ON A CLASSICAL RISK MODEL WITH A CONSTANT DIVIDEND BARRIER

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ABSTRACT

This paper considers a risk model with a constant dividend barrier. It first points out interesting connections between some previous results for this model and those for spectrally negative Lévy processes. An expression is then obtained for the joint distribution of the surplus immediately prior to ruin and the deficit at ruin, discounted from the time of ruin. Such an expression involves known results on the joint distribution at ruin for a classical risk model without barrier. Also discussed are the joint distributions related to the time periods when dividends are paid. In particular, this paper obtains the Laplace transform for the total dividend payments until ruin, and another expression for the expected present value of the total amount of dividend payments until ruin. The results do not require the positive loading condition.

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1. INTRODUCTION

This paper considers the following risk model with a constant dividend barrier. Let $\{N_t\}$ be a Poisson process with intensity λ. Let X_i, $i = 1, 2, \ldots$, be i.i.d. positive random variables with a common density function p, common distribution function P, and finite mean μ. The processes $\{N_t\}$ and $\{X_i\}$ are independent. For $0 \le u \le b$ the surplus process is defined by

$$U_t^b = u + \int_0^t c(U_s^b)ds - \sum_{i=1}^{N_t} X_i, \tag{1.1}$$

where

$$c(x) := \begin{cases} c, & \text{for } x \le b, \\ 0, & \text{for } x > b. \end{cases}$$

Here $c > 0$ is a constant.

In such a surplus process $u \ge 0$ represents the initial surplus. The claims arrive according to $\{N_t\}$. The random variable X_i represents the size of the i-th claim. Premium function $c(x)$ represents the rate at which the premium is collected when the current surplus is x. The process $\{U_t^b\}$ stands for the surplus process for a risk model in which the insurance company would pay a dividend at rate c once the surplus reaches level b and stop the payment whenever the next incoming claim brings the surplus down to below level b. Notice that ruin occurs with probability 1 in such a model. So the *positive safety loading condition* $c > \lambda\mu$ is not required.

Write

$$U_t := u + ct - \sum_{i=1}^{N_t} X_i.$$

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Then $\{U_t\}$ is the surplus process in a classical risk model with a single premium rate. Note that $\{U_t\}$ is just $\{U_t^\infty\}$ by definition. The evolution of $\{U_t^b\}$ can be described intuitively as follows. The process $\{U_t^b\}$ behaves like $\{U_t\}$ when its value is between 0 and b. Whenever $\{U_t^b\}$ reaches level b, it stops growing and keeps its value at b for an exponential time with mean λ^{-1} until the next claim brings it to under b.

Set $T^b := \inf\{t \geq 0 : U_t^b < 0\}$ and $\psi^b(u) := \mathbb{P}\{T^b < \infty | U_0^b = u\}$. The random variable T^b is the *ruin time,* and $\psi^b(u)$ is the *ruin probability* given that the initial surplus is u.

Write dy, $dy > 0$, for the increment on y. Write $\{Y \in dy\}$ for the event $\{y \leq Y < y + dy\}$. For u, δ, x, $y \geq 0$, let

$$W_\delta^b(u; x, y)dxdy := \mathbb{E}[e^{-\delta T^b} 1(U_{T^b-}^b \in dx, -U_{T^b}^b \in dy) | U_0^b = u], \quad 0 \leq x < b,$$

and

$$W_\delta^b(u; b, y)dy := \mathbb{E}[e^{-\delta T^b} 1(U_{T^b-}^b = b, -U_{T^b}^b \in dy) | U_0^b = u].$$

Then write T, $\psi(u)$, and $W_\delta(u; x, y)$ for T^∞, $\psi^\infty(u)$, and $W_\delta^\infty(u; x, y)$.

Risk models with a dividend barrier have been studied by many authors. Refer to Lin, Willmot, and Drekic (2003) and Gerber and Shiu (2004a) for surveys on the previous work. In particular, the distribution of ruin time for a model with a constant dividend barrier was discussed in Gerber (1979). In Lin, Willmot, and Drekic the *Gerber-Shiu discounted penalty function* was further considered for the classical risk models with a constant dividend barrier. More precisely, for $\delta > 0$ the discounted penalty function can be written as

$$m_{b,w}(u) := \mathbb{E}[e^{-\delta T^b} w(U_{T^b-}^b, -U_{T^b}^b) | U_0^b = u],$$

where w is a positive function of two variables. By solving the corresponding integro-differential equations, expressions of $m_{b,w}(u)$ were obtained for certain choices of function w.

Classical risk models with a linear dividend barrier were discussed recently in Albrecher, Hartinger, and Tichy (2005). Work on renewal risk models with a constant dividend barrier can be found in Li and Garrido (2004) and Albrecher, Claramunt, and Mármol (2005). Work on Markov modulated risk models with a constant dividend barrier can be found in Frostig (2005).

A similar model (risk model with a two-step premium rate) was studied in Zhou (2004). In that model the surplus process is also defined by equation (1.1), but with a premium function

$$c(x) := \begin{cases} c_1, & \text{for } x \leq b, \\ c_2, & \text{for } x > b, \end{cases}$$

where $c_i > \lambda\mu$, $i = 1, 2$. The joint distribution at ruin was obtained in Zhou (2004) for such a model. Although the above-mentioned model with a two-step premium rate does not exactly include the risk model with a dividend barrier considered in this paper, nevertheless, the approach there can be implemented under the current setting.

One key in analyzing such models is to know when $\{U_t\}$ first attains level b before ruin. For $0 < v \leq b$ write

$$T(v) := \inf\{t \geq 0 : t < T, U_t = v\}$$

and

$$T^b(v) := \inf\{t \geq 0 : t < T^b, U_t^b = v\}$$

with the convention that $\inf \emptyset := \infty$. $T^b(b)$ is then the time when dividend is first paid before ruin. Notice that both $T(v)$ and $T^b(v)$ are defective random variables.

