

OPTIMAL DIVIDENDS IN AN ORNSTEIN-UHLENBECK TYPE MODEL WITH CREDIT AND DEBIT INTEREST

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ABSTRACT

In the absence of investment and dividend payments, the surplus is modeled by a Brownian motion. But now assume that the surplus earns investment income at a constant rate of credit interest. Dividends are paid to the shareholders according to a barrier strategy. It is shown how the expected discounted value of the dividends and the optimal dividend barrier can be calculated; Kummer's confluent hypergeometric differential equation plays a key role in this context. An alternative assumption is that business can go on after ruin, as long as it is profitable. When the surplus is negative, a higher rate of debit interest is applied. Several numerical examples document the influence of the parameters on the optimal dividend strategy.

1. INTRODUCTION

How should the dividends be paid to the shareholders, if the aim is to maximize the expectation of the discounted dividends until possible ruin of the company? This problem goes back to De Finetti (1957), who solved it in the discrete time model with gains of $+1$ and -1 . A history and set of references can be found in Gerber and Shiu (2004). In their paper the surplus process of the company before dividends is modeled by a Brownian motion with constant drift $\mu > 0$ and variance per unit time σ^2 . Like most other authors, they do not assume that the surplus earns interest. The purpose of this paper is to examine the problem under the assumption that the surplus does earn interest at the constant force $\rho > 0$. It should be distinguished from the constant force of interest $\delta > 0$, which is used to discount the dividends. In fact, we must assume

$$\rho < \delta, \tag{1.1}$$

for the optimization problem. If expression (1.1) does not hold, the expected discounted dividends before ruin can be made as large as possible.

In our search for optimal dividend strategies, we restrict ourselves to barrier strategies and determine the optimal value of the parameter. It is well known that in the classical case ($\rho = 0$), the resulting barrier strategy is optimal among *all* dividend strategies. That this is true for $\rho > 0$ remains a conjecture. In the classical case, the value of a barrier strategy satisfies a linear differential equation with constant coefficients. In this paper ρ is positive, which leads to a linear differential equation with variable coefficients. It is shown that this equation belongs to the class of *Kummer's confluent hypergeometric differential equations*, which lets us calculate the value of a barrier strategy and determine the optimal barrier in numerical examples. Mathematica turns out to be a useful tool in this context.

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In Section 6 we bring the model a step closer to reality. We allow for the possibility that business can go on after “ruin.” When the surplus is negative, interest is charged at a constant force τ . It is assumed that

$$\delta < \tau. \quad (1.2)$$

Thus ρ is the *rate of credit interest*, and τ is the *rate of debit interest*. Now, the business has to stop when the company is not profitable anymore (because of the interest payments on the debt). This model is more general than the first model, which can be considered as the limit $\tau \rightarrow \infty$. Luckily, the mathematics are essentially the same.

In Paulsen and Gjessing (1997) a more general model is analyzed. The surplus process before investment is the sum of a shifted compound Poisson process and an independent Brownian motion, and the stochastic rate of return is an independent process of the same type. In their paper barrier strategies are examined. It is shown that the expected discounted dividends are the solution of an integro-differential equation in conjunction with two boundary conditions. The two examples in Paulsen and Gjessing are of particular interest. In the first example, the surplus process is modeled by a shifted compound Poisson process with exponential claims, and a constant rate of credit interest is assumed. It is shown that the evaluation of a barrier strategy also leads to Kummer’s confluent hypergeometric differential equation (note that in formula [2.8] of Paulsen and Gjessing, ε must be replaced by $-\varepsilon$). In the second example, the surplus process is modeled by a Brownian motion, and the stochastic rate of credit interest is modeled by another Brownian motion. In their paper the numerical implementation and the determination of the optimal barrier have a low priority.

2. THE MODEL

Let $X(t)$ denote the surplus of the company. When no dividends are paid, the surplus process is governed by the following dynamics:

$$dX(t) = (\mu + \rho X(t)) dt + \sigma dW(t), \quad t \geq 0, \quad (2.1)$$

where $\{W(t), t \geq 0\}$ is a standard Wiener process. This means that the surplus process is a *diffusion process*, with variance σ^2 per unit time and drift $\mu + \rho x$, where x is the current surplus. The surplus process resembles the *Ornstein-Uhlenbeck process*, which some readers may have encountered in the context of the *Vasicek model* for the term structure of interest rates. However, note that for the *Ornstein-Uhlenbeck process*, the drift is a linear function of x with negative slope, providing a mean-reversion tendency.

As in Gerber and Shiu (2004), we assume that dividends are paid to the shareholders according to a *barrier strategy*, say, with parameter $b > 0$. Whenever the surplus is about to go above the level b , the excess (or “overflow”) will be paid as dividends.

Let $D(t)$ denote the aggregate dividends by time t , and let

$$T = \inf\{t : X(t) = 0\} \quad (2.2)$$

be the *time of ruin*, which is certain under a barrier strategy. Then

$$D = \int_0^T e^{-\delta t} dD(t) \quad (2.3)$$

is the discounted value of the dividends until ruin. We are interested in $V(x; b)$, $0 \leq x \leq b$, the expectation of D , considered a function of the initial surplus x .

By virtually the same arguments as in Gerber and Shiu (2004), it can be seen that $V(x; b)$ satisfies the second-order differential equation

$$\frac{\sigma^2}{2} V''(x; b) + (\mu + \rho x) V'(x; b) - \delta V(x; b) = 0, \quad 0 < x < b, \quad (2.4)$$

in combination with the boundary conditions

$$\begin{cases} V(0; b) = 0, \\ V'(b; b) = 1. \end{cases} \quad (2.5)$$

It is possible to write $V(x; b)$ in a more transparent form. Let the function $g(x)$ be a nontrivial solution of the differential equation

$$\frac{\sigma^2}{2} g''(x) + (\mu + \rho x)g'(x) - \delta g(x) = 0, \quad x > 0, \quad (2.6)$$

combined with the boundary condition

$$g(0) = 0. \quad (2.7)$$

Such a function $g(x)$ is unique apart from a constant factor. From formulas (2.4)–(2.5) we obtain the *factorization formula*,

$$V(x; b) = \frac{g(x)}{g'(b)}, \quad 0 \leq x \leq b. \quad (2.8)$$

Thus the determination of such a function $g(x)$ is a central problem.

Furthermore,

$$V(x; b) = x - b + V(b; b), \quad x > b, \quad (2.9)$$

which has the following interpretation: If the initial surplus x exceeds the dividend barrier b , the difference is immediately paid out as dividends.

REMARK 2.1

For given $x > 0$, $b > 0$,

$$\lim_{\sigma \rightarrow \infty} V(x; b) = x. \quad (2.10)$$

To see this, we introduce *operational time*, such that with respect to the operational time units, the variance per unit time of the surplus process is 1. Then the drift parameter becomes μ/σ^2 , and the interest rates become δ/σ^2 and ρ/σ^2 . All three vanish in the limit. It follows from equation (2.4) that $V'''(x; b) = 0$ in the limit. (Another way to see that $V'''(x; b)$ vanishes in the limit is to divide equation [2.4] by σ^2 and then to let σ tend to ∞ .) From this and boundary conditions (2.5) we see that $V(x; b) = x$ for $0 \leq x \leq b$ in the limit. Because of formula (2.9), it follows that this limiting equation is also true for $x > b$.

3. KUMMER'S CONFLUENT HYPERGEOMETRIC EQUATION

The differential equation for the function $g(x)$ can be converted into Kummer's confluent hypergeometric equation. We show this by two steps. In each step a new variable is introduced.

