted as the total amount of big stochastic additional changes in interest at some stochastic times up to time t. Due to the ever-changing economic environment, reassessment of the investment portfolio and reallocation of the surplus may be necessary from time to time. When these happen, $Y_{I,k}$ can be used to represent the change in the asset price or to reflect the change in interest in some way. Note that if a risky asset is invested, such an asset may bear negative interest, that is, $P(Y_{I,k} < 0) > 0$.

Assume that S_t and I_t are independent. Suppose that the initial surplus, premium payments, and claims are deposited into or drawn from the investment portfolio. Then the current surplus of the insurer at time t, R_t, is the solution of the following linear stochastic differential equation:

$$R_t = S_t + \int_0^t R_{s^-} dI_s. \tag{1.3}$$

By Theorem 32 of Protter (1992, p. 238), the solution for equation (1.3) is

$$R_t = U_t \left(u + \int_0^t \frac{1}{U_{s^-}} dS_s \right), \tag{1.4}$$

where $U_t = e^{rt} \prod_{k=1}^{N_{I,t}} (1 + Y_{I,k})$. Furthermore, risk process (1.4) is a homogeneous strong Markov process. Denote by $\{T_{S,n}, n \geq 1\}$ and $\{T_{I,n}, n \geq 1\}$ the sequences of the jump times of the Poisson processes $N_{S,t}$ and $N_{I,t}$, respectively.

If $\lambda_I = 0$, then risk process (1.4) is reduced to

$$R_t = e^{rt} \left(u + c \int_0^t e^{-rs} \, ds - \sum_{k=1}^{N_{S,t}} e^{-rT_{S,k}} Y_{S,k} \right). \tag{1.5}$$

Process (1.5) is known as the classical risk process with constant interest rate. The ruin problems for risk process (1.5) have been extensively studied. For example, see Gerber (1971), Harrison (1977), Delbaen and Haezendonck (1987), Sundt and Teugels (1995, 1997), and Cai and Dickson (2002).

The time T_u of ruin for risk process (1.4) is defined by $T_u = \inf\{t : R_t < 0\}$. If $R_t \geq 0$ for all $t \geq 0$, then $T_u = \infty$. The ruin probability of risk process (1.4) is defined as

$$\Psi(u) = P\left(\inf_{t\geq 0} R_t < 0 | R_0 = u\right) = P(T_u < \infty).$$

Let $w = w(x_1, x_2)$ be a measurable function on $[0, \infty) \times [0, \infty)$. Define

$$\begin{aligned}\Phi_\alpha(u) &= \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty)] \\ &\triangleq \mathrm{E}\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) | R_0 = u],\end{aligned} \tag{1.6}$$

with $I(A)$ being the indicator function of set A. Suppose that $w \equiv 1$ and $\alpha = 0$. Then, from equation (1.6), we get $\Phi_\alpha(u) = \Psi(u)$. The function $\Phi_\alpha(u)$ is known as the Gerber-Shiu expected discounted penalty function introduced by Gerber and Shiu (1998). It is obvious that the penalty function w depends on the surplus immediately prior to ruin as well as the deficit at ruin.

In Section 2 we study an integro-differential equation for $\Phi_\alpha(u)$, and in Section 3 we derive an explicit expression for $\Phi_\alpha(u)$. In Section 4 we present two examples in which closed-form solutions for the expected discounted penalty function are derived. In Section 5 we give some remarks on a lower bound for $\Psi(u)$.

2. INTEGRO-DIFFERENTIAL EQUATION

Let $\mu_S = E[Y_{S,1}]$. We assume throughout this paper that $\mu_S < \infty$, $c - \lambda_S\mu_S > 0$, and $F_I(-1) = 0$ so that $\Psi(u) < 1$ for $u \geq 0$, and $\Psi(\infty) = \lim_{u\to\infty} \Psi(u) = 0$.

We now derive an integro-differential equation for $\Phi_\alpha(u)$. For $u \geq 0$, we have

$$\begin{aligned}\Phi_\alpha(u) &= \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) I(T_{S,1} < T_{I,1})] \\ &\quad + \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) I(T_{I,1} < T_{S,1})] \\ &\triangleq \mathrm{E}_1 + \mathrm{E}_2.\end{aligned} \tag{2.1}$$

Since risk process (1.4) has the strong Markov property, the risk process will be restarted at any finite stopping time T from R_T. Therefore, arguments of Sundt and Teugels (1995) can be used to derive an integral equation for $\Phi_\alpha(u)$. If $T_{S,1} = t < T_{I,1}$ with $Y_{S,1} = x$ and $x \leq e^{rt}\ (u + c \int_0^t e^{-rs}\ ds) \triangleq \phi(u, t)$, then ruin does not occur, else ($x > \phi(u, t)$) ruin occurs. Thus, from the independence of $T_{S,1}$ and $T_{I,1}$, we get

$$\begin{aligned}E_1 &= \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) I(T_{S,1} < T_{I,1})] \\ &= \int_0^\infty \int_0^\infty \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) | T_{S,1} = t, Y_{S,1} = x, T_{S,1} < T_{I,1}] \\ &\quad \times P(T_{S,1} \in dt, Y_{S,1} \in dx, T_{S,1} < T_{I,1}) \\ &= \int_0^\infty \lambda_S e^{-\lambda_S t} \int_0^\infty \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) | T_{S,1} = t, Y_{S,1} = x, T_{S,1} < T_{I,1}] e^{-\lambda_I t} dF_S(x) dt \\ &= \int_0^\infty \lambda_S e^{-(\lambda_S+\lambda_I+\alpha)t} \int_0^{\phi(u,t)} \Phi_\alpha(\phi(u, t) - x)\ dF_S(x) dt \\ &\quad + \int_0^\infty \lambda_S e^{-(\lambda_S+\lambda_I+\alpha)t} \int_{\phi(u,t)}^\infty w(\phi(u, t), x - \phi(u, t))\ dF_S(x) dt.\end{aligned} \tag{2.2}$$