For $0 \leq u \leq v$ set

$$B_\delta(v|u) := \mathbb{E}[e^{-\delta T(v)} | U_0 = u].$$

Then $B_\delta(v|u) = \mathbb{E}[e^{-\delta T^b(v)}|U_0^b = u]$ if $v \le b$. Also note that $B_\delta(v|u)$ is the same as $B(0, v|u)$ in Gerber and Shiu (1998).

When ruin occurs, distinguish according to whether the running maximum of the surplus process up to the ruin time is smaller than v or not. Since

$$\mathbb{E}[e^{-\delta T^b}1(T^b(v) = \infty)|U_0^b = u] = \mathbb{E}[e^{-\delta T}1(T(v) = \infty)|U_0 = u],$$

by the Markov property we have for $0 \le u \le v \le b$,

$$\mathbb{E}[e^{-\delta T^b}|U_0^b = u] = \mathbb{E}[e^{-\delta T}|U_0 = u] - B_\delta(v|u)\mathbb{E}[e^{-\delta T}|U_0 = v] + B_\delta(v|u)\mathbb{E}[e^{-\delta T^b}|U_0^b = v]. \tag{1.2}$$

We can rearrange it into a more appealing form:

$$B_\delta(v|u) = \frac{\mathbb{E}[e^{-\delta T^b}|U_0^b = u] - \mathbb{E}[e^{-\delta T}|U_0 = u]}{\mathbb{E}[e^{-\delta T^b}|U_0^b = v] - \mathbb{E}[e^{-\delta T}|U_0 = v]}, \quad 0 \le u \le v \le b. \tag{1.3}$$

Notice that the right-hand side of equation (1.3) depends only on b through $v \le b$.

With a similar argument equation (1.3) can be generalized as

$$B_\delta(v|u) = \frac{m_{b_1,w}(u) - m_{b_2,w}(u)}{m_{b_1,w}(v) - m_{b_2,w}(v)}, \quad 0 \le u \le v \le b_2 < b_1. \tag{1.4}$$

An expression for the Laplace transform of $T(v)$ has been obtained before. In expression (6.25) of Gerber and Shiu (1998) it was shown that, under the condition of positive safety loading,

$$B_\delta(v|u) = \frac{e^{\rho(\delta)u} - \psi_\delta(u)}{e^{\rho(\delta)v} - \psi_\delta(v)}, \quad 0 \le u \le v, \tag{1.5}$$

where

$$\psi_\delta(v) := \mathbb{E}[e^{-\delta T + \rho(\delta)U_T}1(T < \infty)|U_0 = v]$$

is the *generalized ruin probability*, $\rho(\delta)$ is the unique nonnegative solution to *Lundberg's fundamental equation*

$$c\xi + \lambda(\hat{p}(\xi) - 1) = \delta, \tag{1.6}$$

and $\hat{p}$ denotes the *Laplace transform* of p.

Equation (1.5) was also obtained in expression (2.4) of Zhou (2004) for a risk model perturbed by a Brownian motion. Under this setting equation (1.6) becomes

$$\sigma^2\xi^2/2 + c\xi + \lambda(\hat{p}(\xi) - 1) = \delta.$$

Its proof is an application of the Markov property. The positive safety loading condition is, indeed, not required there.

The other key in our approach is to make use of known results on classical risk models without barriers. Since the celebrated work of Gerber and Shiu (1997), the joint distributions of the ruin time, the surplus immediately before ruin, and the deficit at ruin have been studied intensively, often under the condition of positive safety loading. Work along this line also can be found in Chiu and Yin (2003), Wu, Wang, and Wei (2003), and Zhang and Wang (2003).

To find expressions for $W_\delta^b(u; x, y)$ and $W_\delta^b(u; b, y)$ with no restriction on safety loading, a little *fluctuation theory* for Lévy processes with only negative jumps will be introduced. Such processes are also called spectrally negative Lévy processes, or processes with stationary and independent increments and with skip-free upwards sample paths.

The ruin problem for risk models corresponds to the so-called *exit problem* in the theory for Lévy processes. Two excellent references are Chapter VII in Bertoin (1996) and Bertoin (1997). Also see Pistorius (2004) for an introduction on Lévy processes. In the following some related results from Bertoin (1997) will be presented under the simpler setting of this paper.

A spectrally negative Lévy process $\{X_t\}$ with initial value $X_0 = 0$ is considered in Bertoin (1997). Its Laplace transform is given by

$$\mathbb{E}[e^{\xi X_t}] = e^{t\Psi(\xi)}, \quad \xi \geq 0,$$

where the *Laplace exponent* Ψ has the form

$$\Psi(\xi) = m\xi + \sigma^2\xi^2/2 + \int_{-\infty}^{0} (e^{\xi x} - 1 - \xi x 1(x > -1))\Pi(dx),$$

and $\Pi(dx)$ is a measure on $(-\infty, 0)$ satisfying $\int_{-\infty}^{0}(1 \wedge x^2)\Pi(dx) < \infty$. When $\{X_t\}$ is replaced by $\{U_t - U_0\}$, the corresponding Laplace exponent becomes

$$\Psi(\xi) = c\xi + \lambda(\hat{p}(\xi) - 1).$$

Write W_δ, $\delta \geq 0$, for the so-called *scale function* for $\{U_t - U_0\}$. By expression (7) in Bertoin (1997) it is determined by its Laplace transform

$$\int_0^\infty e^{-\xi x} W_\delta(x)dx = \frac{1}{c\xi + \lambda(\hat{p}(\xi) - 1) - \delta}, \quad \xi > \rho(\delta). \tag{1.7}$$

