Option Pricing under Autoregressive Random Variance Models

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Abstract

The autoregressive random variance (ARV) model introduced by Taylor (1980, 1982, 1986) is a popular version of stochastic volatility (SV) models and a discrete-time simplification of the continuous-time diffusion SV models. This paper introduces a valuation model for options under a discrete-time ARV model with general stock and volatility innovations. It employs the discrete-time version of the Esscher transform to determine an equivalent martingale measure under an incomplete market. Various parametric cases of the ARV models, are considered, namely, the log-normal ARV models, the jump-type Poisson ARV models, and the gamma ARV models, and more explicit pricing formulas of a European call option under these parametric cases are provided. A Monte Carlo experiment for some parametric cases is also conducted.

1. Introduction

The autoregressive random variance (ARV) model introduced by Taylor (1980, 1982, 1986) is a popular version of stochastic volatility (SV) models in the finance community. The model supposes that assets' returns dynamics are governed by a discrete-time product process driven by two independent innovations processes, namely, the stock innovations and the volatility innovations. The stock innovations are assumed to be normally distributed, and the logarithmic volatility process is governed by a first-order autoregressive AR(1) model with normal innovations. Other names of the ARV model are the lognormal AR(1) model and the lognormal ARV model. The ARV model is a discrete-time simplification of a continuous-time diffusion model with a stochastic variance process that has been adopted for option valuation by several authors, such as Hull and White (1987), Heston (1993), and Wiggins (1987). Taylor (1994) provided an excellent overview and comprehensive comparative studies on the ARV models and other important volatility models, such as ARCH models by Engle (1982), GARCH models proposed by Bollerslev (1986) and Taylor (1986) independently, and EGARCH models by Nelson (1991). The comparative studies are on several aspects, such as volatility estimation, option valuation, and the ability to study the persistence in volatility shocks. In particular, Taylor (1994) mentioned that it is quicker to evaluate the option prices numerically under ARV models than under ARCH-type models. It has also been said in that paper that the two models give similar pricing results when the parameters of the two models match one another's.

The valuation of derivative instruments under continuous-time SV models has been studied extensively in the literature. The market described by the SV models is incomplete in general. One strand of the literature adopts another option together with the underlying instruments to hedge the option perfectly and complete the market; see, for example, Zhu and Avellaneda (1998), Romano and Touzi (1997), Hobson and Roger (1998), and Davis (2004). Another strand of the literature specifies a criterion for the justification of a particular choice of a martingale pricing measure; see, for example, Laurent and Pham (1999), Biagini, Guasoni, and Pratelli (2000), Heath, Platen, and Schweizer (2001), and Hobson (2004). Basically, there are two popular criteria for choosing martingale pricing measures

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in the context of continuous-time SV models, namely, the variance-optimal martingale measures and the minimal entropy martingale measures. The variance-optimal martingale measure can be related to the quadratic utility functions. The use of the variance-optimal martingale measures has been extensively investigated in Laurent and Pham, Biagini et al., Heath et al., Hobson, and Henderson et al. (2003), among others. The minimal entropy martingale measure can be linked with the option valuation problem under an exponential utility function with constant absolute risk aversion; see, for example, Delbaen et al. (2002), Rheinländer (2002), and Hobson. Shephard (2005) mentioned that most studies focus on pricing options under continuous-time SV models, with an exception being the early work done by Taylor (1986). Relatively little work has been done on pricing options under discrete-time SV models; see, for example, Taylor (1986, 1994) and Amin and Ng (1993).

The Esscher transform is a well-known tool in actuarial science and a popular tool for calculating premiums in the actuarial profession. Gerber and Shiu (1994a) pioneered the use of the Esscher transform for option valuation. Their seminal work highlights the interplay between actuarial pricing and financial pricing. They justified the use of the Esscher transform for determining an equivalent martingale measure by the maximization of the expected power utility. This is consistent with the marginal substitution argument introduced by Davis (1997) when a power utility function is adopted. Carr et al. (2003) discussed the use of the Esscher transform for determining an equivalent martingale measure in the context of continuous-time SV models with Lévy processes. Miyahara (2004) investigated the Esscher transform and the minimal entropy martingale measure (MEMM) in the context of a geometric Lévy process. He pointed out that the Esscher-transformed martingale measure for simple returns corresponding to the MEMM is different from the Esscher-transformed martingale measure for compound returns (ESSMM) adopted in Gerber and Shiu (1994a) for option valuation.

This paper introduces an option valuation model under a general discrete-time ARV model with stock innovations and volatility innovations having infinitely divisible distributions. It employs a modified version of the Esscher transform to determine an equivalent martingale measure under an incomplete market described by the ARV model. In particular, a discrete-time stochastic conditional Esscher transform is presented that modifies the conditional Esscher transform introduced by Bühlmann et al. (1998) for choosing the martingale pricing measure. In a recent paper by Siu, Tong, and Yang (2004), the conditional Esscher transform has been adopted to price derivative instruments under GARCH models. The discrete-time stochastic conditional Esscher transform provides a convenient and flexible way to determine an equivalent martingale measure in the context of the discrete-time ARV model. Some parametric ARV models are considered, namely, the lognormal ARV models, the jump-type Poisson ARV models, and the gamma ARV models, and a Monte Carlo experiment for some of the parametric ARV models is conducted.

The next section presents the general setup of the ARV model and illustrates the use of the discrete-time stochastic conditional Esscher transform for pricing options under the model. Section 3 considers various parametric cases of the present model. In particular, the lognormal ARV models, Poisson ARV models, and gamma ARV models are considered. A Monte Carlo experiment for some of the parametric ARV models is conducted in Section 4. The final section suggests some potential topics for further research.

2. Model Dynamics and Option Valuation

This section considers the option valuation problem under a version of the discrete-time autoregressive random volatility (ARV) model proposed by Taylor (1980, 1982, 1986) with general stock and volatility innovations. The market described by the discrete-time ARV model is incomplete in general. Hence, perfect hedging is impossible, and there is an infinitude of equivalent martingale measures. Different approaches have been proposed in the literature to resolve the puzzle of determining an equivalent martingale measure in the incomplete market setting. A solution is provided here by a discrete-time stochastic conditional Esscher transform since it is a convenient and flexible tool for pricing options under ARV models with different parametric assumptions for stock innovations and volatility innovations. In the sequel, the main setup of the model is presented.