On the other hand, if $T_{I,1} = t < T_{S,1}$ with $Y_{I,1} = x$, ruin does not occur. Therefore,

$$E_2 = \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) I(T_{I,1} < T_{S,1})]$$

$$= \int_0^\infty \lambda_I e^{-\lambda_I t} \int_{-1}^\infty \mathrm{E}^u\ [w(R_{T_u^-}, |R_{T_u}|)e^{-\alpha T_u} I(T_u < \infty) | T_{I,1} = t, Y_{I,1} = x, T_{I,1} < T_{S,1}]\ e^{-\lambda_S t} dF_I(x)dt$$

$$= \int_0^\infty \lambda_I e^{-(\lambda_S+\lambda_I+\alpha)t} \int_{-1}^\infty \Phi_\alpha((1 + x)\phi(u, t))\ dF_I(x)dt. \tag{2.3}$$

From equations (2.1)–(2.3), we get the following integral equation for $\Phi_\alpha(u)$:

$$\Phi_\alpha(u) = \int_0^\infty \lambda_S e^{-(\lambda_S+\lambda_I+\alpha)t} \int_0^{\phi(u,t)} \Phi_\alpha(\phi(u, t) - x)\ dF_S(x)dt$$
$$+ \int_0^\infty \lambda_S e^{-(\lambda_S+\lambda_I+\alpha)t} \int_{\phi(u,t)}^\infty w(\phi(u, t), x - \phi(u, t))\ dF_S(x)dt \tag{2.4}$$
$$+ \int_0^\infty \lambda_I e^{-(\lambda_S+\lambda_I+\alpha)t} \int_{-1}^\infty \Phi_\alpha((1 + x)\phi(u, t))\ dF_I(x)dt.$$

Substituting $y = \phi(u, t)$ in equation (2.4) yields

$$\Phi_\alpha(u) = \lambda_S(ru + c)^{(\lambda_S+\lambda_I+\alpha)r^{-1}} \int_u^\infty (ry + c)^{-(\lambda_S+\lambda_I+\alpha)r^{-1}-1}$$
$$\int_0^y \Phi_\alpha(y - x)dF_S(x)dy + \lambda_S(ru + c)^{(\lambda_S+\lambda_I+\alpha)r^{-1}}$$
$$\int_u^\infty (ry + c)^{-(\lambda_S+\lambda_I+\alpha)r^{-1}-1} \int_y^\infty w(y, x - y)dF_S(x)dy \tag{2.5}$$
$$+ \lambda_I(ru + c)^{(\lambda_S+\lambda_I+\alpha)r^{-1}} \int_u^\infty (ry + c)^{-(\lambda_S+\lambda_I+\alpha)r^{-1}-1}$$
$$\int_{-1}^\infty \Phi_\alpha((1 + x)y)dF_I(x)dy.$$

Denote by f_S and f_I the density of F_S and F_I, respectively. We assume throughout the paper that f_S and f_I are continuous. From equation (2.5), we can verify the following lemma.

Lemma 2.1

Assume that the penalty function w is bounded. Then $\Phi_\alpha(u)$ is continuously differentiable in u on $[0, \infty)$.

Differentiating equation (2.5) with respect to u, we obtain the integro-differential equation for $\Phi_\alpha(u)$:

$$\frac{d}{du}\Phi_\alpha(u) = \frac{\lambda_S + \lambda_I + \alpha}{ru + c}\Phi_\alpha(u) - \frac{\lambda_S}{ru + c}\left(\int_0^u \Phi_\alpha(u - x)dF_S(x) + A(u)\right)$$
$$- \frac{\lambda_I}{ru + c}\int_{-1}^\infty \Phi_\alpha((1 + x)u)dF_I(x), \tag{2.6}$$

where

$$A(u) = \int_u^\infty w(u, x - u)dF_S(x) \tag{2.7}$$

is a known function. Assume that w is bounded. By equation (1.6) and the dominated convergence theorem, we get

$$\lim_{u \to \infty} \Phi_\alpha(u) = 0. \tag{2.8}$$

3. EXACT SOLUTION

In this section we derive an exact solution to the integral equation (2.5) or the integro-differential equation (2.6).

We first examine some properties of risk process (1.4). Recall U_t given in risk process (1.4). For $t \geq s$, it is easy to check that U_s and $U_t U_s^{-1}$ are independent and that $U_t U_s^{-1}$ and U_{t-s} have the same distribution. Set $\tau_{S,n} = T_{S,n} - T_{S,n-1}$ for $n \geq 1$. Then $\{\tau_{S,n}, n \geq 1\}$ is a sequence of i.i.d. random variables with common exponential distribution with parameter λ_S. Put $Y_S = Y_{S,1}$, $Y_I = Y_{I,1}$, and $\tau_S = \tau_{S,1}$. Consider the embedded discrete time process $\{L_n, n \geq 1\}$ defined by $L_n = R_{T_{S,n}}$. For stopping times $S < T$, we have

$$R_T = \frac{U_T}{U_S}\left(R_S + \int_S^T \frac{U_S}{U_{t^-}} dS_t\right),$$

from which we obtain

$$L_n = \xi_n L_{n-1} + \eta_n, \tag{3.1}$$

for $n \geq 1$ with $L_0 = u$, where $\{(\xi_n, \eta_n), n \geq 1\}$ is a sequence of i.i.d. pairs of random variables distributed as (ξ, η) defined by

$$\xi = U_{\tau_S} = e^{r\tau_S} \prod_{k=1}^{N_{I,\tau_S}} (1 + Y_{I,k}), \tag{3.2}$$

$$\eta = c\eta_0 - Y_S \triangleq cU_{\tau_S} \int_0^{\tau_S} \frac{1}{U_t} dt - Y_S. \tag{3.3}$$

By convention, $\Pi_{k=1}^0 (1 + Y_{I,k}) = 1$. These properties can also be found in Kalashnikov and Norberg (2002) and Yuen, Wang, and Ng (2004).