Notice that $W_\delta(0) = 1/c$ and the denominator of the right-hand side of equation (1.7) is related to Lundberg's fundamental equation. It is known from expression (6) in Bertoin that for $0 \leq u \leq v$,

$$B_\delta(v|u) = \frac{W_\delta(u)}{W_\delta(v)}. \tag{1.8}$$

By Theorem 1 and Corollary 2 in Bertoin (1997), we further have, for $x \in (0, b)$ and $y > 0$,

$$\begin{aligned}&\mathbb{E}[e^{-\delta T^*}1(U_{T^*-} \in dx, -U_{T^*} \in dy)|U_0 = u] \\ &\qquad = \left(\frac{W_\delta(u)W_\delta(b - x)}{W_\delta(b)} - 1(u \geq x)W_\delta(u - x)\right) \lambda p(x + y)dxdy,\end{aligned} \tag{1.9}$$

where $T^* := \inf\{t \geq 0 : U_t \notin (0; b)\}$. Moreover, by Theorem 7.1 of Bertoin (1996), we have

$$\mathbb{E}[e^{-\delta\tau(x)}|U_0 = u] = e^{-(x-u)\rho(\delta)}, \quad x \geq u, \tag{1.10}$$

where $\tau(x) := \inf\{t \geq 0 : U_t = x\}$. Letting $b \to \infty$ in equation (1.9), it follows from equations (1.8) and (1.10) that

$$\begin{aligned} W_\delta(u; x, y) &= \lim_{b\to\infty} \left(\frac{W_\delta(u)W_\delta(b - x)}{W_\delta(b)} - 1(u \geq x)W_\delta(u - x)\right) \lambda p(x + y) \\ &= \lim_{b\to\infty} (W_\delta(u)\mathbb{E}[e^{-\delta T(b)}|U_0 = b - x] - 1(u \geq x)W_\delta(u - x))\ \lambda p(x + y) \\ &= (W_\delta(u)e^{-x\rho(\delta)} - 1(u \geq x)W_\delta(u - x))\ \lambda p(x + y). \end{aligned} \tag{1.11}$$

Let $\delta = 0$ in equations (1.11). Then

$$W_0(u; x, y) = (W_0(u)e^{-x\rho(0)} - 1(u \geq x)W_0(u - x))\ \lambda p(x + y). \tag{1.12}$$

Notice that under the positive safety loading condition $\rho(0) = 0$.

The scale function W_δ is closely related to the ruin probability ψ. It is pointed out in expression (9) of Bertoin (1997) that

$$W_\delta(x) = \sum_{k=0}^{\infty} \delta^k W_0^{*(k+1)}(x),$$

where W_0^{*n} stands for the n-fold convolution of W_0. Under the positive loading condition, a comparison of the Laplace transforms for W_0 and for $1 - \psi$ reveals that

$$W_0(x) = \frac{1 - \psi(x)}{c - \lambda\mu}. \tag{1.13}$$

Equation (1.8) had already been pointed out in Gerber (1979). The following identity is found in Section 10.1 there:

$$B_\delta(v|u) = \frac{h(u)}{h(v)}, \tag{1.14}$$

where h solves the integro-differential equation

$$ch'(x) - (\lambda + \delta)h(x) + \lambda \int_0^x h(x - y)dP(y) = 0. \tag{1.15}$$

The Laplace transform $\hat{h}$ for h can be easily determined from equation (1.15) as

$$\hat{h}(\xi) = \frac{ch(0)}{c\xi + \lambda(\hat{p}(\xi) - 1) - \delta}.$$

Then by setting $h(0) = 1/c$ we reach that $h = W_\delta$. The function $h(x)$, which was called $v(x)$, was also analyzed in Section 4 of Lin, Willmot, and Drekic (2003) and Shiu (1998).

We can reconcile equation (1.5) with equation (1.8) by directly showing that $W_\delta(v)$ is proportional to $e^{\rho(\delta)v} - \psi_\delta(v)$. To this end, make use of expression (2.16) in Gerber and Shiu (1998) for penalty function $\omega(x, y) := e^{-\rho(\delta)y}$ and the fact that $\rho(\delta)$ solves equation (1.6). It then follows that $e^{\rho(\delta)x} - \psi_\delta(x)$ solves equation (1.15). Consequently, under the condition of positive safety loading,

$$W_\delta(v) = \frac{W_\delta(0)}{1 - \psi_\delta(0)} (e^{\rho(\delta)v} - \psi_\delta(v)) = \frac{e^{\rho(\delta)v} - \psi_\delta(v)}{c(1 - \psi_\delta(0))}. \tag{1.16}$$

Moreover, from equations (1.11) and (1.16) we can recover expressions (6.34) and (6.35) in Gerber and Shiu. By equation (2.6) in Lin, Willmot, and Drekic (2003) we see that $m_{b_1,w}(x) - m_{b_2,w}(x)$ solves equation (1.15). So we can also reconcile equations (1.4) and (1.14).

The optimal dividend problems have been addressed by several authors. See Gerber and Shiu (1998) for an earlier work and references therein, and Dickson and Waters (2004) for recent work on the classical risk model. Also see Gerber and Shiu (2004a) for work on a Brownian risk model. In particular, $\mathbb{E}[D_\delta(u, b)]$, the expected present value of the dividend payments until ruin when the interest rate is δ, is studied in Gerber and Shiu (1998) (explicit definition of $\mathbb{E}[D_\delta(u, b)]$ will be given in Section 3). More precisely, they proved that

$$\frac{\partial}{\partial u} \mathbb{E}[D_\delta(u, b)]\bigg|_{u=b} = 1. \tag{1.17}$$

Applying equation (1.5), which requires the condition of positive safety loading, and equation (1.17) to the identity

$$\mathbb{E}[D_\delta(u, b)] = B_\delta(b|u)\mathrm{E}[D_\delta(b, b)]$$

yields

$$\mathbb{E}[D_\delta(u, b)] = \frac{e^{\rho(\delta)u} - \psi_\delta(u)}{\rho(\delta)e^{\rho(\delta)b} - \psi_\delta'(b)}, \tag{1.18}$$

where ψ_δ' denotes the derivative of ψ_δ.

