"THE TIME VALUE OF RUIN IN A SPARRE ANDERSEN MODEL," HANS U. GERBER AND ELIAS S. W. SHIU, APRIL 2005

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I wish to congratulate Professors Gerber and Shiu for this interesting paper, which studies the ordinary Sparre Andersen model when the interclaim times are independent and distributed as in (3.1). Here I want to illustrate how the results can be extended to the stationary renewal risk model. First we consider a special Sparre Andersen model with the following structure on the interclaim times:

(1) the interclaim time random variables V_i's are independent;
(2) $V_2, V_3, V_4, \ldots$ have a common distribution F_V and the first arrival time V_1 follows distribution G.

The expected discounted penalty function in this special model is

$$\phi(u; G) = E[e^{-\delta T} w(U(T-), |U(T)|) 1_{(T<\infty)} | \, U(0) = u]$$

with $u \geq 0$ and $\delta > 0$. For the ordinary Sparre Andersen model considered in the paper, V_1 has distribution F_V and hence $\phi(u; F_V) = \phi(u)$. By conditioning on the arrival time and the amount of the first claim, we have

$$\phi(u; G) = \int_0^\infty e^{-\delta t} E[\phi(u + ct - X)] dG(t), \tag{1}$$

where, for $z \geq 0$, $E[\phi(z - X)]$ is interpreted as $\int_0^z \phi(z - y)p(y)dy + \int_z^\infty w(z, y - z)p(y)dy$. If G has density g, then (1) is equivalent to

$$\phi(u; G) = \int_0^\infty g(t) e^{-\delta t} E[\phi(u + ct - X)] dt. \tag{2}$$

Equations (1) and (2) give the relations between the expected discounted penalty functions in the ordinary and the special Sparre Andersen model. Equation (1.5) of the paper can be obtained from (2) by letting

$$g(t) = \frac{f_V(t + \tau)}{1 - F_V(\tau)}.$$

The first equation in the discussion by Dr. Li can be obtained by letting g be the density of V. By letting $G(t) = 1\{t \geq s\}$, one can obtain the expected discounted penalty function in the s-delayed Sparre Andersen model in Ng and Yang (2005). In this case, (1) simplifies as

$$\phi(u; G) = e^{-\delta s} E[\phi(u + cs - X)].$$

Assuming that the density of G exists and using the substitution $z = u + ct$, equation (2) can be rewritten as

$$\phi(u; G) = \frac{1}{c} \int_u^\infty g\left(\frac{z - u}{c}\right) e^{-\delta(z-u)/c} E[\phi(z - X)] dz.$$

Differentiating both sides with respect to u, we get

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$$\begin{aligned}\frac{d}{du}\,\phi(u; G) &= \frac{1}{c}\int_u^\infty \left(-\frac{1}{c}\right)g'\left(\frac{z-u}{c}\right)e^{-\delta(z-u)/c}E[\phi(z-X)]dz \\ &\quad + \frac{1}{c}\int_u^\infty \frac{\delta}{c}\,g\left(\frac{z-u}{c}\right)e^{-\delta(z-u)/c}E[\phi(z-X)]dz - \frac{1}{c}\,g(0)E[\phi(u-X)] \\ &= \frac{1}{c}\int_u^\infty \left(-\frac{1}{c}\right)g'\left(\frac{z-u}{c}\right)e^{-\delta(z-u)/c}E[\phi(z-X)]dz + \frac{\delta}{c}\,\phi(u; G) - \frac{1}{c}\,g(0)E[\phi(u-X)]. \qquad (3)\end{aligned}$$

Now we discuss the stationary Sparre Andersen model. A Sparre Anderson model is stationary if