Put $X_k = 1 + Y_{I,k}$ for $k \geq 1$. Assume that $a_k > 0$ for $k \geq 0$. Define

$$H_n = P\left(a_0 + a_1 X_1 + a_2 X_1 X_2 + \cdots + a_n \prod_{k=1}^n X_k \in dx\right).$$

For $x > a_0$, straightforward calculations give

$$H_n = h_n(a_0, a_1, \ldots, a_n, x)dx, \tag{3.4}$$

where

$$\begin{aligned} h_n(a_0, a_1, \ldots, a_n, x) = &\int_0^{(x-a_0)a_1^{-1}} f_I(z_1 - 1)dz_1 \int_0^{((x-a_0)/z_1 - a_1)a_2^{-1}} f_I(z_2 - 1)dz_2 \cdots \\ &\int_0^{(((x-a_0)/(z_1 \cdots z_{n-2})) - (a_1/(z_2 \cdots z_{n-2})) - \cdots - (a_{n-3}/z_{n-2}) - a_{n-2})a_{n-1}^{-1}} f_I(z_{n-1} - 1) \\ &\times f_I\left(\left(\frac{x - a_0}{z_1 \cdots z_{n-1}} - \frac{a_1}{z_2 \cdots z_{n-1}} - \cdots - \frac{a_{n-2}}{z_{n-1}} - a_{n-1}\right)\left(\frac{1}{a_n}\right) - 1\right) \\ &\times \frac{1}{z_1 \cdots z_{n-1} a_n} dz_{n-1}. \end{aligned}$$

Let

$$\mathrm{E}^u\ [e^{-\alpha\tau_S}\, I\ (R_{\tau_{\bar{S}}} \in dx)] = \mathrm{E}\left[e^{-\alpha\tau_S}\, I\left(U_{\tau_S}\left(u + c\int_0^{\tau_S} U_t^{-1}\, dt\right) \in dx\right)\right] = B_\alpha(u, x)dx.$$

For $x > u \geq 0$, one can show that

$$\begin{aligned} B_\alpha(u, x) = {} & \frac{(ru + c)^{(\lambda_S+\lambda_I+\alpha)r^{-1}}}{(rx + c)^{1+(\lambda_S+\lambda_I+\alpha)r^{-1}}}\,\lambda_S I(u < x < \infty) \\ & + \sum_{n=1}^{\infty}\int_0^\infty \lambda_S e^{-(\lambda_S+\lambda_I+r+\alpha)s}ds\int_0^s dt_1\int_{t_1}^s dt_2\cdots \\ & \int_{t_{n-1}\vee(s-r^{-1}\ln(1+(rc^{-1})x))}^{s} \lambda_I^n h_n\ (cb_{n+1}, cb_n, \ldots, cb_2, c(b_1 + u), xe^{-rs})\ dt_n, \end{aligned} \tag{3.5}$$

where $a \vee b = \max\{a, b\}$, $b_k = \int_{t_{k-1}}^{t_k} e^{-rs}\, ds$ $(k \leq n + 1)$, $t_0 = 0$, and $t_{n+1} = s$.

Let $u \geq 0$, $x_k > y_k > 0$ for $k = 1, \ldots, n - 1$, $x_n > y_n$, and $x_n > 0$. Similar to Wu, Wang, and Zhang (2003), we consider

$$G_n = \mathrm{E}^u\ [e^{-\alpha T_{S,n}}\, I\ (R_{T_{\bar{S},1}} \in dx_1, R_{T_{S,1}} \in dy_1, \ldots, R_{T_{\bar{S},n}} \in dx_n, R_{T_{S,n}} \in dy_n)].$$

Using equation (3.5), we can rewrite G_n as

$$G_n = \prod_{k=1}^{n}\ (B_\alpha(y_{k-1}, x_k)\, f_S\ (x_k - y_k))\ dx_1dy_1\cdots dx_ndy_n, \tag{3.6}$$

for $n \geq 1$, where $y_0 = u$ and B_α is given in equation (3.5).

The proofs of equations (3.5) and (3.6) are given in the Appendix. The form of G_n in equation (3.6) is crucial to the derivation of Theorem 3.1, which provides an explicit expression for $\Phi_\alpha(u)$ and hence an exact solution to equation (2.5) or (2.6).