2.1 Asset Price Dynamics

Let $\mathcal{T}_0$ denote the time index set $\{0, 1, 2, \ldots, T\}$ of the model, and $\mathcal{T}$ the time index set $\{1, 2, \ldots, T\}$. Fix a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is the physical probability measure. Write r_t for the rate of interest of the bond B over the t-th trading period, for each $t \in \mathcal{T}$. The dynamic of the bond price is given by

$$B_t = B_{t-1}e^{r_t}, \qquad t \in \mathcal{T}, \quad B_0 = 1. \tag{2.1}$$

Suppose λ_t denotes the unit risk premium representing a preference parameter over the t-th trading period, for each $t \in \mathcal{T}$. Note that r_t and λ_t are assumed to be deterministic functions of time t and that $r_t > 0$, for each $t \in \mathcal{T}$. In practice, it is more realistic to suppose that the interest rate changes randomly over time and consider the pricing of an option under a discrete-time ARV model with stochastic interest rates.

Let $S := \{S_t\}_{t\in\mathcal{T}_0}$ denote the price process of the underlying risky asset, where $S_0 = s$, $\mathcal{P}$-a.s., and s is a known positive constant. Suppose Y_t denotes the logarithmic return $\ln(S_t/S_{t-1})$ of the underlying risky asset over the t-th trading period, where $Y_0 = 0$, $\mathcal{P}$-a.s. Then assume that the dynamic of $\{Y_t\}_{t\in\mathcal{T}}$ is governed by the following ARV model:

$$Y_t = r_t + \lambda_t V_t - \frac{1}{2}V_t^2 + V_t U_t,$$

$$\ln V_t - \alpha = \phi(\ln V_{t-1} - \alpha) + \sigma\eta_t, \tag{2.2}$$

where

1. $V := \{V_t\}$ is the stochastic volatility process; that is, given $V_t = v_t$, the standard deviation of Y_t is v_t.
2. $\{U_t\}$ and $\{\eta_t\}$ are two independent stochastic processes.
3. $\{U_t\}$ are i.i.d. and $U_t \sim D_U(0, 1)$, where $D_U(0, 1)$ represents a generic distribution with mean zero and unit variance.
4. $\{\eta_t\}$ are i.i.d. and $\eta_t \sim D_\eta(0, 1)$, where $D_\eta(0, 1)$ represents another generic distribution with mean zero and unit variance.
5. $\phi < 1$ for strict stationarity of the logarithmic stochastic volatility process $\{\ln V_t\}$.
6. The mean and variance of $\ln V_t$ given $V_0 = v_0$ are $\phi^t \ln v_0 + \alpha(1 - \phi^t)$ and $\sigma^2 \sum_{k=0}^{t-1} \phi^{2k}$, respectively, where v_0 is assumed to be a fixed positive constant.
7. The stationary unconditional mean and variance of $\ln V_t$ are α and β^2, respectively (see Taylor 1986).
8. σ is interpreted as the volatility of a volatility, and $\sigma = \beta\sqrt{1 - \phi^2}$.

Since Y_t depends on a product of two independent stochastic processes, it is also called a product process in the literature (see Taylor 1980, 1982, 1986). The product process describes the change in conditional volatility levels caused by factors independent of prices. For each $t \in \mathcal{T}$, the innovation η_t will cause the variations in the levels of market activity described by the conditional variance V_t. An independent innovation U_t describes the reaction of the price at a given level of market activity V_t. The underlying concept behind the product process is consistent with the information counting model described in Clark (1973) and Tauchen and Pitts (1983). For details, see Taylor (1986, 1994). Note that the drift term $r_t + \lambda_t V_t - \frac{1}{2}V_t^2$ in equation (2.2) is set to simplify the analysis in the sequel. The same drift term for the logarithmic return has been adopted in the GARCH models in Duan (1995) and Siu, Tong, and Yang (2004).

Option valuation in a discrete-time economy has been studied in the literature. The pioneering work was done in the seminal paper by Rubinstein (1976). He derived the Black-Scholes-Merton option pricing formula in a discrete-time economy by assuming that the stock-price process and the state-price density process are bivariate lognormal. Brennan (1979) extended Rubinstein's model and obtained the risk-neutral valuation relationship (RNVR) by considering the multiperiod framework under either bivariate normality or bivariate lognormality of the stock price and the logarithm of the state-price density. Stapleton and Subrahmanyam (1984) provided a generalization of the results by Rubinstein (1976) and Brennan to the valuation of multistate options under the assumptions of multivariate normality and multivariate lognormality. Amin and Ng (1993) provided a generalization of many option pricing formulas for stochastic interest rate processes and stochastic volatility processes when the bivariate normality assumption for the random shocks is imposed. Elliott and Madan (1998) proposed a method for picking an equivalent martingale measure for a discrete-time and continuous-state economy based on the extended Girsanov's theorem. Schroder (2004) developed a very general discrete-time risk-neutral valuation by imposing a simple restriction on the state-price density process. The derivative pricing model by Schroder is preference-free. Here different specifications of the ARV model are provided from those in Amin and Ng. An equivalent martingale measure by a discrete-time stochastic conditional Esscher transform is determined. The result is applicable when both the random shocks for the stock returns and the logarithmic volatility levels are not normally distributed. In the sequel, the discrete-time stochastic conditional Esscher transform is defined.

2.2 An Equivalent Martingale Measure

First, let us describe the information structure in the model. Let $\mathcal{F}^V := \{\mathcal{F}_t^V\}_{t\in\mathcal{T}_0}$ and $\mathcal{F}^Y := \{\mathcal{F}_t^Y\}_{t\in\mathcal{T}_0}$ denote the $\mathcal{P}$-augmentation of the natural filtrations generated by the volatility process V and the stock return process Y, respectively, and $\mathcal{G}_{t-1}$ the $\mathcal{P}$-augmentation of the σ-fields $\mathcal{F}_{t-1}^Y \vee \mathcal{F}_T^V$, for each $t \in \mathcal{T}$. For each $t \in \mathcal{T}$, let $\mathcal{K}(\Theta_t|\mathcal{F}_T^V)$ denote the cumulant-generating function of the conditional distribution of Y_t given $\mathcal{F}_T^V$ associated with the parameter Θ_t under $\mathcal{P}$, that is,

$$\mathcal{K}(\Theta_t|\mathcal{F}_T^V) := \ln \mathrm{E}[e^{\Theta_t Y_t}|\mathcal{F}_T^V], \qquad t \in \mathcal{T}, \tag{2.3}$$

where $\mathrm{E}[\cdot]$ denotes an expectation with respect to $\mathcal{P}$.

For each $t \in \mathcal{T}$, Θ_t is a function of $\mathcal{F}_T^V$, all the volatilities from equation (2.3). We will assume that there exists a Θ_t such that $\mathcal{K}(\Theta_t|\mathcal{F}_T^V) < \infty$, for each $t \in \mathcal{T}$.