(1) the interclaim time random variables V_i's are independent;
(2) $V_2, V_3, V_4, \ldots$ have a common distribution F_V and the first arrival time V_1 follows the *equilibrium* distribution of V, that is, the density of V_1 is

$$f_{V_1}(x) = \frac{1 - F_V(x)}{E[V]}.$$

Thus the stationary Sparre Andersen model is a particular case of the special Sparre Andersen model discussed earlier. To simplify notation, we let the expected discounted penalty function in the stationary model be ϕ^s. By the definition of the equilibrium distribution,

$$f'_{V_1}(0) = \frac{1 - F_V(0)}{E[V]} = \frac{1}{E[V]} \quad \text{and} \quad f_{V_1}(x) = -\frac{f_V(x)}{E[V]}.$$

Substituting these formulas into (3) and applying (1), we obtain

$$\begin{aligned}\frac{d}{du}\,\phi^s(u) &= \frac{1}{cE[V]}\int_u^\infty \frac{1}{c}f_V\left(\frac{z-u}{c}\right)e^{-\delta(z-u)/c}E[\phi(z-X)]dz + \frac{\delta}{c}\,\phi^s(u) - \frac{1}{cE[V]}\,E[\phi(u-X)] \\ &= \frac{1}{cE[V]}\,\phi(u) + \frac{\delta}{c}\,\phi^s(u) - \frac{1}{cE[V]}\,E[\phi(u-X)] \\ &= \frac{1}{cE[V]}\,\phi(u) + \frac{\delta}{c}\,\phi^s(u) - \frac{1}{cE[V]}\left[\int_0^u \phi(u-y)p(y)dy + \int_u^\infty w(u, y-u)p(y)dy\right]. \qquad (4)\end{aligned}$$

Taking Laplace transforms of both sides of (4) and using (5.10) of the paper, we have

$$\xi\widehat{\phi^s}(\xi) - \phi^s(0) = \frac{1}{cE[V]}\,\hat{\phi}(\xi) + \frac{\delta}{c}\,\widehat{\phi^s}(\xi) - \frac{1}{cE[V]}\,[\hat{\phi}(\xi)\hat{p}(\xi) + \hat{\omega}(\xi)],$$

or

$$\widehat{\phi^s}(\xi) = \frac{[\hat{\phi}(\xi)\hat{p}(\xi) + \hat{\omega}(\xi) - \hat{\phi}(\xi)] - cE[V]\phi^s(0)}{(\delta - c\xi)E[V]}. \qquad (5)$$

By the same argument to obtain (7.3) of the paper, we obtain

$$\phi^s(0) = \frac{1}{cE[V]}\left[\hat{\phi}\left(\frac{\delta}{c}\right)\hat{p}\left(\frac{\delta}{c}\right) + \hat{\omega}\left(\frac{\delta}{c}\right) - \hat{\phi}\left(\frac{\delta}{c}\right)\right], \qquad (6)$$

which is an explicit formula in terms of the Laplace transform of the expected discounted penalty function of the ordinary Sparre Andersen model.

By virtue of (5) and (6), a result in the stationary Sparre Andersen model can be obtained if the corresponding result in the ordinary Sparre Andersen model is available. In view of (7.2) and (7.3) of the paper, we can express $\phi^s(0)$ in terms of γ and the Laplace transform of p and w when the interclaim times $V_2, V_3, V_4, \ldots$ follow the distribution described in (3.1) of the paper. For example, consider

$f^s(x, y|0)$, the discounted joint probability density of surplus just before ruin and the deficit at ruin when initial surplus is 0. We have

$$\phi^s(0) = \int_0^\infty \int_0^\infty w(x, y) f^s(x, y|0)\,dx\,dy,$$

which is analogous to (2.3). By putting (7.2) in the right hand side of (6), we have