Theorem 3.1

For $u \geq 0$, we have

$$\begin{aligned} \Phi_\alpha(u) = {} & \sum_{n=1}^{\infty}\int_0^\infty dx_1\int_0^{x_1} dy_1\int_0^\infty dx_2\int_0^{x_2} dy_2\cdots\int_0^\infty dx_n\int_{-\infty}^0 w(x_n, -y_n) \\ & \times\prod_{k=1}^{n} B_\alpha(y_{k-1}, x_k)f_S(x_k - y_k)dy_n. \end{aligned} \tag{3.7}$$

PROOF

Note that ruin happens only at times $T_{S,n}$ for $n \geq 1$. Following the technique of Wu, Wang, and Zhang (2005), we can decompose $\{T_u < \infty\}$ into $\cup_{n=1}^{\infty}\{T_u = T_{S,n}\}$. It follows that

$$\begin{aligned} \Phi_\alpha(u) & = \sum_{n=1}^{\infty} \mathrm{E}^u\ [w(R_{T_{\bar{S},n}}, |R_{T_{S,n}}|)e^{-\alpha T_{S,n}}I(T_u = T_{S,n})] \\ & \triangleq \sum_{n=1}^{\infty} \phi_n^\alpha(u). \end{aligned} \tag{3.8}$$

Since $\{T_u = T_{S,n}\} = \{R_{T_{\bar{S},1}} > 0, R_{T_{S,1}} > 0, \ldots, R_{T_{\bar{S},n}} > 0, R_{T_{S,n}} < 0\}$, we have

$$\phi_n^\alpha(u) = E^u\ [w(R_{T_{\bar{S},n}}, |R_{T_{S,n}}|)e^{-\alpha T_{s,n}}I(R_{T_{\bar{S},1}} > 0, R_{T_{S,1}} > 0, \ldots, R_{T_{\bar{S},n}} > 0, R_{T_{S,n}} < 0)].$$

Then, using equation (3.6) together with the fact that $R_{T_{\bar{S},n}}$ may take any value in $(0, \infty)$, we get

$$\begin{aligned}\phi_n^\alpha(u) &= \int_0^\infty \int_0^{x_1} \int_0^\infty \int_0^{x_2} \cdots \int_0^\infty \int_{-\infty}^0 \mathrm{E}^u \,[w(R_{T_{\bar{S},n}}, |R_{T_{S,n}}|)e^{-\alpha T_{S,n}} \\ &\quad \times I(R_{T_{\bar{S},1}} \in dx_1, R_{T_{S,1}} \in dy_1, \ldots, R_{T_{\bar{S},n}} \in dx_n, R_{T_{S,n}} \in dy_n)] \\ &= \int_0^\infty \int_0^{x_1} \int_0^\infty \int_0^{x_2} \cdots \int_0^\infty \int_{-\infty}^0 w(x_n, -y_n)\mathrm{E}^u[e^{-\alpha T_{S,n}} I(R_{T_{\bar{S},1}} \in dx_1, \\ &\quad R_{T_{S,1}} \in dy_1, \ldots, R_{T_{\bar{S},n}} \in dx_n, R_{T_{S,n}} \in dy_n)] \\ &= \int_0^\infty dx_1 \int_0^{x_1} dy_1 \int_0^\infty dx_2 \int_0^{x_2} dy_2 \cdots \int_0^\infty dx_n \int_{-\infty}^0 w(x_n, -y_n) \\ &\quad \times \prod_{k=1}^n B_\alpha(y_{k-1}, x_k) f_S(x_k - y_k) dy_n. \qquad (3.9)\end{aligned}$$

Therefore, equation (3.7) holds. □

From equation (3.8), we see that the n-th term in the summation of equation (3.7) represents the expected discounted penalty generated from the n-th claim. In other words, it indicates the contribution of the n-th claim to $\Phi_\alpha(u)$. Thus, equation (3.7) states that $\Phi_\alpha(u)$ can be viewed as an aggregate expected discounted penalty comprising each individual claim's contribution. Moreover, it is easy to see that $\phi_n^\alpha(u)$ of equation (3.9) satisfies the recursive formula

$$\begin{aligned}\phi_n^\alpha(u) &= \int_0^\infty dx_1 \int_0^{x_1} B_\alpha(u, x_1) f_S(x_1 - y_1)\phi_{n-1}^\alpha(y_1) dy_1 \\ &= \int_0^\infty B_\alpha(u, x_1) f_S * \phi_{n-1}^\alpha(x_1) dx_1, \qquad (3.10)\end{aligned}$$

for $n \geq 2$ with

$$\phi_1^\alpha(u) = \int_0^\infty dx_1 \int_{-\infty}^0 B_\alpha(u, x_1) f_S(x_1 - y_1) w(x_1, -y_1) dy_1, \qquad (3.11)$$

where $f_S * \phi_{n-1}^\alpha$ stands for the convolution of f_S and ϕ_{n-1}^α. Hence, $\Phi_\alpha(u)$ of equation (3.7) is completely determined by equations (3.10) and (3.11).

In the case of constant interest (that is, $\lambda_I = 0$ in eq. [1.4]), we write the expected discounted penalty function as $\Phi_\alpha^0(u)$.

Corollary 3.1

For $u \geq 0$, we get

$$\begin{aligned}\Phi_\alpha^0(u) &= \sum_{n=1}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \int_{y_1}^\infty dx_2 \int_0^{x_2} dy_2 \cdots \int_{y_{n-1}}^\infty dx_n \int_{-\infty}^0 w(x_n, -y_n) \\ &\quad \times \prod_{k=1}^n \left(\lambda_S \frac{(ry_{k-1} + c)^{(\lambda_S+\alpha)r^{-1}}}{(rx_k + c)^{1+(\lambda_S+\alpha)r^{-1}}} f_S(x_k - y_k) \right) dy_n. \qquad (3.12)\end{aligned}$$

Similar to equations (3.10) and (3.11), one can derive a recursive formula associated with equation (3.12) for calculating $\Phi_\alpha^0(u)$.