Define a sequence of random variables $\{\Lambda_t^V\}_{t\in\mathcal{T}}$ as follows:

$$\Lambda_t^V := \exp\left\{\sum_{k=1}^{t} [\Theta_k Y_k - \mathcal{K}(\Theta_k|\mathcal{F}_T^V)]\right\}. \tag{2.4}$$

Lemma 2.1

Suppose Y_{t+1} is stochastically independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^V$. Then $\{\Lambda_t^V\}_{t\in\mathcal{T}}$ is a $(\mathcal{G}, \mathcal{P})$-martingale.

PROOF

From the definition of Λ^V, Λ_t^V is $\mathcal{G}_t$-measurable, for each $t \in \mathcal{T}$. Since Y_{t+1} is stochastically independent of $\mathcal{F}_t^Y$ given $\mathcal{F}_T^V$,

$$\mathrm{E}\left[\frac{\Lambda_{t+1}^V}{\Lambda_t^V}\middle|\mathcal{G}_t\right] = \mathrm{E}\left[\frac{e^{\Theta_{t+1}Y_{t+1}}}{\mathrm{E}(e^{\Theta_{t+1}Y_{t+1}}|\mathcal{F}_T^V)}\middle|\mathcal{F}_t^Y \vee \mathcal{F}_T^V\right] = \mathrm{E}\left[\frac{e^{\Theta_{t+1}Y_{t+1}}}{\mathrm{E}(e^{\Theta_{t+1}Y_{t+1}}|\mathcal{F}_T^V)}\middle|\mathcal{F}_T^V\right] = 1,\ \mathcal{P}-\text{a.s.} \tag{2.5}$$

Hence, the result follows. □

Then the stochastic conditional Esscher transform $\mathcal{P}_\Theta \sim \mathcal{P}$ on $\mathcal{G}_t$ with respect to the sequence of random variables $(\Theta_1, \Theta_2, \ldots, \Theta_t)$ is given by

$$\left.\frac{d\mathcal{P}_\Theta}{d\mathcal{P}}\right|_{\mathcal{G}_t} := \Lambda_t^V, \qquad t \in \mathcal{T}. \tag{2.6}$$

Let $\mathcal{K}(u;\, \Theta_t|\mathcal{F}_T^V)$ denote the cumulant-generating function of the conditional distribution of Y_t given $\mathcal{F}_T^V$ under $\mathcal{P}_\Theta$; that is,

$$\mathcal{K}(u;\, \Theta_t|\mathcal{F}_T^V) := \ln \mathrm{E}^{\Theta}[e^{uY_t}|\mathcal{F}_T^V]. \tag{2.7}$$

Then we have the following lemma:

Lemma 2.2

$$\mathcal{K}(u;\, \Theta_t|\mathcal{F}_T^V) = \mathcal{K}(u + \Theta_t|\mathcal{F}_T^V) - \mathcal{K}(\Theta_t|\mathcal{F}_T^V). \tag{2.8}$$

PROOF

By the Bayes rule, the moment-generating function of Y_t given $\mathcal{F}_T^V$ under $\mathcal{P}_\Theta$ is

$$\mathrm{E}^{\Theta}[e^{uY_t}|\mathcal{F}_T^Y] = \mathrm{E}^{\Theta}[e^{uY_t}|\mathcal{F}_{t-1}^Y \vee \mathcal{F}_T^V] = \frac{\mathrm{E}[\Lambda_t^V\, e^{uY_t}|\mathcal{G}_{t-1}]}{\mathrm{E}[\Lambda_t^V|\mathcal{G}_{t-1}]} = \mathrm{E}\left[\frac{\Lambda_t^V}{\Lambda_{t-1}^V}\, e^{uY_t}|\mathcal{G}_{t-1}\right] = \frac{\mathrm{E}[e^{(u+\Theta_t)Y_t}|\mathcal{F}_{t-1}^Y \vee \mathcal{F}_T^V]}{\mathrm{E}[e^{\Theta_tY_t}|\mathcal{F}_T^V]}$$
$$= \frac{\mathrm{E}[e^{(u+\Theta_t)Y_t}|\mathcal{F}_T^V]}{\mathrm{E}[e^{\Theta_tY_t}|\mathcal{F}_T^V]}. \tag{2.9}$$

Hence, the result follows from taking the logarithm. □

Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983) established the relationship between the absence of arbitrage and the existence of an equivalent martingale measure under which the discounted asset price process is a martingale. This is called the fundamental theorem of asset pricing. It was then extended by several authors, including Dybvig and Ross (1987), Back and Pliska (1991), Schachermayer (1992), and Delbaen and Schachermayer (1994). Back and Pliska showed the equivalence between the absence of arbitrage and the existence of an equivalent martingale measure in a discrete-time and infinite-state-space setting. It has been noted in Delbaen and Schachermayer that the equivalence between the absence of arbitrage and the existence of an equivalent martingale measure is not always true in a continuous-time setting. They used the term ''essentially equivalent'' instead of ''equivalent'' to describe a martingale measure.

Now, we will determine an equivalent martingale measure using the stochastic conditional Esscher transform $\mathcal{P}_\Theta$. A suffcient condition on the sequence Θ is provided for $\mathcal{P}_\Theta$ to be an equivalent martingale measure in the following proposition.

Proposition 2.3

Suppose Θ_t satisfies the following condition:

$$\mathcal{K}(\Theta_t + 1|\mathcal{F}_T^V) - \mathcal{K}(\Theta_t|\mathcal{F}_T^V) = r_t, \text{ for each } t \in \mathcal{T}. \tag{2.10}$$

Then the discounted price process $\{S_t/B_t\}_{t\in\mathcal{T}}$ is a $(\mathcal{G}, \mathcal{P}_\Theta)$-martingale.

PROOF

$$\mathrm{E}^{\Theta}\left[\frac{S_{t+1}}{B_{t+1}}\bigg|\mathcal{G}_t\right] = \frac{S_t}{B_t}\,\mathrm{E}^{\Theta}[e^{Y_{t+1}-r_{t+1}}|\mathcal{G}_t] = \frac{S_t}{B_t}\,\mathrm{E}\left[\left(\frac{\Lambda_{t+1}^V}{\Lambda_t^V}\right) e^{Y_{t+1}-r_{t+1}}|\mathcal{G}_t\right] = \frac{S_t}{B_t}\, e^{-r_{t+1}}\, \frac{\mathrm{E}[e^{(\Theta_{t+1}+1)Y_{t+1}}|\mathcal{F}_t^Y \vee \mathcal{F}_T^V]}{\mathrm{E}[e^{\Theta_{t+1}Y_{t+1}}|\mathcal{F}_T^V]}$$
$$= \frac{S_t}{B_t}\, e^{-r_{t+1}}\, \frac{\mathrm{E}[e^{(\Theta_{t+1}+1)Y_{t+1}}|\mathcal{F}_T^V]}{\mathrm{E}[e^{\Theta_{t+1}Y_{t+1}}|\mathcal{F}_T^V]} = \frac{S_t}{B_t}\exp[-r_{t+1} + \mathcal{K}(\Theta_{t+1} + 1|\mathcal{F}_T^V) - \mathcal{K}(\Theta_{t+1}|\mathcal{F}_T^V)]$$
$$= \frac{S_t}{B_t},\ \mathcal{P} - \text{a.s.} \tag{2.11}$$

Hence, the result follows. □

Note that Θ_t satisfying equation (2.10) is unique (see Gerber and Shiu 1994a, b).