In the first step, we define $z = (1/\sigma) \sqrt{2/\rho} (\mu + \rho x)$ and define the function $f(z)$ such that $g(x) = f(z)$. Hence,

$$g'(x) = f'(z) \frac{dz}{dx} = \frac{\sqrt{2\rho}}{\sigma} f'(z),$$

$$g''(x) = \frac{2\rho}{\sigma^2} f''(z).$$

Substitution in equation (2.6) yields the differential equation for $f(z)$:

$$f''(\varkappa) + \varkappa f'(\varkappa) - \left(\frac{\delta}{\rho}\right) f(\varkappa) = 0, \quad \varkappa > \frac{\mu\sqrt{2}}{\sigma\sqrt{\rho}}. \quad (3.1)$$

In the second step, we define $t = -\frac{1}{2}\varkappa^2$ and the function $h(t)$ by the requirement that $f(\varkappa) = h(t)$. Hence,

$$\begin{aligned} f'(\varkappa) &= h'(t) \frac{dt}{d\varkappa} = -\varkappa h'(t), \\ f''(\varkappa) &= \varkappa^2 h''(t) - h'(t) = -2th''(t) - h'(t). \end{aligned}$$

Substitution in equation (3.1) yields the differential equation for the function $h(t)$:

$$th''(t) + \left(\frac{1}{2} - t\right)h'(t) + \frac{\delta}{2\rho} h(t) = 0, \quad t < -\frac{\mu^2}{\rho\sigma^2}. \quad (3.2)$$

Thus the function $h(t)$ indeed satisfies equation (A.1) of the Appendix, with parameters

$$c = \frac{1}{2}, \quad a = -\frac{\delta}{2\rho}. \quad (3.3)$$

We conclude that

$$g(x) = h(t) = \alpha(-t)^{1-c} e^t M(1 - a, 2 - c; -t) + \beta e^t U(c - a, c; -t) \quad (3.4)$$

for certain coefficients α and β , with

$$t = -\frac{1}{\rho\sigma^2} (\mu + \rho x)^2 \quad (3.5)$$

and a and c given by formulas (3.3).

The ratio between α and β follows from condition (2.7). For example, we may set

$$\begin{aligned} \alpha &= U\left(c - a, c; \frac{\mu^2}{\rho\sigma^2}\right), \\ \beta &= -\left(\frac{\mu^2}{\rho\sigma^2}\right)^{1-c} M\left(1 - a, 2 - c; \frac{\mu^2}{\rho\sigma^2}\right). \end{aligned}$$

Furthermore, taking derivatives in formula (3.4), we obtain

$$g'(x) = h'(t) \frac{dt}{dx} = -\frac{2}{\sigma^2} (\mu + \rho x) h'(t). \quad (3.6)$$

Using the product rule combined with equations (A.2) and (A.3), we see that

$$\begin{aligned} h'(t) &= h(t) - \alpha(1 - c)(-t)^{-c} e^t M(1 - a, 2 - c; -t) - \alpha \left(\frac{1 - a}{2 - c}\right) (-t)^{1-c} e^t M(2 - a, 3 - c; -t) \\ &\quad + \beta(c - a) e^t U(c - a + 1, c + 1; -t). \end{aligned} \quad (3.7)$$

Finally, we set $x = b$ in formula (3.6) and substitute it and formula (3.4) in formula (2.8) to obtain $V(x; b)$.

REMARK 3.1

In the limiting situation $\sigma = 0$, we have

$$X(t) = xe^{\rho t} + \mu \bar{s}_\eta,$$

where \bar{s}_η is calculated at the force of interest ρ . Now the determination of $V(x; b)$ is an exercise in compound interest. Let t_0 be the time when $X(t_0) = b$. Then we find that

Table 1
Influence of ρ and x on $V(x; 10)$ with $\delta = 4\%$ and $\mu = 1$

x	$V(x; 10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	13.63 (0.36)	14.44 (0.37)	15.25 (0.38)	16.90 (0.39)	18.57 (0.41)
0.4	16.47 (0.72)	17.44 (0.73)	18.42 (0.75)	20.40 (0.77)	22.41 (0.80)
0.6	17.15 (1.07)	18.16 (1.09)	19.17 (1.11)	21.23 (1.15)	23.31 (1.20)
0.8	17.39 (1.42)	18.42 (1.44)	19.44 (1.47)	21.53 (1.53)	23.63 (1.58)
1.0	17.55 (1.76)	18.58 (1.79)	19.62 (1.82)	21.72 (1.89)	23.85 (1.96)
2.0	18.27 (3.38)	19.34 (3.45)	20.41 (3.51)	22.59 (3.64)	24.78 (3.78)
4.0	19.79 (6.30)	20.92 (6.42)	22.06 (6.53)	24.35 (6.77)	26.67 (7.01)
6.0	21.43 (8.87)	22.61 (9.02)	23.80 (9.17)	26.19 (9.47)	28.59 (9.79)
8.0	23.20 (11.16)	24.42 (11.33)	25.64 (11.50)	28.09 (11.85)	30.54 (12.21)
10	25.12 (13.24)	26.35 (13.42)	27.59 (13.60)	30.05 (13.96)	32.52 (14.34)

$$V(x; b) = e^{-\delta t_0} V(b; b) = \left(\frac{\mu + \rho x}{\mu + \rho b} \right)^{\delta/\rho} \frac{\mu + \rho b}{\delta} \tag{3.8}$$

after simplification.

REMARK 3.2

When $\rho = 0$, by formulas (2.11)–(2.15) of Gerber and Shiu (2004),

$$V(x; b) = \frac{e^{rx} - e^{sx}}{re^{rb} - se^{sb}}, \tag{3.9}$$

where $r = (-\mu + \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$ and $s = (-\mu - \sqrt{\mu^2 + 2\delta\sigma^2})/\sigma^2$.

Now we are prepared to calculate $V(x; b)$. We present two examples to illustrate the dependence of $V(x; b)$ on ρ and σ .

EXAMPLE 3.1

Assume that $\mu = 1, b = 10, \delta = 4\%, \sigma = 0.5$. We calculate $V(x; 10)$ for selected values of x and ρ .

The results are exhibited in Table 1. Of course, $V(x; 10)$ is an increasing function of both x and ρ . Table 1 shows that the influence of ρ is substantial. In the same table, results for $\sigma = 5$ are shown in parentheses.

EXAMPLE 3.2

Assume that $\mu = 1, b = 10, \delta = 4\%, \rho = 2\%$. We calculate $V(x; 10)$ for selected values of x and σ . The results are displayed in Table 2. We note that for $x \leq 0.06, V(x; 10)$ appears to be a decreasing

Table 2
Influence of σ and x on $V(x; 10)$ with $\delta = 4\%$ and $\mu = 1$

x	$V(x; 10)$ with $\rho = 2\%$ ($\rho = 6\%$)				
	$\sigma = 0$	$\sigma = 0.5$	$\sigma = 1$	$\sigma = 3$	$\sigma = 5$
0.2	21.00 (29.47)	16.90 (23.70)	7.28 (10.22)	0.98 (1.34)	0.39 (0.45)
0.4	21.17 (29.71)	20.40 (28.56)	12.17 (17.07)	1.91 (2.61)	0.77 (0.90)
0.6	21.34 (29.94)	21.23 (29.69)	15.47 (21.66)	2.81 (3.83)	1.15 (1.34)
0.8	21.51 (30.17)	21.53 (30.09)	17.71 (24.75)	3.67 (5.00)	1.52 (1.77)
1.0	21.68 (30.40)	21.72 (30.35)	19.25 (26.84)	4.49 (6.12)	1.89 (2.19)
2.0	22.53 (31.53)	22.59 (31.49)	22.42 (31.02)	8.10 (11.00)	3.64 (4.21)
4.0	24.30 (33.75)	24.35 (33.71)	24.50 (33.58)	13.44 (18.01)	6.76 (7.78)
6.0	26.13 (35.89)	26.19 (35.85)	26.34 (35.73)	17.12 (22.55)	9.47 (10.80)
8.0	28.03 (37.97)	28.09 (37.93)	28.24 (37.81)	19.84 (25.63)	11.85 (13.36)
10	30.00 (40.00)	30.05 (39.96)	30.21 (39.84)	22.02 (27.90)	13.96 (15.53)

function of σ . But for $x \geq 0.08$, $V(x; 10)$ first increases with σ , attains a maximum, and decreases thereafter. The following is a tentative attempt to explain this phenomenon. If σ is large, ruin occurs soon. In particular, the probability that the dividend barrier is attained before ruin is small. In this sense, a high value of σ has a negative impact on $V(x; 10)$. On the other hand, if σ is large, dividends are paid at a high rate when the surplus is on the barrier. In this sense, a high value of σ has a positive impact on $V(x; 10)$. Then Table 2 shows that for $x \geq 0.08$, the second factor prevails up to a certain value of σ , where $V(x; 10)$ attains its maximum. Thereafter, the first factor prevails.

We calculate $V(x; 10)$ also for $\rho = 6\%$; the results are shown in parentheses. In this case $V(x; 10)$ appears to be a decreasing function of σ for any x .