$$\begin{aligned} f^s(x, y|0) &= \frac{p(x+y)}{cE[V]} \left[\frac{\hat{p}\left(\frac{\delta}{c}\right) - 1}{\gamma\left(\frac{\delta}{c}\right) - \hat{p}\left(\frac{\delta}{c}\right)} \left(e^{-\delta x/c} - \sum_{j=1}^{n} e^{-\rho_j x} \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\frac{\delta}{c} - \rho_k}{\rho_j - \rho_k} \right) + e^{-\delta x/c} \right] \\ &= \frac{p(x+y)}{cE[V]} \left[\frac{\hat{p}\left(\frac{\delta}{c}\right) - 1}{1 - \hat{p}\left(\frac{\delta}{c}\right)} \left(e^{-\delta x/c} - \sum_{j=1}^{n} e^{-\rho_j x} \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\frac{\delta}{c} - \rho_k}{\rho_j - \rho_k} \right) + e^{-\delta x/c} \right] \\ &= \frac{p(x+y)}{cE[V]} \sum_{j=1}^{n} e^{-\rho_j x} \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\frac{\delta}{c} - \rho_k}{\rho_j - \rho_k}, \end{aligned} \tag{7}$$

where the second equality follows from $\gamma(\xi) = f_V(\delta - c\xi)^{-1}$. Equation (7) gives the corresponding formula for (8.3) of the paper.

Equation (7) holds for $\delta > 0$. The case when $\delta = 0$ follows directly from the theory of general stationary insurance risk model (see, for example, Asmussen and Schmidt 1995). In this case, (7) reduces to the joint density of surplus just before ruin and the deficit at ruin. It is well-known that

$$f^s(x, y|0) = \frac{p(x+y)}{cE[V]}.$$

Integrating with respect to both x and y from 0 to infinity, we get the ruin probability starting with zero initial surplus,

$$\psi^s(0) = \frac{p_1}{cE[V]} = \frac{1}{1+\theta},$$

where p_1 is the mean of the claim sizes and $\theta > 0$ is the security loading. Readers familiar with the Society of Actuaries textbook, *Actuarial Mathematics*, by Bowers et al. (1997) would recognize that Theorem 13.5.1 and the discussions therein are special cases of the two formulas above. The classical compound Poisson risk model is in fact stationary because the equilibrium distribution of exponential distribution is itself.

Finally, we illustrate how (5) and (6) can be combined with the discussion by Professor Schmidli. We use the notation of phase-type distribution as in his discussion. If the interclaim time random variables $V_2, V_3, V_4, \ldots$ follow phase-type distribution $PH(\boldsymbol{\alpha}, \mathbf{B})$, then

$$f_{V_1}(x) = -\frac{\boldsymbol{\alpha}\mathbf{B}^{-1}}{E[V]} e^{x\mathbf{B}}\mathbf{b}$$

where $\mathbf{b} = -\mathbf{Be}$ and $\mathbf{e} = (1, 1, \ldots, 1)'$ is a column vector of 1's. Hence V_1 follows $PH(\boldsymbol{\alpha}^*, \mathbf{B})$ where $\boldsymbol{\alpha}^* = -\boldsymbol{\alpha}\mathbf{B}^{-1}/E[V]$. Note that the initial distribution is changed from $\boldsymbol{\alpha}$ to $-\boldsymbol{\alpha}\mathbf{B}^{-1}/E[V]$, but the generator is unchanged.

Let ϕ_j be defined as in the discussion of Professor Schmidli. By conditioning,

$$\phi(u) = \sum_{j=0}^{n-1} \alpha_{j+1}\phi_j(u) \quad \text{and} \quad \phi^s(u) = \sum_{j=0}^{n-1} \alpha^*_{j+1}\phi_j(u), \tag{8}$$

where α_i^* is the ith component of $\boldsymbol{\alpha}^*$. Hence,

$$\hat{\phi}(u) = \sum_{j=0}^{n-1} \alpha_{j+1}\hat{\phi}_j(u) \quad \text{and} \quad \widehat{\phi^s}(u) = \sum_{j=0}^{n-1} \alpha^*_{j+1}\hat{\phi}_j(u). \tag{9}$$