2.3 Valuation

In this paper we will call Θ_t and $\mathcal{P}_\Theta$ a risk-neutral stochastic conditional Esscher transform and a risk-neutralized stochastic conditional Esscher parameter, respectively. The martingale condition on $\mathcal{G}$ is the one when information about the values of the volatility process V is accessible to the economic agent. If $\mathcal{P}_\Theta$ satisfies the martingale condition on $\mathcal{G}$, it also satisfies that without knowledge of the volatility level. This can be verified by applying the tower property of a conditional expectation.

Suppose that knowledge of the values of the stochastic volatility process is given. Then a price of a European option with payoff $P(S_T)$ at the maturity time T is given by

$$P(t, T|\mathcal{G}_t) = \mathrm{E}^\Theta[e^{-\Sigma_{k=t+1}^T r_k} P(S_T)|\mathcal{G}_t]. \tag{2.12}$$

Let $V_{t,T} := (V_t, V_{t+1}, \ldots, V_T)$, for each $t \in \mathcal{T}$. Because of the Markov property of $\{S_t\}$ and $\{V_t\}$,

$$P(t, T|S_t, V_{t,T}) = \mathrm{E}^\Theta[e^{-\Sigma_{k=t+1}^T r_k} P(S_T)|S_t, V_{t,T}]. \tag{2.13}$$

Note that this pricing by the Esscher transform is not unique. There are other possible ways to determine a price of the option, such as the minimum variance hedging in Duffie and Richardson (1992) and Schweizer (1992).

In practice, one does not have knowledge about future values of the stochastic volatility process and, hence, the price $P(t, T|S_t, V_{t,T})$. One therefore can provide an estimate of the price by taking a second expectation for $P(t, T|S_t, V_{t,T})$ with respect to the joint probability density function (pdf) $\phi_{V_{t+1,T}|v_t}(\mathbf{v}_{t+1,T})$ for $\{V_k\}_{k=t+1,t+2,\ldots,T}$ given $V_t = v_t$ under $\mathcal{P}_\Theta$, where $\mathbf{v}_{t+1,T} := (v_{t+1}, v_{t+2}, \ldots, v_T)$. Because of the Markovian nature of the latent volatility process, $\phi_{V_{t+1,T}|v_t}(\mathbf{v}_{t+1,T})$ can be written as

$$\phi_{V_{t+1,T}|v_t}(\mathbf{v}_{t+1,T}) = \prod_{k=t+1}^{T} \phi_{V_k|v_{k-1}}(v_k), \tag{2.14}$$

where $\phi_{V_k|v_{k-1}}(v_k)$ is the conditional probability density function of V_k given $V_{k-1} = v_{k-1}$ under $\mathcal{P}_\Theta$, for each $k = t + 1, t + 2, \ldots, T$.

Let $\mathcal{R}^+$ denote the positive part of a real line, and $(\mathcal{R}^+)^n$ the n-tuples of $\mathcal{R}^+$. Suppose that the current values of S_t and V_t are given as s_t and v_t, respectively. Then a price of the option is evaluated as

$$\begin{aligned} P(t, T|s_t, v_t) &= \int_{(\mathcal{R}^+)^{T-t}} P(t, T|s_t, v_t, \mathbf{v}_{t+1,T})\phi_{V_{t+1,T}|v_t}(\mathbf{v}_{t+1,T})\, d\mathbf{v}_{t+1,T} \\ &= \int_{(\mathcal{R}^+)^{T-t}} P(t, T|s_t, v_t, \mathbf{v}_{t+1,T}) \times \left[\prod_{k=t+1}^{T} \phi_{V_k|v_{k-1}}(v_k)\right] d\mathbf{v}_{t+1,T}. \end{aligned} \tag{2.15}$$

Now, let us consider a standard European call option with strike price K and maturity at time T. Let $\tilde{f}_{X_{t+1,T}|s_t,v_t,\mathbf{v}_{t+1,T}}(x)$ denote the pdf of $X_{t+1,T} := \Sigma_{k=t+1}^T Y_k$ given $S_t = s_t$, $V_t = v_t$, and $\mathbf{V}_{t+1,T} = \mathbf{v}_{t+1,T}$ under $\mathcal{P}_\Theta$. Write κ_t for $\ln(K/s_t)$. Then it is not difficult to see that the call price given $S_t = s_t$, $V_t = v_t$, and $\mathbf{V}_{t+1,T} = \mathbf{v}_{t+1,T}$ is given by

$$P(t, T|s_t, v_t, \mathbf{v}_{t+1,T}) = s_t e^{-\Sigma_{k=t+1}^T r_k} \int_{\kappa_t}^{\infty} e^x \tilde{f}_{X_{t+1,T}|s_t,v_t,\mathbf{v}_{t+1,T}}(x)\, dx - K e^{-\Sigma_{k=t+1}^T r_k} \int_{\kappa_t}^{\infty} \tilde{f}_{X_{t+1,T}|s_t,v_t,\mathbf{v}_{t+1,T}}(x)\, dx. \tag{2.16}$$

Hence, the call price given $S_t = s_t$ and $V_t = v_t$ is given by

$$\begin{aligned} P(t, T|s_t, v_t) &= s_t e^{-\Sigma_{k=t+1}^T r_k} \int_{(\mathcal{R}^+)^{T-t}} \int_{\kappa_t}^{\infty} e^x \tilde{f}_{X_{t+1,T}|s_t,v_t,\mathbf{v}_{t+1,T}}(x) \times \left[\prod_{k=t+1}^{T} \phi_{V_k|v_{k-1}}(v_k)\right] dx\, d\mathbf{v}_{t+1,T} \\ &\quad - K e^{-\Sigma_{k=t+1}^T r_k} \int_{(\mathcal{R}^+)^{T-t}} \int_{\kappa_t}^{\infty} \tilde{f}_{X_{t+1,T}|s_t,v_t,\mathbf{v}_{t+1,T}}(x) \times \left[\prod_{k=t+1}^{T} \phi_{V_k|v_{k-1}}(v_k)\right] dx\, d\mathbf{v}_{t+1,T}. \end{aligned} \tag{2.17}$$

More explicit formulas for equation (2.17) for certain parametric cases are provided in Section 3.

3. SOME PARAMETRIC CASES

In this section we will consider some parametric cases of the general discrete-time ARV model in the last section, namely, the lognormal ARV model, the gamma ARV model, and the Poisson ARV model.