4. THE OPTIMAL BARRIER

For given $x > 0$, we want to find b that maximizes $V(x; b)$. From formulas (2.8) and (2.9) we obtain

$$\frac{\partial}{\partial b} V(x; b) = \begin{cases} -V(b; b) \frac{g''(b)}{g'(b)} & \text{if } 0 < b < x, \\ -V(x; b) \frac{g''(b)}{g'(b)} & \text{if } b \geq x. \end{cases} \quad (4.1)$$

From formulas (2.6) and (2.7) it follows that

$$\frac{g''(0)}{g'(0)} = -\frac{2\mu}{\sigma^2} < 0.$$

Hence $\frac{\partial}{\partial b} V(x; b)$ is positive for small values of b . Because $V(x; b) \rightarrow 0$ for $b \rightarrow \infty$, we gather that $V(x; b)$ attains its maximum for a finite and positive value of b .

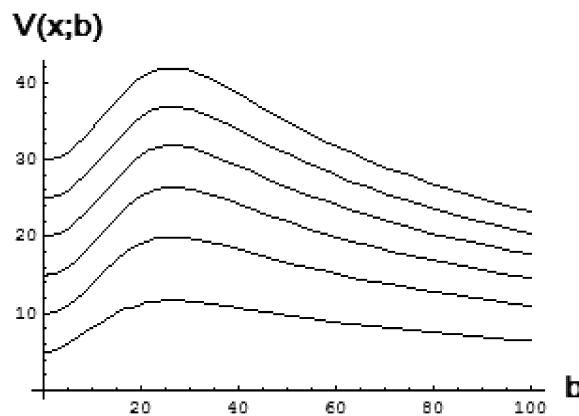
According to equation (4.1), the first-order condition is that

$$g''(b) = 0. \quad (4.2)$$

It turns out that this equation has a unique solution $b = b^*$. Therefore, this is the barrier that maximizes $V(x; b)$, independently of x . This is illustrated by Figure 1.

We note that

Figure 1
Curves of $V(x; b)$ with $\mu = 1$, $\delta = 4\%$, $\sigma = 5$, $\rho = 2\%$, $x = 5, 10, 15, 25, 30$



$$V''(x; b) = \frac{g''(x)}{g'(b)} \text{ for } 0 < x < b$$

and hence

$$V''(b^*; b^*) = 0. \tag{4.3}$$

Thus, if we set $x = b = b^*$ in equation (2.4) and use the second condition in (2.5), we see that

$$\mu + \rho b^* - \delta V(b^*; b^*) = 0,$$

from which it follows that

$$V(b^*; b^*) = \frac{\mu + \rho b^*}{\delta}. \tag{4.4}$$

Thus, $V(b^*; b^*)$ is identical to the present value of a *perpetuity*, where the payment rate is the sum of the drift and the interest on the initial capital. For $\rho = 0$, formula (4.4) can be found as formula (7.1) in Gerber and Shiu (2004). In this case there is a closed-form expression for the optimal barrier:

$$b^* = \frac{2}{r - s} \log \left(-\frac{s}{r} \right), \tag{4.5}$$

where r and s are the same as in equation (3.9).

The difference $V(x; b^*) - x$ is the expected present value of profit on the investment x . Formula (2.10) shows that in the limit $\sigma \rightarrow \infty$ (*extreme volatility business*), the expected present value of profit becomes zero. In particular, we have $V(b^*; b^*) = b^*$ in the limit. From this and equation (4.4) we obtain a linear equation for the limiting value of b^* . We find that

$$\lim_{\sigma \rightarrow \infty} b^* = \frac{\mu}{\delta - \rho}. \tag{4.6}$$

This result generalizes formula (7.3) in Gerber and Shiu (2004).

EXAMPLE 4.1

Assume that $\mu = 1$, $\delta = 4\%$. We calculate the optimal barrier b^* for selected values of ρ and σ . The results are shown in Table 3. We note that b^* is an increasing function of ρ . As ρ tends to δ , b^* tends to ∞ . Moreover, b^* increases from 0 to the value given by formula (4.6) as σ varies between 0 and ∞ . Also, note that Table 3 tells us $b^* = 26.1876$ in Figure 1.

EXAMPLE 4.2

As in Example 3.1, we assume that $\mu = 1$, $\delta = 4\%$, $\sigma = 0.5$ ($\sigma = 5$). Now we calculate $V(x; b^*)$ for the same combinations of x and ρ as in Table 1. The results are shown in Table 4. We observe that

Table 3
Influence of ρ and σ on b^* with $\delta = 4\%$ and $\mu = 1$

σ	Optimal Barrier b^*				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.02476	0.02492	0.02511	0.02562	0.02648
0.10	0.08514	0.08580	0.08656	0.08855	0.09198
0.20	0.28484	0.28739	0.29033	0.29814	0.31161
0.50	1.31399	1.32847	1.34534	1.39034	1.46887
5	19.00860	20.49930	22.17000	26.18760	31.74960
50	24.91700	28.44770	33.13750	49.34760	95.14190
500	24.99920	28.57020	33.33130	49.99330	99.94670
∞	25	28.57140	33.33333	50	100

Table 4
Influence of ρ and x on $V(x; b^*)$ with $\delta = 4\%$ and $\mu = 1$

x	$V(x; b^*)$ with $\sigma = 0.5$ ($\sigma = 5$)				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.2	19.16 (0.42)	19.29 (0.45)	19.42 (0.48)	19.68 (0.56)	19.96 (0.67)
0.4	23.16 (0.84)	23.30 (0.89)	23.45 (0.95)	23.76 (1.11)	24.08 (1.32)
0.6	24.11 (1.25)	24.26 (1.33)	24.41 (1.42)	24.73 (1.65)	25.05 (1.97)
0.8	24.46 (1.66)	24.61 (1.76)	24.76 (1.88)	25.07 (2.18)	25.40 (2.56)
1.0	24.68 (2.06)	24.83 (2.18)	24.99 (2.33)	25.30 (2.70)	25.63 (3.22)
2	25.69 (3.96)	25.84 (4.20)	25.99 (4.48)	26.30 (5.21)	26.63 (6.20)
4	27.69 (7.39)	27.84 (7.82)	27.99 (8.34)	28.30 (9.67)	28.63 (11.51)
6	29.69 (10.39)	29.84 (10.99)	29.99 (11.71)	30.30 (13.55)	30.63 (16.09)
8	31.69 (13.07)	31.84 (13.81)	31.99 (14.69)	32.30 (16.94)	32.63 (20.06)
10	33.69 (15.51)	33.84 (16.36)	33.99 (17.37)	34.30 (19.96)	34.63 (23.55)

$V(x; b^*)$ exceeds $V(x; b)$ significantly. Note that the optimal barrier values can be found in the rows $\sigma = 0.5$ and $\sigma = 5$ in Table 3. In particular, we see that $b^* < 2$ if $\sigma = 0.5$. Therefore, in accordance with formula (2.9) with $b = b^*$, $V(x; b^*)$ is linear for $x \geq 2$.

5. THE DISTRIBUTION OF THE TIME OF RUIN UNDER A BARRIER STRATEGY

In this section we assume that dividends are paid according to a barrier strategy with parameter b , which may be different from the optimal value b^* . Consider the expected present value of a payment of 1, due at the time of ruin. We use the notation

$$L(x; b) = E[e^{-\delta T}] \quad (5.1)$$

to indicate that this expression will be treated as a function of the initial capital x . Hence, $L'(x; b)$ denotes the derivative with respect to x . As a function of δ , $L(x; b)$ is the *Laplace transform* of the probability density function of the time of ruin T .