As noted in the discussion by Professor Schmidli, $\hat{\phi}_j$'s can be obtained analytically. On combining the first part of (9) with (6), we have an analytic expression of $\phi^s(0)$ for any p and w. In particular, (9) gives two methods to obtain the Laplace transform of ϕ^s. The first method is to use the first part of (9) to obtain the Laplace transform of ϕ and then substitute the result into (5). The second method is to obtain $\boldsymbol{\alpha}^*$ and use the second part of (9). For a simple example of the second method, let V_2, V_3, V_4, . . . be distributed as in (3.1). It can be shown that V_1 follows $PH(\boldsymbol{\alpha}^*, \mathbf{B})$ where $\boldsymbol{\alpha}^*$ is a 1 by n row vector with elements

$$\alpha_i^* = \frac{1/\lambda_i}{\sum_{j=1}^{n} 1/\lambda_j}. \tag{10}$$

Thus, $\widehat{\phi^s}$ can be obtained by substituting (10) into the second part of (9).

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The ruin problem for a renewal risk model is often difficult to solve; Professors Gerber and Shiu have elegantly extended the results for the classical model in Gerber and Shiu (1998) to a Sparre Andersen risk model with generalized Erlang interclaim times. The purpose of this discussion is to present explicit formulas for $\phi(u)$ and $f(x, y|u)$ for the special case in which $\hat{p}(\xi)$ is a rational function.

Gerber and Shiu express $\hat{\phi}$ by (eq. 7.2)

$$\hat{\phi}(\xi) = \frac{\hat{\omega}(\xi) - q(\xi)}{\gamma(\xi) - \hat{p}(\xi)}. \tag{1}$$

It is shown by Gerber and Shiu in Section 4 that, in the right half of the complex plane, the function in the denominator of (1) has n zeros $\rho_1, \rho_2, \ldots, \rho_n$. For simplicity, we assume that they are distinct. By (9.2) and (11.1), we can write

$$\hat{\phi}(\xi) = \frac{\widehat{S\omega}(\xi)}{1 - \widehat{Sp}(\xi)},$$

where the operator S is defined by

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$$S = \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{j=1}^{n} T_{\rho_j}.$$

Moreover, by (9.10), $(Sp)(y)$ can be written as a linear function of $T_{\rho_j}p(y)$'s:

$$(Sp)(y) = g(y) = \frac{\Pi_{i=1}^n \lambda_i}{c^n} \left[\sum_{j=1}^{n} \left(\prod_{k=1,k\neq j}^{n} \frac{1}{\rho_k - \rho_j} \right) T_{\rho_j} p \right] (y).$$

It follows that the denominator of (1) is a rational function if and only if $\hat{p}(\xi)$ is a rational function. Let $\theta_1, \ldots, \theta_k$ be the distinct roots with negative real parts of the equation $\hat{g}(\xi) = 1$ and $n_1, \ldots, n_k$ be their multiplicities, respectively. From equation (11.4) we can see that the θ_k's are exactly the zeros with negative real parts of equation (4.2). By the principle of partial fractions we may write

$$\frac{1}{1 - \hat{g}(\xi)} = P_1 \left(\frac{1}{\xi - \theta_1} \right) + \cdots + P_k \left(\frac{1}{\xi - \theta_k} \right) + 1,$$

where

$$P_j(\alpha) = c_{j,n_j}\alpha^{n_j} + c_{j,n_j-1}\alpha^{n_j-1} + \cdots + c_{j,1}\alpha, \; j = 1, 2, \ldots, k.$$

Denote by

$$Q_j(u) = c_{j,n_j} \frac{1}{(n_j - 1)!} u^{n_j-1} e^{\theta_j(\delta)u} + c_{j,n_j-1} \frac{1}{(n_j - 2)!} u^{n_j-2} e^{\theta_j(\delta)u} + \cdots + c_{j,1} e^{\theta_j(\delta)u},$$

we have

$$\hat{Q}_j(\xi) = P_j \left(\frac{1}{\xi - \theta_j} \right), \; j = 1, 2, \cdots, k.$$

Hence (1) can be written as

$$\hat{\phi}(\xi) = \sum_{j=1}^{k} (\hat{Q}_j(\xi) \cdot \widehat{S\omega}(\xi)) + \widehat{S\omega}(\xi).$$