First, we will consider the lognormal ARV model in which the innovations of the stock returns $\{U_t\}$ are independent and identically distributed with the common distribution being a standard normal distribution. The pricing result for the lognormal ARV model does not depend on any assumptions about the innovations $\{\eta_t\}$ of the stochastic volatility process. It holds true for any parametric assumptions for the innovations $\{\eta_t\}$. In the case that $\{\eta_t\}$ are normally distributed, the lognormal ARV model coincides with the first-order lognormal ARV model proposed by Taylor. Shephard (2005) pointed out that the lognormal ARV model by Taylor can be considered an Euler discretization of the continuous-time Gaussian Ornstein-Uhlenbeck (OU) process for stochastic volatility. Hence, the pricing result in this case is a discrete-time version of the pricing result under the corresponding continuous-time OU process for stochastic volatility. Then, we will discuss the gamma ARV model in which $\{U_t\}$ are independent and identically distributed with the common distribution being a shifted-gamma distribution with zero mean and unit variance. The pricing result for the gamma ARV model holds true for any parametric assumptions for the innovations $\{\eta_t\}$. Finally, we will consider the jump-type ARV model, in particular, the Poisson ARV model, in which $\{U_t\}$ are independent and identically distributed with the common distribution being a shifted-Poisson distribution with zero mean and unit variance.

3.1 Lognormal ARV Model

We will first consider the pricing of a European call option under the lognormal ARV model without imposing any parametric assumptions for the innovations $\{\eta_t\}$ of the stochastic volatility process. Then we will derive the pricing formulas for the call under the lognormal ARV models with different parametric assumptions on the innovations $\{\eta_t\}$, namely, the normal distribution, the shifted-gamma distribution, and the shifted-Poisson distribution. First, we will suppose that the dynamic of the logarithmic return $\{Y_t\}_{t\in\mathcal{T}}$ is governed by the nonzero drift lognormal ARV model.

For each $t \in \mathcal{T}$, $Y_t|\mathcal{F}_T^V \sim N(r_t + \lambda_t V_t - \frac{1}{2}V_t^2, V_t^2)$ under $\mathcal{P}$, where $N(r_t + \lambda_t V_t - \frac{1}{2}V_t^2, V_t^2)$ is a normal distribution with mean $r_t + \lambda_t V_t - \frac{1}{2}V_t^2$ and variance V_t^2. Hence, the cumulant-generating function $\mathcal{K}(\Theta_t|\mathcal{F}_T^V)$ is given by

$$\mathcal{K}(\Theta_t|\mathcal{F}_T^V) = \Theta_t\left(r_t + \lambda_t V_t - \frac{1}{2} V_t^2\right) + \frac{1}{2} \Theta_t^2 V_t^2. \tag{3.1}$$

Then the martingale condition implies that the risk-neutral stochastic conditional Esscher parameter Θ_t at time $t \in \mathcal{T}$ can be determined uniquely as follows:

$$\Theta_t = -\frac{\lambda_t}{V_t}. \tag{3.2}$$

Now, the cumulant-generating function $\mathcal{K}(u; \Theta_t|\mathcal{F}_T^V)$ is given by

$$\mathcal{K}(u; \Theta_t|\mathcal{F}_T^V) = u\left(r_t - \frac{1}{2} V_t^2\right) + \frac{1}{2} u^2 V_t^2. \tag{3.3}$$

Hence, the conditional distribution of Y_t given $\mathcal{F}_T^V$ under $\mathcal{P}_\Theta$ is a normal distribution with mean $r_t - \frac{1}{2}V_t^2$ and variance V_t^2. Note that the dynamic of the logarithmic volatility level is unaltered by changing the measures from $\mathcal{P}$ to $\mathcal{P}_\Theta$ since $\{U_t\}$ and $\{\eta_t\}$ are independent under $\mathcal{P}$.

Conditional on $\mathcal{F}_T^V$, $Y_1, Y_2, \ldots, Y_T$ are independent under $\mathcal{P}_\Theta$. For each $t \in \mathcal{T}_0$ and $k = t + 1, t + 2, \ldots, T$, $Y_k|\mathcal{G}_t \sim N(r_k - \frac{1}{2}V_k^2, V_k^2)$ under $\mathcal{P}_\Theta$. Then $Y_k|\mathbf{V}_{t+1,T} = \mathbf{v}_{t+1,T} \sim N(r_k - \frac{1}{2}v_k^2, v_k^2)$. Let $H_k := \ln V_k$. Write $\mathbf{H}_{t+1,T} := (H_{t+1}, H_{t+2}, \ldots, H_T)$ and $\mathbf{h}_{t+1,T} := (h_{t+1}, h_{t+2}, \ldots, h_T)$. Then $Y_k|\mathbf{H}_{t+1,T} = \mathbf{h}_{t+1,T} \sim N(r_k - \frac{1}{2}e^{2h_k}, e^{2h_k})$ under $\mathcal{P}_\Theta$. The result of the risk-neutral dynamic of the logarithmic return $\{Y_t\}$ holds true for any parametric assumptions of the innovations $\{\eta_t\}$ for the stochastic volatility process. It can be used for the pricing of an option under the lognormal ARV model with different parametric assumptions for $\{\eta_t\}$. We will consider the pricing of a European call option. In this case, the call price given $S_t = s_t$, $H_t = h_t$, $\mathbf{H}_{t+1,T} = \mathbf{h}_{t+1,T}$ is given by

$$P^C(t, T|s_t, h_t, \mathbf{h}_{t+1,T}) = s_t\Phi(d_1(\mathbf{h}_{t+1,T})) - Ke^{-\sum_{k=t+1}^{T} r_k}\Phi(d_2(\mathbf{h}_{t+1,T})), \tag{3.4}$$

where

$$d_1(\mathbf{h}_{t+1,T}) = \frac{\ln(s_t/K) + \sum_{k=t+1}^{T}(r_k + \frac{1}{2}e^{2h_k})}{\sqrt{\sum_{k=t+1}^{T} e^{2h_k}}}, \tag{3.5}$$

and

$$d_2(\mathbf{h}_{t+1,T}) = d_1(\mathbf{h}_{t+1,T}) - \sqrt{\sum_{k=t+1}^{T} e^{2h_k}}. \tag{3.6}$$

The call price given $S_t = s_t$ and $H_t = h_t$ is given by

$$\begin{aligned} P^C(t, T|s_t, h_t) = {} & s_t \int_{\mathcal{R}^{T-t}} \Phi(d_1(\mathbf{h}_{t+1,T})) \prod_{k=t+1}^{T} \phi_{H_k|h_{k-1}}(h_k)\, d\mathbf{h}_{t+1,T} \\ & - K \int_{\mathcal{R}^{T-t}} e^{-\sum_{k=t+1}^{T} r_k}\Phi(d_2(\mathbf{h}_{t+1,T})) \times \prod_{k=t+1}^{T} \phi_{H_k|h_{k-1}}(h_k)\, d\mathbf{h}_{t+1,T}\,. \end{aligned} \tag{3.7}$$

Then we can derive the pricing formulas for the call under the lognormal ARV model with different parametric assumptions for the innovations $\{\eta_t\}$.