By analogy with equation (3.4) in Gerber and Shiu (2004), $L(x; b)$ can be characterized as the solution of second-order differential equation

$$\frac{\sigma^2}{2} L''(x; b) + (\mu + \rho x)L'(x; b) - \delta L(x; b) = 0, \quad 0 < x < b, \quad (5.2)$$

in conjunction with the boundary conditions

$$\begin{cases} L(0; b) = 1, \\ L'(b; b) = 0. \end{cases} \quad (5.3)$$

Let the function $g(x)$ be a nontrivial solution of the differential equation (2.6), subject to the boundary condition

$$g'(b) = 0. \quad (5.4)$$

Then it follows from formulas (5.2) and (5.3) that

$$L(x; b) = \frac{g(x)}{g(0)}, \quad 0 \leq x \leq b. \quad (5.5)$$

Such a function $g(x)$ is obtained by the same method as in formula (3.4), except that the coefficients α and β in formula (3.4) have different values. Their ratio now follows from condition (5.4), which is equivalent to the condition

$$h' \left(-\frac{(\mu + \rho b)^2}{\rho \sigma^2} \right) = 0.$$

Table 5
Influence of ρ and x on $E[T]$ with $\delta = 4\%$, $\mu = 1$, $\sigma = 3$

x	$E[T]$					
	$\rho = 0$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 4\%$	$\rho = 6\%$	$\rho = 8\%$
0.2	1.605	1.701	1.805	2.039	2.314	2.637
0.4	3.132	3.320	3.523	3.981	4.517	5.148
0.6	4.584	4.859	5.157	5.827	6.614	7.538
0.8	5.963	6.322	6.710	7.583	8.608	9.811
1.0	7.274	7.713	8.166	9.252	10.502	11.970
2.0	12.900	13.676	14.514	16.398	18.604	21.193
4.0	20.454	21.656	22.952	25.857	29.243	33.199
6.0	24.579	25.973	27.473	30.823	34.711	39.234
8.0	26.507	27.962	29.525	30.010	37.045	41.728
10	27.025	28.488	30.058	33.559	37.611	42.311

From this and formula (3.7) with (3.4), we see that we may set

$$\alpha = (c - a)U\left(c - a + 1, c + 1; \frac{(\mu + \rho b)^2}{\rho\sigma^2}\right) + U\left(c - a, c; \frac{(\mu + \rho b)^2}{\rho\sigma^2}\right), \quad (5.6)$$

$$\begin{aligned} \beta = & \frac{1 - a}{2 - c} \left(\frac{(\mu + \rho b)^2}{\rho\sigma^2}\right)^{1-c} M\left(2 - a, 3 - c; \frac{(\mu + \rho b)^2}{\rho\sigma^2}\right) \\ & - \left(\frac{(\mu + \rho b)^2}{\rho\sigma^2}\right)^{-c} \left(\frac{(\mu + \rho b)^2}{\rho\sigma^2} - (1 - c)\right) M\left(1 - a, 2 - c; \frac{(\mu + \rho b)^2}{\rho\sigma^2}\right). \end{aligned} \quad (5.7)$$

Then formula (3.4) with this choice of α and β must be substituted in formula (5.5) to obtain $L(x; b)$.

As an application, we can calculate the expectation of the time of ruin by the formula

$$E[T] = \left. -\frac{dL(x, b)}{d\delta} \right|_{\delta=0},$$

if $\rho > 0$. If $\rho = 0$, we have

$$E[T] = \frac{\sigma^2}{2\mu^2} \left(e^{2\mu b/\sigma^2} - e^{2\mu(b-x)/\sigma^2} - \frac{2\mu x}{\sigma^2} \right), \quad 0 \leq x \leq b;$$

see, for example, formula (3.8) in Gerber and Shiu (2004).

EXAMPLE 5.1

Assume $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 3$. We calculate the expected time of ruin $E[T]$ for different values of ρ and x . The results are displayed in Table 5. Not surprisingly, $E[T]$ is an increasing function of both x and ρ .

6. LIFE AFTER RUIN

As before, we assume that dividends are paid according to a barrier strategy with parameter b . But now we allow for the possibility that the business goes on after “ruin,” and hence that dividends can be paid after ruin. Whenever the surplus is negative, interest is debited at a force $\tau > \delta$. Now we assume that the company has to go out of business, when the surplus is at the critical level

$$\lambda = -\frac{\mu}{\tau}. \quad (6.1)$$

If the surplus is at this level, the company is not profitable anymore in that the instantaneous drift of the surplus process is 0.

Let $V(x; b)$, $\lambda \leq x \leq b$, denote the expectation of the discounted dividends until the company has to go out of business. The differential equation (2.4) and the second boundary condition in (2.5) remain valid. The first boundary condition in (2.5) is replaced by the condition

$$V(\lambda; b) = 0, \quad (6.2)$$

and for $\lambda < x < 0$, $V(x; b)$ satisfies

$$\frac{\sigma^2}{2} V''(x; b) + (\mu + \tau x)V'(x; b) - \delta V(x; b) = 0, \quad \lambda < x < 0. \quad (6.3)$$

Furthermore, $V(x; b)$ and $V'(x; b)$ are continuous at the junction $x = 0$.

From these considerations we obtain the factorization formula,

$$V(x; b) = \frac{g(x)}{g'(b)}, \quad \lambda \leq x \leq b. \quad (6.4)$$

Here $g(x)$ is a continuously differentiable function, satisfying equation (2.6),

$$\frac{\sigma^2}{2} g''(x) + (\mu + \tau x)g'(x) - \delta g(x) = 0, \quad \lambda < x < 0, \quad (6.5)$$

and

$$g(\lambda) = 0. \quad (6.6)$$

Again, such a function $g(x)$ is unique apart from a constant factor.

By introducing the new variable

$$\tilde{t} = -\frac{1}{\tau\sigma^2} (\mu + \tau x)^2, \quad (6.7)$$

equation (6.5) can be converted into Kummer's confluent hypergeometric equation for the function $\tilde{h}(\tilde{t}) = g(x)$. The parameters are now

$$c = \frac{1}{2}, \quad \tilde{a} = -\frac{\delta}{2\tau}. \quad (6.8)$$

We conclude that $g(x)$, $\lambda \leq x \leq 0$, must be a linear combination of the two functions $(-\tilde{t})^{1-c} e^{\tilde{t}} M(1 - \tilde{a}, 2 - c; -\tilde{t})$ and $e^{\tilde{t}} U(c - \tilde{a}, c; -\tilde{t})$. Because $U(c - \tilde{a}, c; 0) \neq 0$, it follows from equation (6.6) that $g(x)$, $\lambda \leq x \leq 0$, is a multiple of the first function. We may set

$$g(x) = \tilde{h}(\tilde{t}) = (-\tilde{t})^{1-c} e^{\tilde{t}} M(1 - \tilde{a}, 2 - c; -\tilde{t}), \quad \lambda \leq x \leq 0, \quad (6.9)$$

with \tilde{t} given by expression (6.7).

For $x > 0$, $g(x)$ is of the form (3.4). To determine the coefficients α and β , we use the continuity conditions,

$$g(0+) = g(0-), \quad g'(0+) = g'(0-), \quad (6.10)$$

which lead to the conditions

$$h\left(-\frac{\mu^2}{\rho\sigma^2}\right) = \tilde{h}\left(-\frac{\mu^2}{\tau\sigma^2}\right), \quad h'\left(-\frac{\mu^2}{\rho\sigma^2}\right) = \tilde{h}'\left(-\frac{\mu^2}{\tau\sigma^2}\right),$$

resulting in the equations

$$\begin{aligned} \alpha \left(\frac{\mu^2}{\rho\sigma^2} \right)^{1-c} \exp \left\{ -\frac{\mu^2}{\rho\sigma^2} \right\} M \left(1 - a, 2 - c; \frac{\mu^2}{\rho\sigma^2} \right) + \beta \exp \left\{ -\frac{\mu^2}{\rho\sigma^2} \right\} U \left(c - a, c; \frac{\mu^2}{\rho\sigma^2} \right) \\ = \left(\frac{\mu^2}{\tau\sigma^2} \right)^{1-c} \exp \left\{ -\frac{\mu^2}{\tau\sigma^2} \right\} M \left(1 - \tilde{a}, 2 - c; \frac{\mu^2}{\tau\sigma^2} \right), \end{aligned}$$

and

$$\begin{aligned} \alpha(1 - c) \left(\frac{\mu^2}{\rho\sigma^2} \right)^{-c} \exp \left\{ -\frac{\mu^2}{\rho\sigma^2} \right\} M \left(1 - a, 2 - c; \frac{\mu^2}{\rho\sigma^2} \right) + \alpha \left(\frac{\mu^2}{\rho\sigma^2} \right)^{1-c} \exp \left\{ -\frac{\mu^2}{\rho\sigma^2} \right\} \left(\frac{1 - a}{2 - c} \right) \\ \times M \left(2 - a, 3 - c; \frac{\mu^2}{\rho\sigma^2} \right) + \beta \exp \left\{ -\frac{\mu^2}{\rho\sigma^2} \right\} (a - c) U \left(c - a + 1, c + 1; \frac{\mu^2}{\rho\sigma^2} \right) = (1 - c) \left(\frac{\mu^2}{\tau\sigma^2} \right)^{-c} \\ \times \exp \left\{ -\frac{\mu^2}{\tau\sigma^2} \right\} M \left(1 - \tilde{a}, 2 - c; \frac{\mu^2}{\tau\sigma^2} \right) + \left(\frac{\mu^2}{\tau\sigma^2} \right)^{1-c} \exp \left\{ -\frac{\mu^2}{\tau\sigma^2} \right\} \left(\frac{1 - \tilde{a}}{2 - c} \right) M \left(2 - \tilde{a}, 3 - c; \frac{\mu^2}{\tau\sigma^2} \right). \end{aligned}$$

These are two linear equations for α and β .