Combing equations (1.16) and (1.18) yields

$$\mathbb{E}[D_\delta(u, b)] = \frac{W_\delta(u)}{W'_\delta(b)} = \frac{h(u)}{h'(b)} \tag{1.19}$$

under the positive safety loading condition. Identity (1.19) was already obtained in expression (8.9) of Gerber (1972) for a perturbed risk model. Also see expression (10.1.13) of Gerber (1979).

In this paper a different approach is adopted. Either by studying the joint distribution on the durations of dividend payments or by conditioning on the first claim, we could express $\mathbb{E}[D_\delta(b, b)]$ in terms of p and $B_\delta(b|v)$. In this way we could eventually express $\mathbb{E}[D_\delta(u, b)]$ in terms of p and W_δ. Again, the positive safety loading condition is not required.

The rest of this paper is arranged as follows. In Section 2, first, an expression for $W^b_\delta(u; x, y)$ is found. Since

$$m_{b,w}(u) = \int_0^b dx \int_0^\infty dy w(x, y) W^b_\delta(u; x, y) + \int_0^\infty dy w(b, y) W^b_\delta(u; b, y),$$

an expression of $m_{b,w}(u)$ follows readily for any w. In this way we generalize the results in Lin, Willmot, and Drekic (2003) to all the possible choices of function w. The proof is an adaptation of that in Zhou (2004), which involves intensive applications of the Markov property. The Laplace transform is also found in Section 2 for the last time when the dividend is paid before ruin.

In Section 3 the Laplace transform is obtained for the total amount of dividends before ruin. Another formula is derived for the expected present value of the total amount of dividend payments until ruin. Explicit results are found for a model with exponential claims in Section 4.

Our approach is different from that used in Lin, Willmot, and Drekic (2003), Gerber and Shiu (1998), and some other related work that relies on solving certain integro-differential equations. One advantage is that it is closely connected to the previous work on Gerber-Shiu functions for the classical risk models and the general theory for Lévy processes. This approach does not require the positive safety loading condition. It also goes around some technicalities such as the differentiability when dealing with differential equations.

It should be pointed out that the continuity assumption on the distribution for X_i in equation (1.1) is not necessary. In Bertoin's original work there is no assumption on the distribution for X_i. But this paper will not get into the details in this respect.

Of course, this approach has its own shortcoming, mainly because equation (1.7) cannot always be inverted analytically. In many cases we have to count on numerical inversions.

2. JOINT DISTRIBUTION AT RUIN

To find an expression for W^b we first need to find an expression for

$$\bar{W}_\delta(u; v, x, y)dxdy := \mathbb{E}[e^{-\delta T} 1(T(v) = \infty, U_{T-} \in dx, -U_T \in dy)|U_0 = u],$$

where $0 \le u \le v$, $0 < x < v$, $y > 0$.

Note that $\bar{W}_\delta(u; v, x, y)dxdy$ gives the discounted joint distribution at ruin when the surplus process $\{U_t\}$ never can reach level v before ruin. Applying the strong Markov property at $T(v)$, we have

$$\begin{aligned}\bar{W}_\delta(u; v, x, y)dxdy &= \mathbb{E}[e^{-\delta T} 1(T(v) = \infty, U_{T-} \in dx, -U_T \in dy)|U_0 = u] \\ &= \mathbb{E}[e^{-\delta T} 1(U_{T-} \in dx, -U_T \in dy)|U_0 = u] \\ &\quad - \mathbb{E}[e^{-\delta T} 1(T(v) < \infty, U_{T-} \in dx, -U_T \in dy)|U_0 = u] \\ &= W_\delta(u; x, y)dxdy - \mathbb{E}[e^{-\delta T(v)}|U_0 = u]\mathbb{E}[e^{-\delta T} 1(U_{T-} \in dx, -U_T \in dy)|U_0 = v] \\ &= W_\delta(u; x, y)dxdy - B_\delta(v|u) W_\delta(v; x, y)dxdy.\end{aligned}$$

Therefore,

$$\bar{W}_\delta(u; v, x, y) = W_\delta(u; x, y) - B_\delta(v|u)W_\delta(v; x, y). \tag{2.1}$$

The next proposition is one of our main results, in which W^b can be expressed in terms of p and W_δ.

Proposition 2.1

Given $0 \le u < b$, $0 < x < b$, and $y > 0$, we have

$$W_\delta^b(b; b, y) = \frac{\lambda p(b + y)}{\delta + \lambda - \lambda \int_0^b p(z)B_\delta(b|b - z)dz}, \tag{2.2}$$

$$W_\delta^b(b; x, y) = \frac{\lambda \int_0^b p(z)\bar{W}_\delta(b - z; b, x, y)dz}{\delta + \lambda - \lambda \int_0^b p(z)B_\delta(b|b - z)dz}, \tag{2.3}$$

and

$$W_\delta^b(u; x, y) = \bar{W}_\delta(u; b, x, y) + B_\delta(b|u)W_\delta^b(b; x, y). \tag{2.4}$$

Proof

Starting from level b the surplus process $\{U_t^b\}$ spends an exponential time at level b until the first claim arrives. For the event $\{U_{T^b-}^b = b, -U_{T^b}^b \in dy\}$ to occur, either that claim causes ruin, or else $\{U_t^b\}$ first jumps to between 0 and b, then comes back to level b before ruin and starts all over again. Applying the Markov property at the jumping time, we have

$$W_\delta^b(b; b, y) = \frac{\lambda}{\lambda + \delta}\left(p(b + y) + W_\delta^b(b; b, y)\int_0^b p(z)B_\delta(b|b - z)dz\right).$$

Then equation (2.2) follows by solving the above equation.