First, we will consider the case of normal innovations $\{\eta_t\}$. In this case the conditional pdf of H_k given $H_{k-1} = h_{k-1}$ is given by

$$\phi_{H_k|h_{k-1}}(h_k) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}\,[h_k - \alpha - \phi(h_{k-1} - \alpha)]^2\right\}. \tag{3.8}$$

Hence,

$$\begin{aligned} P^C(t, T|s_t, h_t) = {} & \frac{s_t}{(2\pi\sigma^2)^{(T-t)/2}} \int_{\mathcal{R}^{T-t}} \Phi(d_1(\mathbf{h}_{t+1,T})) \\ & \times \exp\left\{\frac{-1}{2\sigma^2} \sum_{k=t+1}^{T} [h_k - \alpha - \phi(h_{k-1} - \alpha)]^2\right\} d\mathbf{h}_{t+1,T} \\ & - \frac{Ke^{-\sum_{k=t+1}^{T} r_k}}{(2\pi\sigma^2)^{(T-t)/2}} \int_{\mathcal{R}^{T-t}} \Phi(d_2(\mathbf{h}_{t+1,T})) \\ & \times \exp\left\{\frac{-1}{2\sigma^2} \sum_{k=t+1}^{T} [h_k - \alpha - \phi(h_{k-1} - \alpha)]^2\right\} d\mathbf{h}_{t+1,T}\,. \end{aligned} \tag{3.9}$$

Now, suppose $\{\eta_t\}$ are independent and identically distributed with the common distribution being a shifted-Poisson distribution with zero mean and unit variance. In this case, we can capture the purely jump behavior in the stochastic volatility level. Let $\{X_t\}$ be a sequence of independent and identically distributed Poisson random variables with the common intensity parameter λ_p. Then

$$\eta_t := \frac{X_t - \lambda_p}{\sqrt{\lambda_p}}, \qquad t \in \mathcal{T}. \tag{3.10}$$

Hence, we can write the dynamic of the logarithmic volatility H in the following form:

$$H_t = \alpha + \phi(H_{t-1} - \alpha) - \sqrt{\lambda_p}\sigma + \frac{\sigma}{\sqrt{\lambda_p}} X_t,\ t \in \mathcal{T}. \tag{3.11}$$

Let $\tilde{x}(h_k)$ denote the greatest integral part of the value of the following function $x(h_k)$ of h_k:

$$x(h_k) = \left(\frac{\sqrt{\lambda_p}}{\sigma}\right)[h_k - \alpha - \phi(h_{k-1} - \alpha) + \sqrt{\lambda_p}\sigma]. \tag{3.12}$$

Then the conditional pdf of H_k given $H_{k-1} = h_{k-1}$ is given by

$$\phi_{H_k|h_{k-1}}(h_k) = \frac{(\lambda_p)^{\tilde{x}(h_k)+1/2}e^{-\lambda_p}}{\sigma\tilde{x}(h_k)!}. \tag{3.13}$$

Hence,

$$\begin{aligned} P^C(t, T|s_t, h_t) &= s_t e^{-(T-t)\lambda_p} \int_{\mathcal{R}^{T-t}} \Phi(d_1(\mathbf{h}_{t+1,T})) \times \left\{\frac{(\lambda_p)^{\Sigma_{k=t+1}^{T}\tilde{x}(h_k)+(T-t)/2}}{\sigma^{T-t}\prod_{k=t+1}^{T}\tilde{x}(h_k)!}\right\} d\mathbf{h}_{t+1,T} \\ &\quad - Ke^{-\Sigma_{k=t+1}^{T} r_k-(T-t)\lambda_p} \int_{\mathcal{R}^{T-t}} \Phi(d_2(\mathbf{h}_{t+1,T})) \times \left\{\frac{(\lambda_p)^{\Sigma_{k=t+1}^{T}\tilde{x}(h_k)+(T-t)/2}}{\sigma^{T-t}\prod_{k=t+1}^{T}\tilde{x}(h_k)!}\right\} d\mathbf{h}_{t+1,T}. \end{aligned} \tag{3.14}$$

Finally, we will assume that $\{\eta_t\}$ are independent and identically distributed with the common distribution being a shifted-gamma distribution with zero mean and unit variance. Let $\{X_t\}$ be a sequence of independent and identically distributed gamma random variables with the shape parameter a_1 and scale parameter b_1. Then

$$\eta_t := \frac{X_t - \dfrac{a_1}{b_1}}{\sqrt{\dfrac{a_1}{b_1^2}}}, \qquad t \in \mathcal{T}. \tag{3.15}$$

Hence, we can write the dynamic of the logarithmic volatility H in the following form:

$$H_t - \alpha = \phi(H_{t-1} - \alpha) - \sqrt{a_1}\sigma + \frac{b_1\sigma}{\sqrt{a_1}} X_t. \tag{3.16}$$

Now, let $x(h_k)$ denote a function of h_k given as follows:

$$x(h_k) = \frac{\sqrt{a_1}}{b_1\sigma}[h_k - \alpha - \phi(h_{k-1} - \alpha) + \sqrt{a_1}\sigma]. \tag{3.17}$$

Then the conditional pdf of H_k given $H_{k-1} = h_{k-1}$ is given by

$$\phi_{H_k|h_{k-1}}(h_k) = \frac{\sqrt{a_1}b_1^{a_1-1}}{\sigma\Gamma(a_1)}(x(h_k))^{a_1-1}e^{-b_1 x(h_k)}. \tag{3.18}$$

Hence,

$$\begin{aligned} P^C(t, T|s_t, h_t) &= s_t\left(\frac{\sqrt{a_1}b_1^{a_1-1}}{\sigma\Gamma(a_1)}\right)^{T-t} \int_{\mathcal{R}^{T-t}} \Phi(d_1(\mathbf{h}_{t+1,T})) \times \left[\prod_{k=t+1}^{T}(x(h_k))^{a_1-1}\right] e^{-b_1\Sigma_{k=t+1}^{T} x(h_k)}\, d\mathbf{h}_{t+1,T} \\ &\quad - Ke^{\Sigma_{k=t+1}^{T} r_k}\left(\frac{\sqrt{a_1}b_1^{a_1-1}}{\sigma\Gamma(a_1)}\right)^{T-t} \int_{\mathcal{R}^{T-t}} \Phi(d_2(\mathbf{h}_{t+1,T})) \\ &\quad \times \left[\prod_{k=t+1}^{T}(x(h_k))^{a_1-1}\right] e^{-b_1\Sigma_{k=t+1}^{T} x(h_k)}\, d\mathbf{h}_{t+1,T}. \end{aligned} \tag{3.19}$$