REMARK 6.1

In the limiting situation $\sigma = 0$, formula (3.8) is still valid for $0 \leq x \leq b$. Furthermore,

$$V(x; b) = \left(\frac{\mu + \tau x}{\mu} \right)^{\delta/\tau} V(0; b) \quad \text{for } \lambda \leq x < 0. \quad (6.11)$$

To show this is an exercise in compound interest.

REMARK 6.2

The calculations are somewhat simpler when $\rho = 0$. Then formulas (6.4) and (6.9) are still valid. But now $g(x)$ for $x \geq 0$ is of the form

$$g(x) = \alpha e^{rx} + \beta e^{sx}, \quad x \geq 0, \quad (6.12)$$

where r and s are the same as in equation (3.9). From equation (6.12) we obtain the conditions

$$\alpha + \beta = \left(\frac{\mu^2}{\tau\sigma^2} \right)^{1-c} \exp \left\{ -\frac{\mu^2}{\tau\sigma^2} \right\} M \left(1 - \tilde{a}, 2 - c; \frac{\mu^2}{\rho\sigma^2} \right)$$

and

$$r\alpha + s\beta = - \left(\frac{2\mu}{\sigma^2} \right) \tilde{h}' \left(-\frac{\mu^2}{\tau\sigma^2} \right),$$

from which the coefficients α and β can be determined.

REMARK 6.3

For $x > \lambda$, $b > 0$,

$$\lim_{\alpha \rightarrow \infty} V(x; b) = x + \frac{\mu}{\tau}. \quad (6.13)$$

To see this, we first use the same reasoning as in Remark 2.1 to conclude that $V''(x; b) = 0$ in the limit. In view of the boundary conditions at $x = \lambda$ and $x = b$, we gather that $V(x; b) = x - \lambda$, which is equivalent to formula (6.13).

Now we are prepared to calculate $V(x; b)$ when dividends can be paid after ruin. We present two examples to illustrate the dependence of $V(x; b)$ on ρ , τ , and x .

EXAMPLE 6.1

Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5$ ($\sigma = 5$), and $\tau = 6\%$. We calculate $V(x; 10)$ for the same combinations of x and ρ as in Table 1.

The results are exhibited in Table 6. Because of the dividends that are paid after ruin, the values of $V(x; 10)$ in Table 6 are higher than the corresponding values in Table 1. However, we note that if $\sigma = 0.5$, the two values are practically the same, if x is sufficiently positive. Because the difference between the values in Tables 6 and 1 is the expectation of the discounted dividends after ruin, it can be written as the product $L(x; 10)V(0; 10)$. Thus, in a given column of Table 6, this difference is proportional to $L(x; 10)$.

Further, to see the influence of τ on $V(x; b)$, we reconsider Example 6.1 as follows.

EXAMPLE 6.2

Assume that $\mu = 1$, $b = 10$, $\delta = 4\%$, $\sigma = 0.5$ ($\sigma = 5$), and $\rho = 2\%$. We calculate $V(x; 10)$ for selected values of x and τ .

The results are exhibited in Table 7. Of course, $V(x; 10)$ is an increasing function of x and a decreasing function of τ . Table 7 shows that the influence of τ is substantial for large σ . However, for small σ and x positive, the influence of τ on $V(x; 10)$ is modest. In the limiting situation $\sigma = 0$, $V(x; 10)$ does not depend on τ , provided that x is positive.

Let b^* denote the optimal barrier. It is the value of b that maximizes $V(x; b)$, independently of x ; this is illustrated in Figure 2. As in Section 4, b^* can be obtained from the condition that $g''(b^*) = 0$. Furthermore, formulas (4.3) and (4.4) remain valid in this section.

There are no closed-form expressions for b^* , with one noteworthy exception. If $\rho = 0$, it follows from Remark 6.2 that

$$b^* = \frac{1}{r - s} \log \left(\frac{-\beta s^2}{\alpha r^2} \right), \quad (6.14)$$

which generalizes formula (4.5). Furthermore,

$$\lim_{\sigma \rightarrow \infty} b^* = \frac{\mu}{\delta - \rho} \left(1 - \frac{\delta}{\tau} \right). \quad (6.15)$$

Table 6
Influence of ρ and x on $V(x; 10)$ with $\delta = 4\%$, $\mu = 1$, $\tau = 6\%$

x	$V(x; 10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
-10	9.12 (8.09)	9.65 (8.22)	10.19 (8.35)	11.29 (8.61)	12.39 (8.88)
-8	10.89 (10.44)	11.53 (10.60)	12.17 (10.77)	13.47 (11.11)	14.79 (11.46)
-6	12.52 (12.73)	13.25 (12.93)	13.99 (13.13)	15.49 (13.54)	17.01 (13.97)
-4	14.04 (14.95)	14.87 (15.18)	15.70 (15.42)	17.38 (15.91)	19.09 (16.41)
-2	15.49 (17.09)	16.40 (17.36)	17.32 (17.63)	19.17 (18.19)	21.05 (18.76)
0	16.87 (19.16)	17.87 (19.46)	18.86 (19.77)	20.88 (20.39)	22.93 (21.03)
0.2	17.01 (19.36)	18.01 (19.67)	19.02 (19.98)	21.05 (20.61)	23.11 (21.25)
0.4	17.15 (19.56)	18.15 (19.87)	19.17 (20.18)	21.22 (20.82)	23.30 (21.48)
0.6	17.28 (19.76)	18.30 (20.08)	19.32 (20.39)	21.39 (21.03)	23.48 (21.70)
0.8	17.42 (19.96)	18.44 (20.28)	19.47 (20.60)	21.56 (21.25)	23.67 (21.92)
1.0	17.56 (20.16)	18.59 (20.48)	19.63 (20.80)	21.73 (21.46)	23.85 (22.13)
2.0	18.27 (21.15)	19.34 (21.49)	20.41 (21.82)	22.59 (22.51)	24.78 (23.22)
4.0	19.79 (23.10)	20.92 (23.46)	22.06 (23.83)	24.35 (24.57)	26.67 (25.34)
6.0	21.43 (25.04)	22.61 (25.42)	23.80 (25.81)	26.19 (26.60)	28.59 (27.41)
8.0	23.20 (26.98)	24.42 (27.38)	25.64 (27.78)	28.09 (28.60)	30.54 (29.44)
10	25.12 (28.96)	26.35 (29.36)	27.59 (29.77)	30.05 (30.60)	32.52 (31.45)