REMARK 3.1

Equation (2.8) of Cai and Dickson (2002) provides another solution expression for $\Phi_\alpha^0(u)$ that is rather different from equation (3.12). However, their expression for $\Phi_\alpha^0(u)$ involves an unknown factor, namely, $\Phi_\alpha^0(0)$. As was mentioned in their paper, if $\Phi_\alpha^0(0)$ is available, then one can find the form of solution for $\Phi^0_\alpha(u)$ and can approximate $\Phi^0_\alpha(u)$ recursively. Cai and Dickson were able to find an exact solution for $\Phi^0_0(0)$, but $\Phi^0_\alpha(0)$ remains unsolved unfortunately. In this paper the result of Theorem 3.1 provides an exact solution for a more general problem, and $\Phi^0_\alpha(0)$ of equation (3.12) is simply a special case of equation (3.7). Suppose that F_S has a continuous density. Then the solution to the Volterra integral equation (2.7) of Cai and Dickson is unique. This means that $\Phi^0_\alpha(u)$ of equation (3.12) is equivalent to that of equation (2.8) of Cai and Dickson. □

Assume that $w(x_1, x_2) \equiv 1$. In this case, $\Phi_\alpha(u)$ is the Laplace transform of the time of ruin T_u, and equation (2.7) becomes $A(u) = 1 - F_S(u)$. From equations (2.6) and (2.8), we see that $\Phi_\alpha(u)$ is the solution to

$$\frac{d}{du}\Phi_\alpha(u) = \frac{\lambda_S + \lambda_I + \alpha}{ru + c}\Phi_\alpha(u) - \frac{\lambda_S}{ru + c}\int_0^u \Phi_\alpha(u - x)dF_S(x) - \frac{\lambda_I}{ru + c}\int_{-1}^\infty \Phi_\alpha((1 + x)u)dF_I(x) - \frac{\lambda_S}{ru + c}(1 - F_S(u)),$$

with $\lim_{u\to\infty}\Phi_\alpha(u) = 0$. By Theorem 3.1, we have the following corollary.

Corollary 3.2

Let $u \geq 0$. Assume that $w(x_1, x_2) \equiv 1$. Then we have the following results.

1. *The expected discounted penalty function takes the form*

$$\Phi_\alpha(u) = \sum_{n=1}^\infty \int_0^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_0^\infty dx_n \int_{-\infty}^0 \prod_{k=1}^n B_\alpha(y_{k-1}, x_k) f_S(x_k - y_k)dy_n.$$

2. *If $\alpha = 0$, then the explicit expression for the ruin probability $\Psi(u) = \Phi_0(u)$ of risk process (1.4) is*

$$\Psi(u) = \sum_{n=1}^\infty \int_0^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_0^\infty dx_n \int_{-\infty}^0 \prod_{k=1}^n B_0(y_{k-1}, x_k) f_S(x_k - y_k)dy_n.$$

3. *If $\alpha = 0$ and $\lambda_I = 0$, then the explicit expression for the ruin probability $\Psi(u) = \Phi^0_0(u)$ of risk process (1.5) is*

$$\Psi(u) = \sum_{n=1}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-1}}^\infty dx_n \int_{-\infty}^0 \prod_{k=1}^n \left(\lambda_S \frac{(ry_{k-1} + c)^{\lambda_S r^{-1}}}{(rx_k + c)^{1+\lambda_S r^{-1}}} f_S(x_k - y_k)\right) dy_n.$$

Let $w(x_1, x_2) = I(x_1 \leq x)I(x_2 \leq y)$ and $\alpha = 0$. Then

$$H(u, x, y) \triangleq \Phi_0(u) = P(R_{T_u^-} \leq x, |R_{T_u}| \leq y, T_u < \infty | R_0 = u)$$

is the joint distribution of the surplus immediately before ruin and the deficit at ruin when ruin occurs. The function $A(u)$ of equation (2.7) becomes

$$A(u) = \int_u^\infty w(u, x - u)dF_S(x) = I(u \leq x)(F_S(u + y) - F_S(u)).$$

Again, equations (2.6) and (2.8) imply that $H(u, x, y)$ is the solution to the following problem:

$$\frac{d}{du}H(u, x, y) = \frac{\lambda_S + \lambda_I}{ru + c}H(u, x, y) - \frac{\lambda_S}{ru + c}\int_0^u H(u - z, x, y)dF_S(z) - \frac{\lambda_I}{ru + c}\int_{-1}^\infty H((1 + z)u, x, y)\, dF_I(z) - \frac{\lambda_S}{ru + c} I(u \leq x)(F_S(u + y) - F_S(u)),$$

with $\lim_{u\to\infty} H(u, x, y) = 0$. Applying Theorem 3.1 yields the following corollary.

Corollary 3.3

Let $u \geq 0$ and $\alpha = 0$. Then

$$H(u, x, y) = \sum_{n=1}^{\infty} \int_0^{\infty} dx_1 \int_0^{x_1} dy_1 \cdots \int_0^{x} dx_n \int_{-y}^{0} \prod_{k=1}^{n} B_0(y_{k-1}, x_k) f_S(x_k - y_k) dy_n.$$

It is clear that the exact expressions for the distribution of the deficit at ruin and that of the surplus immediately prior to ruin are given by $H(u, \infty, y)$ and $H(u, x, \infty)$, respectively. For details of these distributions, see, for example, Gerber, Goovaerts, and Kaas (1987), Dickson (1992), Wang and Wu (2001), and Yang and Zhang (2001).

4. EXAMPLES

In this section we present two examples in which closed-form expressions for $\Phi_\alpha^0(u)$ are obtained by solving equations (2.6) and (2.8) under certain assumptions.