3.2 Gamma ARV Model

In this subsection we will consider the pricing of a European call option under the gamma ARV model. First, we will not impose any parametric assumptions for $\{\eta_t\}$. Then we will derive the pricing formulas for the call under the gamma ARV models with different parametric assumptions on $\{\eta_t\}$, namely, the normal distribution, the shifted-gamma distribution, and the shifted-Poisson distribution. Note that the gamma ARV model can incorporate the skewed behavior of the underlying risky asset's returns. The inverse Gaussian distribution is a popular distribution, which can incorporate the skewed behavior of the asset's returns. Since the treatment of the inverse-Gaussian ARV model is similar to that of the gamma ARV model, we will not repeat here the derivation of the pricing results for the inverse-Gaussian ARV model.

First, suppose that random variables $\{X_t\}_{t\in\mathcal{T}}$ are i.i.d. and follow a gamma distribution $Ga(a, b)$ with shape parameter a and scale parameter b. Further assume that

$$U_t = \frac{X_t - \frac{a}{b}}{\sqrt{\frac{a}{b^2}}}, \qquad t \in \mathcal{T}. \tag{3.20}$$

Note that $\{U_t\}$ are i.i.d. and that U_t follows a shifted-gamma distribution with zero mean and unit variance under $\mathcal{P}$. Then assume that the dynamic of $\{Y_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}$ is governed by the ARV model with U_t having a shifted-gamma distribution.

In this case we also can write the dynamic of $\{Y_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}$ as follows:

$$Y_t = r_t + \lambda_t V_t - \frac{1}{2} V_t^2 - \sqrt{a} V_t + b \frac{V_t}{\sqrt{a}} X_t. \tag{3.21}$$

Given $\mathcal{F}_T^V$, $b\, V_t/\sqrt{a}\, X_t \sim Ga(a, \sqrt{a}/V_t)$ under $\mathcal{P}$, for each $t \in \mathcal{T}$. Hence, $\mathcal{K}(\Theta_t|\mathcal{F}_T^V)$ is given by

$$\mathcal{K}(\Theta_t|\mathcal{F}_T^V) = a\left[\ln\left(\frac{\sqrt{a}}{V_t}\right) - \ln\left(\frac{\sqrt{a}}{V_t} - \Theta_t\right)\right] + \left(r_t + \lambda_t V_t - \frac{1}{2} V_t^2 - \sqrt{a} V_t\right)\Theta_t. \tag{3.22}$$

Then the martingale condition implies that the risk-neutral stochastic conditional Esscher parameter Θ_t at time $t \in \mathcal{T}$ can be determined uniquely as follows:

$$\Theta_t = \frac{\sqrt{a}}{V_t} - \left[1 - \exp\left(\frac{\lambda_t V_t - \frac{1}{2}V_t^2 - \sqrt{a}V_t}{a}\right)\right]^{-1}. \tag{3.23}$$

For each $t \in \mathcal{T}$, define b_t^q as follows:

$$b_t^q := \frac{\sqrt{a}}{V_t} - \Theta_t = \left[1 - \exp\left(\frac{\lambda_t V_t - \frac{1}{2}V_t^2 - \sqrt{a}V_t}{a}\right)\right]^{-1}. \tag{3.24}$$

Then the cumulant-generating function $\mathcal{K}(u; \Theta_t|\mathcal{F}_T^V)$ is given by

$$\mathcal{K}(u; \Theta_t|\mathcal{F}_T^V) = a[\ln(b_t^q) - \ln(b_t^q - u)] + \left(r_t + \lambda_t V_t - \frac{1}{2} V_t^2 - \sqrt{a} V_t\right)u. \tag{3.25}$$

Hence, the conditional distribution of Y_t given $\mathcal{F}_T^V$ under $\mathcal{P}_\Theta$ is a shifted-gamma distribution with shape parameter a, scale parameter b_t^q, and shifted parameter $-r_t - \lambda_t V_t + \frac{1}{2}V_t^2 + \sqrt{a}V_t$. The shape parameter and the shifted parameter are invariant with respect to the change of measures by the Esscher transform. The risk-neutral and the real-world dynamics of the asset's returns do not depend on the real-world scale parameter b. This means that only one parameter, namely, the shape parameter a, is required to incorporate various degrees of skewness in the distribution of the asset's returns. Again, the dynamic of the logarithmic volatility level is unaltered by changing the measures from $\mathcal{P}$ to $\mathcal{P}_\Theta$.

Conditional on $\mathcal{F}_T^V$, $Y_1, Y_2, \ldots, Y_T$ are independent under $\mathcal{P}_\Theta$. For each $t \in \mathcal{T}_0$ and $k = t + 1, t + 2, \ldots, T$, $Y_k|\mathcal{G}_t$ follows a shifted-gamma distribution with shape parameter a, scale parameter b_k^q, and shifted parameter $-r_k - \lambda_k V_k + \frac{1}{2}V_k^2 + \sqrt{a}V_k$ under $\mathcal{P}_\Theta$. This implies that $Y_k|\mathbf{V}_{t+1,T} = \mathbf{v}_{t+1,T}$ follows a shifted-gamma distribution with shape parameter a, scale parameter b_k^q, and shifted parameter $-r_k -$

$\lambda_k V_k + \frac{1}{2}V_k^2 + \sqrt{a}V_k$ under $\mathcal{P}_\Theta$. This is true for any parametric assumptions of $\{\eta_t\}$. The analysis of different parametric cases for $\{\eta_t\}$ is the same as that in Section 3.1.

3.3 Poisson ARV Model

In this subsection we will consider the pricing of a European call option under the Poisson ARV model. This model can describe the purely jump behavior of the underlying risky asset's returns. First, we will derive the pricing result without imposing any parametric assumptions on $\{\eta_t\}$. Then we will derive the pricing formulas for the call under the Poisson ARV models with different parametric assumptions on $\{\eta_t\}$, namely, the normal distribution, the shifted-gamma distribution, and the shifted-Poisson distribution. In the case when both $\{U_t\}$ and $\{\eta_t\}$ are i.i.d. shifted-Poisson random variables, the purely jump behavior in both the asset's returns and the stochastic volatility levels can be incorporated. This case is called the double-Poisson ARV model or Poisson-Poisson ARV model.