Table 7
Influence of ρ and x on $V(x; 10)$ with $\delta = 4\%$, $\mu = 1$, $\rho = 2\%$

x	$V(x; 10)$ with $\sigma = 0.5$ ($\sigma = 5$)				
	$\tau = 5\%$	$\tau = 6\%$	$\tau = 7\%$	$\tau = 8\%$	$\tau = 10\%$
-10	11.98 (11.36)	11.29 (8.61)	10.37 (6.06)	8.89 (3.77)	0 (0)
-8	13.87 (13.56)	13.47 (11.11)	13.00 (8.82)	12.41 (6.74)	10.19 (3.27)
-6	15.69 (15.72)	15.49 (13.54)	15.27 (11.49)	15.01 (9.63)	14.35 (6.47)
-4	17.47 (17.85)	17.38 (15.91)	17.29 (14.08)	17.20 (12.40)	16.98 (9.56)
-2	19.19 (19.93)	19.17 (18.19)	19.15 (16.55)	19.13 (15.04)	19.08 (12.48)
0	20.89 (21.97)	20.88 (20.39)	20.88 (18.90)	20.88 (17.53)	20.88 (15.20)
0.2	21.05 (22.18)	21.05 (20.61)	21.05 (19.12)	21.05 (17.77)	21.05 (15.46)
0.4	21.22 (22.38)	21.22 (20.82)	21.22 (19.35)	21.22 (18.01)	21.22 (15.71)
0.6	21.39 (22.58)	21.39 (21.03)	21.39 (19.58)	21.39 (18.24)	21.39 (15.97)
0.8	21.56 (22.78)	21.56 (21.25)	21.56 (19.80)	21.56 (18.48)	21.56 (16.22)
1.0	21.73 (22.98)	21.73 (21.46)	21.73 (20.03)	21.73 (18.71)	21.73 (16.47)
2.0	22.59 (23.98)	22.59 (22.51)	22.59 (21.13)	22.59 (19.86)	22.59 (17.70)
4.0	24.35 (25.96)	24.35 (24.57)	24.35 (23.27)	24.35 (22.07)	24.35 (20.04)
6.0	26.19 (27.93)	26.19 (26.60)	26.19 (25.34)	26.19 (24.19)	26.19 (22.23)
8.0	28.09 (29.90)	28.09 (28.60)	28.09 (27.37)	28.09 (26.25)	28.09 (24.33)
10	30.05 (31.89)	30.05 (30.60)	30.05 (29.38)	30.05 (28.26)	30.05 (26.36)

To see this, we combine formula (6.13) with $x = b = b^*$ and formula (4.4) to obtain a linear equation for the limiting value of b^* . Note that equation (6.15) generalizes equation (4.6).

We use the following example to illustrate what optimal barriers are when dividends can be paid after ruin.

EXAMPLE 6.3

As in Example 4.1, we assume that $\mu = 1$ and $\delta = 4\%$. But now business (and dividends) can go on after ruin according to a debit rate of interest of $\tau = 6\%$. In Table 8, the optimal barrier is shown for the same combinations of σ and ρ as in Table 3. We note that for the same pairs of σ and ρ , b^* in Table 8 is smaller than b^* in Table 3. In a given column, b^* increases from 0 to the limiting value in equation (6.15) as σ varies from 0 to ∞ . Also, Table 8 reveals that $b^* = 8.72959$ in Figure 2.

To see the influence of τ on b^* , we reconsider Example 6.3 as follows.

Figure 2

Curves of $V(x; b)$ with $\mu = 1$, $\delta = 4\%$, $\sigma = 5$, $\rho = 2\%$, $\tau = 6\%$, $x = 5, 10, 15, 25, 30$

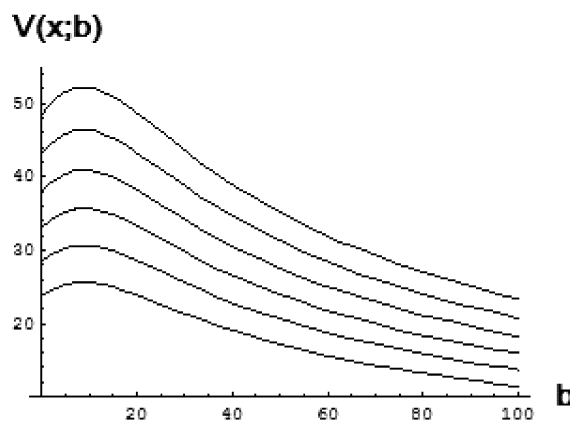


Table 8
Influence of ρ and σ on b^* with $\delta = 4\%$, $\mu = 1$, $\tau = 6\%$

σ	Optimal Barrier b^*				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
0.05	0.00051	0.00057	0.00064	0.00087	0.00137
0.10	0.00203	0.00226	0.00256	0.00347	0.00549
0.20	0.00812	0.00905	0.01023	0.01388	0.02199
0.50	0.05113	0.05698	0.06439	0.08731	0.13817
5	5.11239	5.70392	6.45109	8.72959	13.49200
50	8.28724	9.46708	11.03840	16.51990	32.75470
500	8.33287	9.52324	11.11030	16.66520	33.32740
∞	8.33333	9.52381	11.11111	16.66667	33.33333

Table 9
Influence of τ and σ on b^* with $\delta = 4\%$, $\mu = 1$, $\rho = 2\%$

σ	Optimal Barrier b^*				
	$\tau = 5\%$	$\tau = 6\%$	$\tau = 7\%$	$\tau = 8\%$	$\tau = 10\%$
0.05	0.00051	0.00087	0.00115	0.00137	0.00173
0.10	0.00203	0.00347	0.00458	0.00550	0.00693
0.20	0.00812	0.01388	0.01835	0.02201	0.02778
0.50	0.05101	0.08731	0.11556	0.13872	0.17547
5	5.28134	8.72959	11.17560	13.00690	15.57390
50	9.92057	16.51990	21.22670	24.75300	29.68440
500	9.99920	16.66520	21.42650	24.99750	29.99680
∞	10	16.66667	21.42860	25	30

Table 10
Influence of τ and ρ on b^* with $\delta = 4\%$, $\mu = 1$, $\sigma = 5$

τ	Optimal Barrier b^*				
	$\rho = 0$	$\rho = 0.5\%$	$\rho = 1\%$	$\rho = 2\%$	$\rho = 3\%$
5%	2.9176	3.2850	3.7591	5.2813	8.8752
10%	10.0780	11.0680	12.2608	15.5739	21.2945
20%	14.3007	15.5484	17.0031	20.7685	26.5588
50%	17.0589	18.4467	20.0405	23.9767	29.6566
100%	18.0216	19.4630	21.0932	25.0730	30.6977
200%	18.5119	19.9778	21.6284	25.6278	31.2220
500%	18.8092	20.2896	21.9525	25.9631	31.5381
∞	19.0086	20.4993	22.1700	26.1876	31.7496

EXAMPLE 6.4

Assume that $\mu = 1$, $\delta = 4\%$, and $\rho = 2\%$. We calculate the optimal barrier b^* for selected values of τ and σ . The results are shown in Table 9. As in Example 6.3, b^* is an increasing function of both τ and σ . If σ is small, b^* is close to 0.

Again, if σ is very large, b^* in this example is close to the limiting value given in formula (6.15).

Further, to see the influence of τ and σ on b^* , we consider the following example.

EXAMPLE 6.5

Assume that $\mu = 1$, $\delta = 4\%$, and $\sigma = 5$. We calculate the optimal barrier for selected values of τ and ρ . The results are shown in Table 10, which shows that b^* is an increasing function of both τ and ρ . Note that the last line is from Table 3.

APPENDIX

A second-order differential equation of the form

$$x \frac{d^2y}{dx^2} + (c - x) \frac{dy}{dx} - ay = 0 \quad (\text{A.1})$$

is called Kummer's confluent hypergeometric equation; see Seaborn (1991), Slater (1960), and <http://mathworld.wolfram.com/confluenthypergeometricdifferenialequation.html>. This equation is also an associated Laguerre differential equation; see <http://mathworld.wolfram.com/laguerredifferenialequation.html>.

For $x > 0$, the general solution of equation (A.1) can be expressed as a linear combination of the functions $M(a, c; x)$ and $U(a, c; x)$, called the confluent hypergeometric functions of the first and second kinds, respectively.

There are useful formulas for the derivatives of these functions:

$$\frac{d}{dx} M(a, c; x) = \frac{a}{c} M(a + 1, c + 1; x), \quad (\text{A.2})$$

$$\frac{d}{dx} U(a, c; x) = -aU(a + 1, c + 1; x). \quad (\text{A.3})$$

These formulas and a wealth of other results can be found in Abramowitz and Stegun (1972) or Slater (1960). It is important to know that $M(a, c; 0) = 1$, $M(a, c; x) \rightarrow \infty$ for $x \rightarrow \infty$, $U(a, c; x) \rightarrow 0$ for $x \rightarrow \infty$, and, for $0 < c < 1$, $U(a, c; 0)$ is different from 0.

In this paper we are to solve equation (A.1) when $x < 0$. Then the general solution is a linear combination of the functions

$$e^x U(c - a, c; -x), \quad x < 0, \quad (\text{A.4})$$

and

$$(-x)^{1-c} e^x M(1 - a, 2 - c; -x), \quad x < 0. \quad (\text{A.5})$$

See formulas (13.1.15) and (13.1.18) in Abramowitz and Stegun (1972).