Example 4.1

Let $u \geq 0$, $\alpha = \lambda_I = 0$, and $F_S(x) = 1 - e^{-\beta x}$ with $\beta > 0$. Assume that $A'(u)$ is continuous on $[0, \infty)$. Then, from equations (2.6) and (2.8), we conclude that $\Phi_0^0(u)$ is the solution to the following problem:

$$\Phi_0^{0\prime\prime}(u) + f(u)\Phi_0^{0\prime}(u) = g(u), \tag{4.1}$$

with

$$\lim_{u\to\infty} \Phi_0^0(u) = 0 \quad \text{and} \quad c\Phi_0^{0\prime}(0) = \lambda_S\Phi_0^0(0) - \lambda_S A(0), \tag{4.2}$$

where

$$f(u) = \frac{\beta(ru + c) + r - \lambda_S}{ru + c} \quad \text{and} \quad g(u) = -\frac{\lambda_S\beta A(u) - \lambda_S A'(u)}{ru + c}.$$

Solving equations (4.1) and (4.2), we obtain

$$\Phi_0^0(u) = -\int_u^{\infty} \frac{1}{b_1(x)} \left(b_2(x) + \frac{\lambda_S}{c} (\Phi_0^0(0) - A(0)) \right) dx, \tag{4.3}$$

with

$$\Phi_0^0(0) = \frac{\lambda_S A(0)c^{-1} \int_0^{\infty} b_1^{-1}(x)dx - \int_0^{\infty} b_2(x)b_1^{-1}(x)dx}{1 + \lambda_S c^{-1} \int_0^{\infty} b_1^{-1}(x)dx}, \tag{4.4}$$

where $b_1(x) = e^{\int_0^x f(p)dp}$ and $b_2(x) = \int_0^x b_1(z)g(z)dz$.

Let $w(x_1, x_2) = e^{-\delta x_2}$ with $\delta > 0$. Then we have $A(0) = \beta(\delta + \beta)^{-1}$ and $g(u) \equiv 0$. From equations (4.3) and (4.4), one can show that the exact expression for the Laplace transform of the deficit at ruin has the form

$$\begin{aligned}\Phi(u) &= \mathrm{E}^u[e^{-\delta|R_{T_u}|} I(T_u < \infty)] \\ &= \frac{\lambda_S}{c} \left(\frac{\beta}{\delta + \beta} - \frac{\lambda_S\beta}{c(\delta + \beta)} \frac{\int_0^{\infty} b_1^{-1}(x)dx}{1 + \lambda_S c^{-1} \int_0^{\infty} b_1^{-1}(x)dx} \right) \int_u^{\infty} \frac{1}{b_1(x)} dx.\end{aligned}$$

Example 4.2

Let $u \geq 0$, $\lambda_I = 0$, $\alpha > 0$, $w(x_1, x_2) = e^{-\delta x_2}$ and $F_S(x) = 1 - e^{-\beta x}$ with $\delta > 0$ and $\beta > 0$. Again, from equations (2.6) and (2.8), we conclude that $\Phi_\alpha^0(u)$ is the solution to the following problem:

$$(ru + c)\Phi_{\alpha}^{0\prime\prime}(u) + (\beta(ru + c) + r - \lambda_S - \alpha)\Phi_{\alpha}^{0\prime}(u) - \alpha\beta\Phi_{\alpha}^{0}(u) = 0, \tag{4.5}$$

with

$$\lim_{u\to\infty} \Phi_{\alpha}^{0}(u) = 0 \quad \text{and} \quad c\Phi_{\alpha}^{0\prime}(0) = (\lambda_S + \alpha)\Phi_{\alpha}^{0}(0) - \lambda_S A(0), \tag{4.6}$$

where $A(0) = \beta(\delta + \beta)^{-1}$.

Put

$$\xi_1(u) = e^{-\beta u}\left(u + \frac{c}{r}\right)^{(\alpha+\lambda_S)/r} U\left(1 + \frac{\alpha}{r}, 1 + \frac{\alpha + \lambda_S}{r}, \beta\left(u + \frac{c}{r}\right)\right), \tag{4.7}$$

$$\xi_2(u) = e^{-\beta u}\left(u + \frac{c}{r}\right)^{(\alpha+\lambda_S)/r} F\left(1 + \frac{\alpha}{r}, 1 + \frac{\alpha + \lambda_S}{r}, \beta\left(u + \frac{c}{r}\right)\right), \tag{4.8}$$

where

$$F(a, b, u) = \frac{\Gamma(b)}{\Gamma(b - a)\Gamma(a)} \int_0^1 e^{ut} t^{a-1}(1 - t)^{b-a-1} dt, \quad b > a > 0, u \geq 0,$$

is the standard confluent hypergeometric function, and its second form is

$$U(a, b, u) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-ut} t^{a-1}(1 + t)^{b-a-1} dt, \quad a > 0, u \geq 0.$$

For details of confluent hypergeometric functions, see Slater (1960, p. 5). From Example 2.1 of Paulsen and Gjessing (1997), we see that the general solution of equation (4.5) is of the form

$$\Phi_{\alpha}^{0}(u) = c_1\xi_1(u) + c_2\xi_2(u). \tag{4.9}$$

Using equations (10.01) and (10.07) of Olver (1974, pp. 256–57), one can show that $\lim_{u\to\infty} \xi_1(u) = 0$ and $\lim_{u\to\infty} \xi_2(u) = \infty$. It follows from the boundary condition (4.6) that

$$c_2 = 0 \tag{4.10}$$

and

$$c_1 = \frac{\lambda_S}{c(\delta + \beta)}\left(\frac{r}{c}\right)^{(\alpha+\lambda_S)r^{-1}}\left(U\left(1 + \frac{\alpha}{r}, 1 + \frac{\alpha + \lambda_S}{r}, \frac{c\beta}{r}\right) + \frac{\Gamma(2 + \alpha r^{-1})}{\Gamma(1 + \alpha r^{-1})} U\left(2 + \frac{\alpha}{r}, 2 + \frac{\alpha + \lambda_S}{r}, \frac{c\beta}{r}\right)\right)^{-1}. \tag{4.11}$$

Finally, equations (4.7)–(4.11) give the exact expression for $E^u\,[e^{-\alpha T_u} e^{-\delta|R_{T_u}|} I(T_u < \infty)]$.