Similarly, starting from b, for the event $\{U_{T^b-}^b \in dx, -U_{T^b}^b \in dy\}$ to occur, the first claim cannot cause ruin. Consequently $\{U_t^b\}$ first jumps to somewhere between 0 and b. Then ruin occurs either before $\{U_t^b\}$ comes back to level b, or else $\{U_t^b\}$ comes back to level b and starts all over again. It follows that

$$W_\delta^b(b; x, y) = \frac{\lambda}{\lambda + \delta}\left(\int_0^b p(z)\bar{W}_\delta(b - z; b, x, y)dz + W_\delta^b(b; x, y)\int_0^b p(z)B_\delta(b|b - z)dz\right).$$

Thus equation (2.3) also follows.

Starting from u, ruin occurs either before $\{U_t^b\}$ ever reaches level b, or else $\{U_t^b\}$ reaches b before ruin. Identity (2.4) also follows from the strong Markov property. □

We can also find an expression for the Laplace transform of T^b. Because of equation (1.2) we have only to find $\mathbb{E}[e^{-\delta T^b}|U_0^b = b]$.

Proposition 2.2

We have

$$\mathbb{E}[e^{-\delta T^b}|U_0^b = b] = \frac{\lambda(1 - P(b)) + \lambda \int_0^b p(z)(\mathbb{E}[e^{-\delta T}|U_0 = b - z] - B_\delta(b|b - z)\mathbb{E}[e^{-\delta T}|U_0 = b])\,dz}{\delta + \lambda - \lambda \int_0^b p(z)B_\delta(b|b - z)dz}. \tag{2.5}$$

Proof

Integrating equation (2.1) yields

$$\int_0^b dx \int_0^\infty \bar{W}_\delta(u; v, x, y)dy = \mathbb{E}[e^{-\delta T}|U_0 = u] - B_\delta(v|u)\mathbb{E}[e^{-\delta T}|U_0 = v].$$

Then integrating equations (2.2) and (2.3), and summing them up, we obtain equation (2.5). □

Formulas of $W_0(u; x, y)$ for a classical risk model with either positive, negative, or zero safety loading also can be found in Schmidli (1999). By letting $\delta \to 0+$ in equation (1.8), we have

$$\mathbb{P}\{T(v) < \infty | U_0 = u\} = \frac{W_0(u)}{W_0(v)}.$$

Moreover,

$$\begin{aligned}\bar{W}_0(u; v, x, y) &= W_0(u; x, y) - \mathbb{P}\{T(v) < \infty | U_0 = u\} W_0(v; x, y)\\ &= W_0(u; x, y) - \frac{W_0(u)}{W_0(v)} W_0(v; x, y).\end{aligned}$$

We can then obtain the joint distribution of $(U^b_{T^b-}, U^b_{T^b})$ by letting $\delta \to 0+$ in Proposition 2.1.

Proposition 2.3

Given $0 \le u < b$, $0 < x < b$, *and* $y > 0$, *we have*

$$W^b_0(b; b, y) = \frac{\lambda p(b + y) W_0(b)}{\lambda W_0(b) - \lambda \int_0^b p(z) W_0(b - z) dz},$$

$$W^b_0(b; x, y) = \frac{\lambda \int_0^b p(z) [W_0(b) W_0(b - z; x, y) - W_0(b - z) W_0(b; x, y)]\, dz}{\lambda W_0(b) - \lambda \int_0^b p(z) W_0(b - z) dz},$$

and

$$W^b_0(u; x, y) = W_0(u; x, y) - \frac{W_0(u)}{W_0(b)} W_0(b; x, y) + \frac{W_0(u)}{W_0(b)} W^b(b; x, y).$$

The last result in this section concerns the last time when a dividend is paid before ruin. Set

$$T'(b) := \sup\{t \ge 0 : t \le T,\ U^b_t = b\}$$

with the convention that $\sup \emptyset = 0$.

Proposition 2.4

For $0 \le u \le b$, *we have*

$$\mathbb{E}[e^{-\delta T'(b)} | U^b_0 = u] = \frac{\lambda (1 - P(b)) B_\delta(b|u)}{\lambda + \delta - \lambda \int_0^b p(z) B_\delta(b|b - z) dz}.$$

PROOF

Starting from b, the next claim arrives after an exponential time. Conditioning on the size of that claim, we have that

$$\mathbb{E}[e^{-\delta T'(b)} | U^b_0 = b] = \frac{\lambda}{\lambda + \delta} \left(1 - P(b) + \mathbb{E}[e^{-\delta T'(b)} | U^b_0 = b] \int_0^b p(z) B_\delta(b|b - z) dz \right).$$

Moreover, $T'(b) \ge T(b)$ if and only if $T(b) < \infty$. An application of the strong Markov property at time $T(b)$ gives

$$\mathbb{E}[e^{-\delta T'(b)} | U^b_0 = u] = B_\delta(b|u) \mathbb{E}[e^{-\delta T'(b)} | U^b_0 = b].$$

The assertion of this proposition thus follows. □

3. PRESENT VALUE OF DIVIDENDS UNTIL RUIN

Write R_i (resp., L_i) for the time when $\{U^b_t\}$ reaches (resp., leaves) level b from below (resp., above) for the i-th time before ruin. Then $0 \le R_1 < L_1 < R_2 < L_2 < \ldots$, and $L_i - R_i$ represents the duration of the i-th period of dividend payment. The Laplace transform of R_1 is given by either equation (1.5) or (1.8). Let $M := \sup\{i : R_i < T(u)\}$ be the total number of dividend payment periods before ruin.