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DISCUSSIONS

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I truly enjoyed reading this paper. The purpose of my discussion is twofold. First, in example 2.1 of Paulsen and Gjessing (1997) it is indicated how the calculations can be done, if the aggregate claims process is a compound Poisson process with exponential claims. In this discussion the calculations are implemented with Mathematica. Moreover, as in Gerber, Shiu, and Smith (2006), a variant of the problem is analyzed assuming that the company is liable to cover the deficit at ruin. This idea is due to Dickson and Waters (2004). Second, it is well known that a Brownian motion can be obtained as a limit of an appropriate family of shifted compound Poisson processes. Hence, the optimal barrier in the paper also can be considered as a limit of optimal barriers in shifted compound Poisson models. The techniques developed in the first part of this discussion help us to illustrate this numerically.

PART 1

In the compound Poisson model, we will use same notation as in Gerber, Shiu, and Smith (2006), but we will assume that the surplus earns interest at a constant rate $\rho > 0$. Suppose that the dividends are paid according to a barrier strategy with parameter b . Let $V(x; b)$ denote the expectation of the discounted dividends until ruin and $R(x; b)$ the expectation of the discounted deficit at ruin. There are two possible values for b : the traditional, which is b^* , the value that maximizes $V(x; b)$, or alternatively, according to Dickson and Waters (2004), the optimal value b° maximizes $W(x; b) = V(x; b) - R(x; b)$. The functions $V(x; b)$ and $W(x; b)$ satisfy the integro-differential equations

$$(c + \rho x)V'(x; b) - (\lambda + \delta)V(x; b) + \lambda \int_0^x V(y; b)p(x - y) dy = 0, \quad 0 < x < b, \quad (\text{D.1})$$

and

$$(c + \rho x)W'(x; b) - (\lambda + \delta)W(x; b) + \lambda \int_0^x W(y; b)p(x - y) dy = \lambda \int_x^\infty (1 - P(y)) dy, \quad 0 < x < b. \quad (\text{D.2})$$

They should be combined with the boundary conditions

$$V'(b; b) = 1 \quad (\text{D.3})$$

and

$$W'(b; b) = 1, \quad (\text{D.4})$$

respectively.

Consider the special case of an exponential claim amount distribution,

$$p(y) = \beta e^{-\beta y}, \quad 1 - P(y) = e^{-\beta y}, \quad y > 0. \quad (\text{D.5})$$

By applying the operator $(d/dx + \beta)$ to the integro-differential equations (D.1) and (D.2), we can eliminate the integrals. In either case we obtain a differential equation of the form

$$(c + \rho x)f''(x) + (\beta(c + \rho x) + \rho - \lambda - \delta)f'(x) - \beta\delta f(x) = 0, \quad 0 < x < b, \quad (\text{D.6})$$

where $f(x)$ is $V(x; b)$ or $W(x; b)$. Note that the differential equation (D.6) has a two-parameter family of solutions, while the solutions of the integro-differential equations (D.1) and (D.2) are only one-

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parameter families. To make up for this, we need an additional condition in connection with equation (D.6). It is obtained by setting $x = 0$ in equations (D.1) and (D.2), respectively. Hence,

$$cV'(0; b) - (\lambda + \delta)V(0; b) = 0 \tag{D.7}$$

and

$$cW'(0; b) - (\lambda + \delta)W(0; b) = \frac{\lambda}{\beta}. \tag{D.8}$$

As shown by Paulsen and Gjessing (1997), equation (D.6) can be converted into Kummer's confluent hypergeometric equation. They introduce the new variable

$$z = -\frac{\beta}{\rho} (c + \rho x) \tag{D.9}$$

and the function $h(z)$, which is defined by the condition that $f(x) = h(z)$. Hence,

$$f'(x) = h'(z) \frac{dz}{dx} = -\beta h'(z), \tag{D.10}$$

$$f''(x) = \beta^2 h''(z). \tag{D.11}$$

With this substitution, equation (D.6) becomes the differential equation

$$zh''(z) + \left(1 - \frac{\lambda + \delta}{\rho} - z\right) h'(z) + \frac{\delta}{\rho} h(z) = 0, \quad -\frac{\beta}{\rho} (c + \rho b) < z < -\frac{\beta}{\rho} c. \tag{D.12}$$

Thus the function $h(z)$ satisfies equation (A.1) of the Appendix, with parameters

$$a_A = -\frac{\delta}{\rho}, \tag{D.13}$$

$$c_A = 1 - \frac{\lambda + \delta}{\rho}. \tag{D.14}$$

We can conclude that

$$V(x; b) = \alpha_V (-z)^{1-c_A} e^{sz} M(1 - a_A, 2 - c_A; -z) + \beta_V e^{sz} U(c_A - a_A, c_A; -z), \tag{D.15}$$

$$W(x; b) = \alpha_W (-z)^{1-c_A} e^{sz} M(1 - a_A, 2 - c_A; -z) + \beta_W e^{sz} U(c_A - a_A, c_A; -z), \tag{D.16}$$

for certain coefficients α_V, β_V and α_W, β_W , respectively. To determine α_V, β_V , we will substitute formula (D.15) in conditions (D.3) and (D.7). To determine α_W, β_W , substitute formula (D.16) in conditions (D.4) and (D.8). In each case we obtain two linear equations for the pair of coefficients. Thanks to Mathematica, $V(x; b)$ and $W(x; b)$ and the optimal barriers b^* and b^o can be calculated.

In Tables 1 and 2 the optimal barriers b^* and b^o are shown as well as the corresponding expectations of the discounted dividends if $x = 1$. The values in the first lines ($\rho = 0$) are calculated according to

Table 1
Influence of ρ and δ on b^* and $V(1; b^*)$, $c = 2, \lambda = 1, \beta = 1$

ρ	b^* and $(V(1; b^*))$			
	$\delta = 2.5\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
0	9.96 (22.65)	7.00 (10.68)	4.21 (5.36)	1.83 (3.16)
0.5%	10.45 (23.90)	7.26 (11.08)	4.34 (5.47)	1.88 (3.18)
1.0%	11.04 (25.23)	7.53 (11.50)	4.47 (5.58)	1.93 (3.20)
2.0%	13.13 (28.23)	8.16 (12.41)	4.74 (5.82)	2.04 (3.24)
3.0%		8.98 (13.43)	5.05 (6.08)	2.16 (3.29)

Table 2
Influence of ρ and δ on b° and $W(1; b^\circ)$, $c = 2$, $\lambda = 1$, $\beta = 1$

ρ	b° and $(W(1; b^\circ))$			
	$\delta = 2.5\%$	$\delta = 5\%$	$\delta = 10\%$	$\delta = 20\%$
0	10.03 (22.33)	7.13 (10.33)	4.47 (4.97)	2.28 (2.68)
0.5%	10.51 (23.58)	7.38 (10.74)	4.59 (5.08)	2.33 (2.71)
1.0%	11.09 (24.92)	7.65 (11.17)	4.72 (5.20)	2.38 (2.74)
2.0%	13.18 (27.94)	8.27 (12.10)	4.99 (5.46)	2.49 (2.80)
3.0%		9.09 (13.14)	5.29 (5.74)	2.61 (2.86)

Gerber, Shiu, and Smith (2006); see in particular formulas (5.12) and (6.2) or Tables 1 and 2 with $\theta = 1$ and $\alpha = \delta$. Note that $b^\circ > b^*$ consistently, and that the optimal barriers are increasing functions of ρ as in the paper.

PART 2

For given $\mu > 0$, $\sigma > 0$, consider the family of shifted compound Poisson processes with exponential claim amounts such that the expected gain per unit time is μ and the variance of the gain per unit time is σ^2 . Hence, for the three parameters c , λ , β of the model, we obtain two conditions:

$$\begin{aligned} c - \frac{\lambda}{\beta} &= \mu, \\ \frac{2\lambda}{\beta^2} &= \sigma^2. \end{aligned} \quad (\text{D.17})$$

Thus λ and c are the following functions of β :

$$\begin{aligned} \lambda &= \frac{1}{2}\sigma^2\beta^2, \\ c &= \mu + \frac{1}{2}\sigma^2\beta. \end{aligned} \quad (\text{D.18})$$

In the limit $\beta \rightarrow \infty$, the shifted compound Poisson process becomes the Brownian motion with parameters μ and σ , which is considered in the paper. Tables 3 and 4 illustrate the convergence of b^* , $V(1; b^*)$, b° , and $W(1; b^\circ)$. The entries in the bottom line are for the Brownian motion. They are taken from Tables 3 and 4 of the paper. Evidently, in the limit, there is no difference between b^* and b° , and $V(1; b^*)$ and $W(1; b^\circ)$. Furthermore, note that b^* , $V(1; b^*)$, b° and $W(1; b^\circ)$ are essentially linear in $1/\beta$ near the limit.