5. CONCLUDING REMARKS

Yuen, Wang, and Ng (2004) studied a more complex risk process in which the stochastic return follows a compound Poisson process plus a Brownian motion with positive drift. They extended the result of Kalashnikov and Norberg (2002) to the risk process and came up with a lower bound for the ruin probability. With assumptions slightly different from those in Yuen, Wang, and Ng, one can also derive a similar lower bound for the ruin probability of risk process (1.4).

Recall ξ of equation (3.2) and η of equation (3.3). To establish the main result of this section, we need to assume that $0 < F_S(z) < 1$ and $F_I(z - 1) > 0$ for any $z > 0$. Then it can be shown that

$$P(\xi < 1) > 0 \quad \text{and} \quad P(\eta \leq u\xi) > 0 \quad \text{for all } -\infty < u < \infty. \tag{5.1}$$

The two inequalities in expression (5.1) imply that interest may be negative and the loss from the investments may take place. Hence, risk process (1.4) involves not only the traditional liability risk related to the insurance portfolio (insurance risk), but also the asset risk related to the investment portfolio (financial risk). Also, expression (5.1) implies that there exist $0 < \bar{\xi} < 1$ and $\bar{\eta} > 0$ such that

$$\Gamma_0 = P(\xi \le \bar{\xi}, \eta \le \bar{\eta}) > 0 \quad \text{and} \quad \Gamma_1 = P\left(\eta \le -\frac{2\bar{\eta}\xi}{1-\bar{\xi}}\right) > 0.$$

The following theorem shows that the lower bound for the ruin probability $\Psi(u)$ can be expressed in terms of these probabilities.

Theorem 5.1

For $u \ge 0$, we have

$$\Psi(u) \ge \begin{cases} Au^{-B}, & u > C_0, \\ \Gamma_1, & 0 \le u \le C_0, \end{cases} \tag{5.2}$$

where

$$C_0 = \frac{\bar{\eta}}{1-\bar{\xi}}, \qquad B = \frac{\ln \Gamma_0}{\ln \bar{\xi}}, \qquad \textit{and} \qquad A = \Gamma_0 \Gamma_1 \left(\frac{\bar{\eta}}{1-\bar{\xi}}\right)^B. \tag{5.3}$$

The proof of expression (5.1) is similar to that of expression (3.5) of Yuen, Wang, and Ng (2004). Theorem 5.1 is analogous to Theorem 3.1 of Yuen, Wang, and Ng and Theorem 1 of Kalashnikov and Norberg (2002). It should be pointed out that the parameters in expression (5.2) are very difficult to compute.

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APPENDIX

In this Appendix we prove that equations (3.5) and (3.6) hold true.

PROOF OF EQUATION (3.5)

By the law of total probability, we obtain

$$\begin{aligned} &\mathrm{E}\left[e^{-\alpha\tau_S} I\left(U_{\tau_S}\left(u + c\int_0^{\tau_S} \frac{1}{U_t}\,dt\right) \in dx\right)\right] \\ &= \sum_{n=0}^{\infty} \mathrm{E}\left[e^{-\alpha\tau_S} I\left(U_{\tau_S}\left(u + c\int_0^{\tau_S} \frac{1}{U_t}\,dt\right) \in dx, N_{I,\tau_S} = n\right)\right] \\ &\triangleq \sum_{n=0}^{\infty} Q_n. \end{aligned} \tag{A.1}$$

It can be shown that

$$Q_0 = \mathrm{E}\left[e^{-\alpha\tau_S} I\left(U_{\tau_S}\left(u + c\int_0^{\tau_S} \frac{1}{U_t}\,dt\right) \in dx,\, N_{I,\tau_S} = 0\right)\right]$$
$$= \frac{(ru + c)^{(\lambda_S+\lambda_I+\alpha)r^{-1}}}{(rx + c)^{1+(\lambda_S+\lambda_I+\alpha)r^{-1}}}\,\lambda_S I_{(u,\infty)}(x)dx. \tag{A.2}$$

Moreover, from the independence assumption, we get

$$Q_n = \int_0^\infty \lambda_S e^{-(\lambda_S+\alpha)s}\,\frac{(\lambda_I s)^n}{n!}\,e^{-\lambda_I s}\,P\left(U_s\left(u + c\int_0^s \frac{1}{U_t}\,dt\right) \in dx \mid N_{I,s} = n\right) ds$$
$$\triangleq \int_0^\infty \lambda_S e^{-(\lambda_S+\alpha+\lambda_I)s}\,\frac{(\lambda_I s)^n}{n!}\,J_n ds, \tag{A.3}$$

for $n \geq 1$.