Let $\delta \geq 0$ be the force of interest for valuation. Given $U_0^b = u$, the present value of the total amount of dividend payments until ruin is defined by

$$D_\delta(u, b) := 1_{\{M\geq 1\}} \sum_{i=1}^{M} \int_{R_i}^{L_i} ce^{-\delta t}dt = \frac{c}{\delta} 1_{\{M\geq 1\}} \sum_{i=1}^{M} (e^{-\delta R_i} - e^{-\delta L_i}). \tag{3.1}$$

Write $p_0 := \mathbb{P}\{R_1 < \infty | U_0 = u\}$ and $p_1 := \mathbb{P}\{R_2 < \infty | R_1 < \infty\}$. Then

$$p_0 = \mathbb{P}\{T(b) < \infty | U_0 = u\} = \frac{W_0(u)}{W_0(b)}.$$

We will derive an expression for p_1.

Starting from level b, after the first downward jump, the overall probability of ruin before $\{U_t^b\}$ ever climbs back to b again is

$$1 - P(b) + \int_0^b p(z)\mathbb{P}\{T(b) = \infty | U_0 = b - z\}dz = 1 - P(b) + \int_0^b p(z) \left(1 - \frac{W_0(b - z)}{W_0(b)}\right) dz.$$

So

$$\begin{aligned} p_1 &= 1 - \left(1 - P(b) + \int_0^b p(z)\mathbb{P}\{T(b) = \infty | U_0 = b - z\}dz\right) \\ &= \int_0^b p(z) \frac{W_0(b - z)}{W_0(b)} dz. \end{aligned}$$

We want to understand the joint distribution for $\{R_i\}$ and $\{L_i\}$. Our next result is a consequence of the Markov property applied to $\{U_t^b\}$.

Proposition 3.1

The random variable M follows a distribution such that $\mathbb{P}\{M = 0\} = 1 - p_0$ and $\mathbb{P}\{M = k\} = p_0 p_1^{k-1}(1 - p_1)$ for $k \geq 1$. Given $M = k$, $\{L_i - R_i, i = 1, \ldots, k\}$ are i.i.d. exponential random variables with mean λ^{-1}; $\{R_{i+1} - L_i, i = 1, \ldots, k - 1\}$ are i.i.d. random variables with a common conditional Laplace transform

$$\mathbb{E}[e^{-\delta(R_2 - L_1)} | M = k] = \frac{1}{p_1} \int_0^b p(y)B_\delta(b - y, b)dy, \quad \delta \geq 0;$$

and R_1 has a conditional Laplace transform $B_\delta(b|u)/p_0$. Moreover, the two sequences and R_1 are independent.

REMARK 3.2

It follows from Proposition 3.1 that the total dividend time

$$\sum_{i=1}^{M} (L_i - R_i) = \frac{D_0(u, b)}{c}$$

is a (defective) geometric summation of i.i.d. exponential random variables. A somewhat similar work on duration of the time in red for the classical risk model can be found in Dos Reis (1993) and Dickson and Dos Reis (1996).

We can further compute the Laplace transform of the undiscounted dividends until ruin.

Proposition 3.3

Given $\gamma \geq 0$, we have

$$\mathbb{E}[e^{-\gamma D_0(u,b)}] = 1 - p_0 + \frac{p_0 c\lambda(1 - p_1)}{\gamma + c\lambda(1 - p_1)}. \tag{3.2}$$

PROOF

Proposition 3.1 gives

$$\mathbb{E}[e^{-\gamma D_0(u,b)}] = 1 - p_0 + \sum_{k=1}^{\infty} p_0(1 - p_1)p_1^{k-1}\left(\frac{c\lambda}{c\lambda + \gamma}\right)^k.$$

Then equation (3.2) follows readily. □

REMARK 3.4

Note that $D_0(b, b)$ follows an exponential distribution, and conditional on $D_0(u, b) > 0$, $D_0(u, b)$ also follows an exponential distribution.

REMARK 3.5

The distribution for $D_0(u, b)$ was considered in Section 5 of Gerber and Shiu (2004a) for the Brownian risk model. Since the Brownian risk model arises as a time-space scaling limit of the Poisson risk model (see Gerber and Shiu 2004b), expression (5.2) in Gerber and Shiu (2004a) can be obtained from equation (3.2) by taking an appropriate limit.

Observe that

$$\mathbb{E}[D_\delta(u, b)] = B_\delta(b|u)\mathbb{E}[D_\delta(b, b)].$$

Therefore, we need to find only an expression for $\mathbb{E}[D_\delta(b, b)]$. Proposition 3.1 essentially gives an explicit description on the distribution of $D_\delta(b, b)$. It allows us to compute directly the expected value of $D_\delta(u, b)$ similar to the proof for Proposition 3.3. Hans Gerber points out a different proof, which this paper will borrow.