Inverting the Laplace transform yields

$$\phi(u) = \sum_{j=1}^{k} \int_0^u Q_j(u - x) \cdot (S\omega)(x) dx + (S\omega)(u). \tag{2}$$

In particular, when $n_1 = \cdots = n_k = 1$, we have

$$\phi(u) = \sum_{j=1}^{k} A_j \int_0^u e^{\theta_j(u-x)} \cdot (S\omega)(x) dx + (S\omega)(u), \tag{3}$$

where

$$A_j = -\frac{1}{\hat{g}'(\theta_j)}, \; j = 1, 2, \ldots, k.$$

Moreover, the discounted joint probability density function $f(x, y|u)$ defined by (2.1) can be obtained from (2) or (3). For simplicity, we assume that all roots with negative real parts of $\hat{g}(\xi) = 1$ are simple, that is, $n_1 = \cdots = n_k = 1$. It follows from (9.11) that

$$(S\omega)(u) = \int_0^\infty \int_0^\infty w(x, y) 1(x \geq u) f(x - u, y + u|0) dx dy,$$

from which we obtain

$$\int_0^u e^{\theta_j(u-z)}(S\omega)(z)dz = \int_0^\infty \int_0^\infty w(x, y)\left(\int_0^{u\wedge x} e^{\theta_j(u-z)}f(x - z, y + z|0)dz\right) dxdy.$$

Consequently,

$$\phi(u) = \int_0^\infty \int_0^\infty w(x, y) \sum_{j=1}^k A_j \left(\int_0^{u\wedge x} e^{\theta_j(u-z)}f(x - z, y + z|0)dz\right) dxdy$$
$$+ \int_0^\infty \int_0^\infty w(x, y)1(x \geq u)f(x - u, y + u|0)dxdy. \tag{4}$$

Recall that $\phi(u) = \int_0^\infty \int_0^\infty w(x, y)f(x, y|u)dxdy$; by comparing this with (4) we find that

$$f(x, y|u) = \sum_{j=1}^k A_j \left(\int_0^{u\wedge x} e^{\theta_j(u-z)}f(x - z, y + z|0)dz\right) + 1(x \geq u)f(x - u, y + u|0)$$
$$= \frac{\Pi_{l=1}^n \lambda_l}{c^n} p(x + y) \sum_{j=1}^k A_j \left[e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{(1 - e^{-(u\wedge x)(\theta_j-\rho_i)})}{\theta_j - \rho_i} \prod_{k=1,k\neq i}^n \frac{1}{\rho_k - \rho_i}\right]$$
$$+ 1(x \geq u)f(x - u, y + u|0),$$

with (see eq. 8.3)

$$f(x, y|0) = \frac{\Pi_{l=1}^n \lambda_l}{c^n} p(x + y) \sum_{j=1}^n \left[e^{-\rho_j x} \prod_{k=1,k\neq j}^n \frac{1}{\rho_k - \rho_j}\right].$$

In particular, we get the marginal discounted probability density function

$$f_{U(T-)}(x|u) = \frac{\Pi_{l=1}^n \lambda_l}{c^n} \bar{P}(x) \sum_{j=1}^k A_j \left[e^{\theta_j u} \sum_{i=1}^n e^{-\rho_i x} \frac{(1 - e^{-(u\wedge x)(\theta_j-\rho_i)})}{\theta_j - \rho_i} \prod_{k=1,k\neq i}^n \frac{1}{\rho_k - \rho_i}\right]$$
$$+ 1(x \geq u) \frac{\Pi_{l=1}^n \lambda_l}{c^n} \bar{P}(x) \sum_{j=1}^n \left[e^{-\rho_j(x-u)} \prod_{k=1,k\neq j}^n \frac{1}{\rho_k - \rho_j}\right].$$

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AUTHORS' REPLY

We are very grateful to receive two more discussions on our paper. The six discussions have much enhanced the value of our paper. Beginning this year, ruin theory is deleted from the actuarial examination syllabus. These discussions show that ruin theory remains a vibrant and exciting area of actuarial research.