First, suppose that $\{X_t\}_{t\in\mathcal{T}}$ are i.i.d. and follow a Poisson distribution with intensity parameter λ^U. Then further assume that

$$U_t = \frac{X_t - \lambda^U}{\sqrt{\lambda^U}}, \qquad t \in \mathcal{T}. \tag{3.26}$$

Note that $\{U_t\}$ are i.i.d. and that U_t follows a shifted-Poisson distribution with zero mean and unit variance under $\mathcal{P}$. Then, assume that the dynamic of $\{Y_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}$ is governed by the ARV model with U_t having a shifted-Poisson distribution.

In this case, one can also write the dynamic of $\{Y_t\}_{t\in\mathcal{T}}$ under $\mathcal{P}$ as follows:

$$Y_t = r_t + \lambda_t V_t - \frac{1}{2}V_t^2 - \sqrt{\lambda^U}V_t + \frac{V_t}{\sqrt{\lambda^U}}X_t. \tag{3.27}$$

Then $\mathcal{K}(\Theta_t|\mathcal{F}_T^V)$ is given by

$$\mathcal{K}(\Theta_t|\mathcal{F}_T^V) = \lambda^U\left[\exp\left(\frac{\Theta_t V_t}{\sqrt{\lambda^U}}\right) - 1\right] + \left(r_t + \lambda_t V_t - \frac{1}{2}V_t^2 - \sqrt{\lambda^U}V_t\right)\Theta_t. \tag{3.28}$$

From the martingale condition, the risk-neutral stochastic conditional Esscher parameter Θ_t satisfies

$$\lambda^U \exp\left(\frac{\Theta_t V_t}{\sqrt{\lambda^U}}\right)\left[\exp\left(\frac{V_t}{\sqrt{\lambda^U}}\right) - 1\right] = \frac{1}{2}V_t^2 + \sqrt{\lambda^U}V_t - \lambda_t V_t, \qquad t \in \mathcal{T}. \tag{3.29}$$

Let $\lambda_t^\Theta := \lambda^U \exp(\Theta_t V_t/\sqrt{\lambda^U})$, for each $t \in \mathcal{T}$, which represents the risk-neutral jump-intensity parameter at time t. Then,

$$\lambda_t^\Theta = \frac{\frac{1}{2}V_t^2 + \sqrt{\lambda^U}V_t - \lambda_t V_t}{\exp\left(\frac{V_t}{\sqrt{\lambda^U}}\right) - 1}, \tag{3.30}$$

which depends on the volatility level V_t at time t and the real-world jump-intensity parameter λ^U.

Hence,

$$\mathcal{K}(u;\Theta_t|\mathcal{F}_T^V) = \lambda_t^\Theta\left[\exp\left(\frac{uV_t}{\sqrt{\lambda^U}}\right) - 1\right] + \left(r_t + \lambda_t V_t - \frac{1}{2}V_t^2 - \sqrt{\lambda^U}V_t\right)u. \tag{3.31}$$

This implies that the conditional distribution of Y_t given $\mathcal{F}_T^V$ under $\mathcal{P}_\Theta$ is a shifted-Poisson distribution with the risk-neutral jump-intensity parameter λ_t^Θ, scale parameter $V_t/\sqrt{\lambda^U}$, and shifted parameter $-r_t - \lambda_t V_t + \frac{1}{2}V_t^2 + \sqrt{\lambda^U}V_t$. The dynamic of the logarithmic volatility level is unaltered by the change of measures.

For each $t \in \mathcal{T}_0$ and $k = t + 1, t + 2, \ldots, T$, $Y_k|\mathbf{V}_{t+1,T} = \mathbf{v}_{t+1,T}$ follows a shifted-Poisson distribution with the jump-intensity parameter λ_k^Θ, scale parameter $V_k/\sqrt{\lambda^U}$, and shifted parameter $-r_k - \lambda_k V_k +$

Table 1
Prices of European Options with Different Strikes and Same Time to Maturity, 84 Trading Days (4 Months)

Strike	BS	LL	LP	PL	PP
80	22.6875	25.3943	25.1173	25.9102	25.1791
85	18.1762	21.2223	21.1366	21.1811	21.9399
90	14.0128	17.7697	17.5394	17.5495	16.9549
95	10.3523	14.5122	14.6278	14.6266	14.9299
100	7.31023	11.616	11.9836	11.6784	11.7188
105	4.92956	9.10148	9.44389	9.64402	9.59863
110	3.17599	7.49992	7.76673	7.75241	7.75697
115	1.95803	5.63113	5.99675	6.22163	6.31268
120	1.15774	4.95769	4.99943	4.81438	5.06476
125	0.658303	3.7788	4.18244	4.10745	3.99899

$\frac{1}{2}V_k^2 + \sqrt{\lambda^U}V_k$ under $\mathcal{P}_\Theta$. The analysis of different parametric cases for $\{\eta_t\}$ is the same as that in Section 3.1.

4. MONTE CARLO EXPERIMENT

In this section we will conduct a Monte Carlo experiment for some of the parametric ARV models in Section 3. In particular, we will consider the lognormal-lognormal ARV model, lognormal-Poisson model, Poisson-lognormal model, and Poisson-Poisson model. We will compare the simulated option prices predicted from these parametric ARV models with those predicted from the celebrated Black-Scholes model. In this section we will suppose that the annual risk-free interest rate is 10%. The current price of the underlying risky asset is assumed to be 100. We will consider European call options with a fixed maturity of four months, that is, 84 trading days, but a range of different strike prices from 80 to 125, with an increment of 5. Monte Carlo simulation coupled with the control variate technique in Boyle (1977) is employed to simulate the call prices under different parametric ARV models. Ten thousand simulation runs will be generated for computing each option price. We will compute the call prices for different configurations of the specimen values of the parameters for each parametric ARV model. The computations were done using C++ codes with GSL functions. The simulation results will be presented in the sequel.

We will consider the specimen values of the model parameters and suppose that $\beta = 0.5$, $\sigma_v = 0.02$, $\phi = 0.98$, and the current value of the conditional volatility $V_t = 0.0156$. Hence, $\alpha = -3.40768$. For the lognormal-Poisson ARV model, assume that $\lambda_p = 1000$; for the Poisson-lognormal ARV model, suppose that $\lambda^U = 1.0$ and that the constant unit risk premium $\lambda = 0.000378$; for the Poisson-Poisson ARV model, assume that $(\lambda^U, \lambda_p) = (1.0, 1000)$ and that the constant unit risk premium $\lambda = 0.000378$. Table 1 presents the Black-Scholes call prices (BS), lognormal-lognormal ARV call prices (LL), lognormal-Poisson ARV call prices (LP), Poisson-lognormal ARV call prices (PL), and Poisson-Poisson ARV call prices (PP) with a fixed maturity of four months, that is, 84 trading days, but a range of different strike prices from 80 to 125, with an increment of 5.

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Discussions on this paper can be submitted until October 1, 2006. The author reserves the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.