Proposition 3.6

The expected present value of dividends until ruin is

$$\mathbb{E}[D_\delta(b, b)] = \frac{cB_\delta(b|b)}{\lambda + \delta - \lambda \int_0^b p(y)B_\delta(b|b - y)dy}. \tag{3.3}$$

PROOF

Consider the surplus process $\{U_t^b\}$, starting at b, up to time t. Conditioning on the arrival time and the size of the first claim, we have

$$D_\delta(b, b) = e^{-\lambda t}\left(\frac{c}{\delta}(1 - e^{-\delta t}) + e^{-\delta t}D_\delta(b, b)\right) + \int_0^t \lambda e^{-\lambda s}\left[\frac{c}{\delta}(1 - e^{-\delta s}) + e^{-\delta s}\int_0^b p(y)D_\delta(b - y, b)dy\right]ds. \tag{3.4}$$

Differentiating equation (3.4) with respect to t, we have

$$0 = -\lambda e^{-\lambda t}\left(\frac{c}{\delta}(1 - e^{-\delta t}) + e^{-\delta t}D_\delta(b, b)\right) + e^{-\lambda t}(ce^{-\delta t} - \delta e^{-\delta t}D_\delta(b, b)) + \lambda e^{-\lambda t}\left[\frac{c}{\delta}(1 - e^{-\delta t}) + e^{-\delta t}\int_0^b p(y)D_\delta(b - y, b)dy\right]. \tag{3.5}$$

Letting $t = 0$ in equation (3.5) yields

$$\begin{aligned} 0 &= -\lambda D_\delta(b, b) + c - \delta D_\delta(b, b) + \lambda \int_0^b p(y) D_\delta(b - y, b) dy \\ &= -\lambda D_\delta(b, b) + c - \delta D_\delta(b, b) + \lambda D_\delta(b, b) \int_0^b p(y) B_\delta(b | b - y) dy. \end{aligned} \tag{3.6}$$

Finally, equation (3.3) is obtained by solving equation (3.6) for $D_\delta(b, b)$. □

REMARK 3.7

Combining equations (1.8) and (3.3), we have

$$\mathbb{E}[D_\delta(u, b)] = \frac{c W_\delta(u)}{(\lambda + \delta) W_\delta(b) - \lambda \int_0^b p(y) W_\delta(b - y) dy}. \tag{3.7}$$

REMARK 3.8

Since $W_\delta = h$ satisfies equation (1.15), then equation (3.7) can be rewritten as

$$\mathbb{E}[D_\delta(u, b)] = \frac{W_\delta(u)}{W'_\delta(b)},$$

and we have recovered equation (1.19).

This paper assumes that dividends are paid according to a *barrier strategy* described earlier. By an *optimal dividend barrier* is meant a value b^* such that b^* is the value of $b \geq 0$ that minimizes the denominator of equation (3.7),

$$(\lambda + \delta) W_\delta(b) - \lambda \int_0^b p(y) W_\delta(b - y) dy. \tag{3.8}$$

We first consider the case that $\delta = 0$. With the positive safety loading condition, it is evident from equation (1.13) that

$$\lim_{b \to \infty} W_0(b) = \frac{1}{c - \lambda \mu} > 0.$$

Consequently,

$$\lim_{b \to \infty} \left(\lambda W_0(b) - \lambda \int_0^b p(y) W_0(b - y) dy \right) = 0.$$

So there is no meaningful optimal dividend barrier in this situation.

On the other hand, for $\delta > 0$, with the positive safety loading condition or not, we always have that, for fixed u, $T(b)$ increases to ∞ as $b \to \infty$. By equation (1.8) we thus have that $W_\delta(b)$ increases to ∞ as $b \to \infty$. Consequently,

$$\lim_{b \to \infty} \left((\lambda + \delta) W_\delta(b) - \lambda \int_0^b p(y) W_\delta(b - y) dy \right) = \infty.$$

So the optimal dividend barrier always exists for $\delta > 0$.

To find the optimal dividend barrier when $\delta > 0$, we can take derivatives on expression (3.8) and solve the following equation for b:

$$(\lambda + \delta) W'_\delta(b) - \lambda p(b) W_\delta(0) - \lambda \int_0^b p(y) W'_\delta(b - y) dy = 0, \tag{3.9}$$

where W'_δ denotes the derivative of W_δ. We might not have the closed-form expression for W'_δ in general. Then we have to settle for a numerical solution to equation (3.9).

4. AN EXAMPLE

This section considers a model in which the claim size X_i follows an exponential distribution with mean β^{-1}. The positive loading condition is not assumed.

Because $p(x) = \beta e^{-\beta x}$ and $\hat{p}(\xi) = \beta/(\beta + \xi)$, equation (1.6) then becomes

$$c\xi^2 + (c\beta - \lambda - \delta)\xi - \delta\beta = 0,$$

which has a positive solution

$$\rho(\delta) = \frac{\lambda + \delta - c\beta + \sqrt{(c\beta - \lambda - \delta)^2 + 4c\beta\delta}}{2c}$$

and a negative solution

$$\bar{\rho}(\delta) = \frac{\lambda + \delta - c\beta - \sqrt{(c\beta - \lambda - \delta)^2 + 4c\beta\delta}}{2c}.$$

Inverting the Laplace transform (1.7), we have

$$W_\delta(x) = \frac{(\beta + \rho(\delta))e^{\rho(\delta)x} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)x}}{\sqrt{(c\beta - \lambda - \delta)^2 + 4c\beta\delta}}.$$

We thus have an expression for $W_\delta(u;\, x,\, y)$ by equation (1.11). It follows from equation (1.8) (or Example 2.6 in Zhou 2004) that

$$B_\delta(v|u) = \frac{(\beta + \rho(\delta))e^{\rho(\delta)u} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)u}}{(\beta + \rho(\delta))e^{\rho(\delta)v} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)v}}.$$

Then by equation (2.1) we can reach an expression for $\bar{W}_\delta(u;\, v,\, x,\, y)$ that leads to explicit, but rather complicated, expressions for $W_\delta^b(u;\, x,\, y)$ by Proposition 2.1.