Given $N_{I,s} = n \geq 1$, we have (1) the joint distribution of $(T_{I,1}, \ldots, T_{I,n})$ is the same as that of the order statistics of n uniformly distributed random variables on $(0, s]$; (2) $\Pi_{i=1}^{N_{I,t}} X_i = \Pi_{i=1}^{k-1} X_i$ for $t \in [T_{I,k-1}, T_{I,k})$ and $k = 1, \ldots, n + 1$; and (3) $\int_0^s = \Sigma_{k=1}^n \int_{T_{I,k-1}}^{T_{I,k}} + \int_{T_{I,k}}^s$. Then (1)–(3) imply that

$$J_n = P\left(e^{rs}\prod_{i=1}^n X_i\left(u + c\sum_{k=1}^n \int_{T_{I,k-1}}^{T_{I,k}} e^{-rt}\prod_{i=1}^{k-1}\frac{1}{X_i}\,dt\right.\right.$$
$$\left.\left. + c\int_{T_{I,n}}^s e^{-rt}\prod_{i=1}^n \frac{1}{X_i}\,dt\right) \in dx \mid N_{I,s} = n\right)$$
$$= \int_0^s dt_1 \int_{t_1}^s dt_2 \cdots \int_{t_{n-1}}^s \frac{n!}{s^n}\,P\left((u + cb_1)\prod_{i=1}^n X_i + cb_2\prod_{i=2}^n X_i\right.$$
$$\left. + \cdots + cb_n X_n + cb_{n+1} \in e^{-rs}dx\right)dt_n. \tag{A.4}$$

Note that if $xe^{-rs} < cb_{n+1}$, that is, $t_n < s - r^{-1}\ln(1 + (rc^{-1})x)$, then the probability in the integrand on the right-hand side of equation (A.4) equals zero. Since $(X_1, \ldots, X_n)$ and $(X_n, \ldots, X_1)$ have the same distribution, we obtain from equations (3.4) and (A.4) that

$$J_n = dx\int_0^s dt_1 \int_{t_1}^s dt_2 \cdots \int_{t_{n-1}\vee(s-r^{-1}\ln(1+(rc^{-1})x))}^s \frac{n!}{s^n}\,e^{-rs}$$
$$\times\, h_n(cb_{n+1}, cb_n, \ldots, cb_2, c(b_1 + u), xe^{-rs})dt_n. \tag{A.5}$$

Hence, equations (A.1)–(A.5) yield equation (3.5). This completes the proof. □

To prove equation (3.6), we need the following lemma.

Lemma A.1

Let $\eta_n^0 = \eta_n + Y_{S,n}$. Then, (ξ_{n+1}, η_{n+1}) and $(\xi_{n+1}, \eta_{n+1}^0)$ are independent of L_k and $R_{T_{\bar{S},k}}$ for $k = 1, \ldots, n$.

PROOF

The lemma can be easily proved by induction and equation (3.1). □

PROOF OF EQUATION (3.6)

Rewrite G_n as

$$G_n = \mathrm{E}^u \,[\mathrm{E}^u \,[e^{-\alpha T_{S,n-1}} I(R_{T_{\bar{S},1}} \in dx_1, \ldots, R_{T_{S,n-1}} \in dy_{n-1})$$
$$\times\, e^{-\alpha(T_{S,n}-T_{S,n-1})} I(R_{T_{\bar{S},n}} \in dx_n, R_{T_{S,n}} \in dy_n) | T_{S,n-1}, R_{T_{\bar{S},1}}, \ldots, R_{T_{S,n-1}}]]$$
$$= \int_0^\infty e^{-\alpha s}\, P(T_{S,n-1} \in ds, R_{T_{\bar{S},1}} \in dx_1, \ldots, R_{T_{S,n-1}} \in dy_{n-1} | R_0 = u)$$
$$\times\, \mathrm{E}^u \,[e^{-\alpha(T_{S,n}-T_{S,n-1})} I(\xi_n y_{n-1} + \eta_n^0 \in dx_n, Y_{S,n} \in x_n - dy_n) | T_{S,n-1} = s,$$
$$R_{T_{\bar{S},1}} = x_1, \ldots, R_{T_{S,n-1}} = y_{n-1}].$$

Lemma A.1 together with the independence assumption imply that the expectation in the integrand on the right-hand side of the above equality equals

$$\mathrm{E}\,[e^{-\alpha(T_{S,n}-T_{S,n-1})} I(\xi_n y_{n-1} + \eta_n^0 \in dx_n, Y_{S,n} \in x_n - dy_n)]$$
$$= \mathrm{E}^{y_{n-1}}\,[e^{-\alpha\tau_S} I(R_{\tau_{\bar{S}}} \in dx_n)\, I(Y_S \in x_n - dy_n)].$$

It follows that

$$G_n = \int_0^\infty e^{-\alpha s}\, P(T_{S,n-1} \in ds, R_{T_{\bar{S},1}} \in dx_1, \ldots, R_{T_{S,n-1}} \in dy_{n-1} | R_0 = u)$$
$$\times\, \mathrm{E}^{y_{n-1}}\,[e^{-\alpha\tau_S} I(R_{\tau_{\bar{S}}} \in dx_n)\, I(Y_S \in x_n - dy_n)]$$
$$= \mathrm{E}^u\,[e^{-\alpha T_{S,n-1}} I(R_{T_{\bar{S},1}} \in dx_1, \ldots, R_{T_{S,n-1}} \in dy_{n-1})]$$
$$\times\, \mathrm{E}^{y_{n-1}}\,[e^{-\alpha\tau_S} I(R_{\tau_{\bar{S}}} \in dx_n) I(Y_S \in x_n - dy_n)].$$

Then, by induction, we get

$$G_n = \prod_{k=1}^n E^{y_{k-1}}\,[e^{-\alpha\tau_S} I(R_{\tau_{\bar{S}}} \in dx_k)\, I(Y_S \in x_k - dy_k)]. \tag{A.6}$$

By independence and equations (A.6) and (3.5), we obtain

$$G_n = \prod_{k=1}^n (B_\alpha(y_{k-1}, x_k) f_S(x_k - y_k) dx_k dy_k).$$

This ends the proof. □

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