Let us continue to work out an expression for $\mathbb{E}[e^{-\delta T^b}|U_0^b = u]$. Since

$$\int_0^b p(z)B_\delta(b|b - z)dz = \int_0^b \beta e^{-\beta z}\, \frac{(\beta + \rho(\delta))e^{\rho(\delta)(b-z)} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)(b-z)}}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}\, dz$$
$$= \frac{\beta e^{\rho(\delta)b} - \beta e^{\bar{\rho}(\delta)b}}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}$$

and

$$\int_0^b p(z)\mathbb{E}[e^{-\delta T}|U_0 = b - z]\; dz = \int_0^b \beta e^{-\beta z} e^{\bar{\rho}(\delta)(b-z)} \left(1 + \frac{\bar{\rho}(\delta)}{\beta}\right) dz$$
$$= e^{\bar{\rho}(\delta)b} - e^{-\beta b},$$

then by equation (2.5),

$$\mathbb{E}[e^{-\delta T^b}|U_0^b = b] = \frac{\lambda\left\{e^{-\beta b} + e^{\bar{\rho}(\delta)b} - e^{-\beta b} - e^{\bar{\rho}(\delta)b}\left(1 + \frac{\bar{\rho}(\delta)}{\beta}\right)\frac{\beta e^{\rho(\delta)b} - \beta e^{\bar{\rho}(\delta)b}}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}\right\}}{\delta + \lambda - \frac{\lambda\beta(e^{\rho(\delta)b} - e^{\bar{\rho}(\delta)b})}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}}$$
$$= \frac{\lambda(\rho(\delta) - \bar{\rho}(\delta))e^{(\rho(\delta)+\bar{\rho}(\delta))b}}{(\delta\beta + (\delta + \lambda)\rho(\delta))e^{\rho(\delta)b} - (\delta\beta + (\delta + \lambda)\bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}.$$

Hence, $\mathbb{E}[e^{-\delta T^b}|U_0^b = u]$ follows from equation (1.2) readily. Such an expression was obtained in expression (6.3) of Lin, Willmot, and Drekic (2003).

By Proposition 2.4, we have

$$\mathbb{E}[e^{-\delta T'(b)}|U_0^b = u] = \frac{\lambda\beta e^{-\beta(b+y)}[(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}]}{[\delta\beta + (\lambda + \delta)\rho(\delta)]e^{\rho(\delta)b} - [\delta\beta + (\lambda + \delta)\bar{\rho}(\delta)]e^{\bar{\rho}(\delta)b}}\,.$$

In addition,

$$p_0 = \frac{\lambda e^{(\lambda - c\beta)u/c} - c\beta}{\lambda e^{(\lambda - c\beta)b/c} - c\beta}$$

and

$$p_1 = \frac{c\beta\ (e^{(\lambda - c\beta)b/c} - 1)}{\lambda e^{(\lambda - c\beta)b/c} - c\beta}.$$

We can easily find the Laplace transform for $D_0(u, b)$ by Proposition 3.3.

Moreover, by Proposition 3.6 we have

$$\begin{aligned}\mathbb{E}[D_\delta(u, b)] &= \frac{c\,\dfrac{(\beta + \rho(\delta))e^{\rho(\delta)u} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)u}}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}}{\lambda + \delta - \lambda \int_0^b \beta e^{-\beta y}\,\dfrac{(\beta + \rho(\delta))e^{\rho(\delta)(b-y)} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)(b-u)}}{(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}}\,dy} \\ &= \frac{c[(\beta + \rho(\delta))e^{\rho(\delta)u} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)u}]}{(\lambda + \delta)[(\beta + \rho(\delta))e^{\rho(\delta)b} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)b}] - \lambda\beta(e^{\rho(\delta)b} - e^{\bar{\rho}(\delta)b})} \\ &= \frac{c[(\beta + \rho(\delta))e^{\rho(\delta)u} - (\beta + \bar{\rho}(\delta))e^{\bar{\rho}(\delta)u}]}{[\delta\beta + (\lambda + \delta)\rho(\delta)]e^{\rho(\delta)b} - [\delta\beta + (\lambda + \delta)\bar{\rho}(\delta)]e^{\bar{\rho}(\delta)b}}\,. \end{aligned} \tag{4.1}$$

By expressions (5.39) and (5.40) in Gerber and Shiu (1998) it is easy to check that equation (4.1) is exactly the same as expression (7.8) in Gerber and Shiu under the positive safety loading condition.

To find the optimal dividend barrier for $\delta > 0$, we differentiate (with respect to b) the denominator of equation (4.1) and obtain

$$[\delta\beta + (\lambda + \delta)\rho(\delta)]\rho(\delta)e^{\rho(\delta)b} - [\delta\beta + (\lambda + \delta)\bar{\rho}(\delta)]\bar{\rho}(\delta)e^{\bar{\rho}(\delta)b},$$

which we now call $G(b)$. If $[\delta\beta + (\lambda + \delta)\bar{\rho}(\delta)]\bar{\rho}(\delta) \le 0$, then $G(b) > 0$ for all b and $b^* = 0$. Otherwise, $G(b) = 0$ has a unique solution

$$b_0 = \frac{1}{\rho(\delta) - \bar{\rho}(\delta)} \ln \frac{[\delta\beta + (\lambda + \delta)\bar{\rho}(\delta)]\bar{\rho}(\delta)}{[\delta\beta + (\lambda + \delta)\rho(\delta)]\rho(\delta)}\,.$$

Then $b^* = b_0$ if $b_0 \ge 0$ and $b^* = 0$ if $b_0 < 0$. Note further that $b_0 \ge 0$ if and only if

$$(\lambda + \delta)(\lambda + \delta - c\beta) + c\delta\beta \le 0.$$

In particular, b^* is always 0 if the positive safety loading condition fails.

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