Table 3
From the Shifted Compound Poisson Process to Brownian Motion, $\delta = 4\%$, $\rho = 2\%$, $\mu = 1$, $\sigma = 5$

β	λ	c	b^*	$V(1; b^*)$	b°	$W(1; b^\circ)$
1	25/2	27/2	25.79	4.82	26.24	3.99
2	50	26	26.03	3.81	26.24	3.38
4	200	51	26.12	3.27	26.22	3.05
8	800	101	26.16	2.99	26.21	2.88
16	3,200	201	26.17	2.85	26.20	2.79
∞	∞	∞	26.19	2.70	26.19	2.70

Table 4
From the Shifted Compound Poisson Process to Brownian Motion,
 $\delta = 4\%$, $\rho = 2\%$, $\mu = 1$, $\sigma = 0.5$

β	λ	c	b^*	$V(1; b^*)$	b^o	$W(1; b^o)$
1	1/8	9/8	5.100	22.297	5.150	22.235
2	1/2	5/4	3.952	23.036	3.965	23.012
4	2	3/2	2.948	23.943	2.952	23.935
8	8	2	2.268	24.622	2.269	24.621
16	32	3	1.861	24.978	1.862	24.978
32	128	5	1.635	25.144	1.635	25.144
64	512	9	1.515	25.223	1.515	25.223
128	2,048	17	1.454	25.262	1.454	25.262
∞	∞	∞	1.390	25.300	1.390	25.300

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ANDREW C. Y. NG*

I wish to congratulate Professors Cai, Gerber, and Yang for this interesting paper. Here I would like to present an alternative method for solving equation (2.6) and derive an integro-differential equation analogous to equation (2.4) in a more general setting.

AN ALTERNATIVE SOLUTION OF EQUATION (2.6)

In their paper Professors Cai, Gerber, and Yang present a clever method to convert equation (2.6) into Kummer's confluent hypergeometric equation (3.2). While equation (3.2) appears more complicated than the original equation, it admits an analytic solution in terms of the confluent hypergeometric functions of the first and second kinds. An alternative method to solve equation (2.6) is to convert it into the parabolic cylinder equation

$$\frac{d^2y}{dx^2} + (ax^2 + bx + c)y = 0,$$

which admits an analytic solution in terms of the confluent hypergeometric function of the first kind; see, for example, Section 19 of Abramowitz and Stegun (1972) or <http://mathworld.wolfram.com/paraboliccylinderfunction.html>.

First, we let $z = (1/\sigma) \sqrt{2/\rho} (\rho x + \mu)$ and define the function $f(z)$ such that $g(x) = f(z)$. Substitution in equation (2.6) yields

$$\rho f''(z) + z\rho f'(z) - \delta f(z) = 0, \quad z > \frac{\mu}{\sigma} \sqrt{\frac{2}{\rho}}. \quad (D.1)$$

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This is the same procedure as in the first step of the derivation of equation (3.1). Then we define $f(z) = v(z) \exp(-z^2/4)$. Hence,

$$\begin{aligned} f'(z) &= \left(v'(z) - \frac{z}{2} v(z) \right) \exp(-z^2/4), \\ f''(z) &= \left(v''(z) - z v'(z) + \frac{z^2 - 2}{4} v(z) \right) \exp(-z^2/4). \end{aligned}$$

Substitution in equation (D.1) yields

$$v''(z) - \left(\frac{z^2}{4} + \frac{1}{2} + \frac{\delta}{\rho} \right) v(z) = 0, \quad z > \frac{\mu}{\sigma} \sqrt{\frac{2}{\rho}}.$$

The solution of the parabolic cylinder equation above is

$$v(z) = C_1 e^{-z^2/4} M\left(\frac{1}{2} + \frac{\delta}{2\rho}, \frac{1}{2}; \frac{1}{2} z^2\right) + C_2 z e^{-z^2/4} M\left(1 + \frac{\delta}{2\rho}, \frac{3}{2}; \frac{1}{2} z^2\right),$$

where C_1 and C_2 are constants; see Section 19.2 of Abramowitz and Stegun (1972). As a result,

$$g(x) = C_1 e^t M\left(\frac{1}{2} + \frac{\delta}{2\rho}, \frac{1}{2}; -t\right) + C_2 (-2t)^{1/2} e^t M\left(1 + \frac{\delta}{2\rho}, \frac{3}{2}; -t\right), \quad (\text{D.2})$$

where $t = -(1/(\rho\sigma^2)) (\mu + \rho x)^2$ as in equation (3.5). By using the identity

$$U(a, c; x) = \pi \csc(\pi c) \left[\frac{M(a, c; x)}{\Gamma(a - c + 1)} - \frac{x^{1-c} M(a - c + 1, 2 - c; x)}{\Gamma(a)} \right],$$

it can be shown that equation (3.4) is the same as equation (D.2). The values of C_1 and C_2 can be obtained from condition (2.5). For example, we may set

$$C_1 = -\frac{\mu}{\sigma} \sqrt{\frac{2}{\rho}} M\left(1 + \frac{\delta}{2\rho}, \frac{3}{2}; \frac{\mu^2}{\rho\sigma^2}\right), \quad C_2 = M\left(\frac{1}{2} + \frac{\delta}{2\rho}, \frac{1}{2}; \frac{\mu^2}{\rho\sigma^2}\right).$$

The first derivative of $g(x)$ can be obtained by using the product rule combined with equation (A.2) similar to the derivation of equation (3.7). A numerical study using MATLAB was performed, and it is found that equation (D.2) with the choices of C_1 and C_2 as above gives the same values of $V(x; 10)$ for $\sigma = 3, 5, 7$, and 10 .

A GENERALIZATION OF EQUATION (2.4)

Now we derive equation (2.4) in a more general setting. Let $\{J(t)\}_{t \geq 0}$ be a homogeneous continuous-time Markov chain taking values in $\mathcal{M} = \{1, 2, \dots, n\}$ with generator $\Lambda = (\lambda_{ij})$, which is assumed to be irreducible. $J(t)$ governs the state of economy. When the state of economy is i , the surplus of the company $X(t)$ is governed by the following dynamics:

$$dX(t) = (\mu_i + \rho_i X(t)) dt + \sigma_i dW(t) + dC_i(t), \quad t \geq 0. \quad (\text{D.3})$$

Here $\{C_i(t)\}_{t \geq 0}$ are pure jump processes for $i \in \mathcal{M}$, and each $\{C_i(t)\}$ is independent of both the Brownian motion $\{W(t)\}$ and the pure jump processes in other states. This means that when the state of economy is i , $X(t)$ evolves with a linear drift $\mu_i + \rho_i X(t)$, but is perturbed by a Brownian motion with variance per unit time σ_i^2 and jumps. Equation (D.3) reduces to equation (2.1) when there is only one state of economy and no jumps.

Let $b > 0$ be the barrier, which does not vary with time, and $D(t)$ be the aggregate dividends by time t . The discounted value of the dividends until ruin is

$$D = \int_0^T \exp \left(- \int_0^s \delta_{J(t)} dt \right) dD(s)$$

where T is the time of ruin and δ_i is the constant force of interest when the state of economy is i . The quantity of interest is

$$V_i(x; b) = E(D|X(0) = x, J(0) = i),$$

the expected discounted dividends until ruin, given that the barrier is b , the initial surplus is x , and the initial state of economy is i .

An equation satisfied by $V_i(x; b)$, $i \in \mathcal{M}$, can be derived by Itô's lemma for diffusions with jumps. (See, for example, Proposition 8.14 of Cont and Tankov [2004]). Since a pure jump process can be thought of as a limit of compound Poisson processes, we will assume in the following that $C_i(t) = \sum_{k=1}^{N_i(t)} Y_k^i$, where $\{N_i(t)\}$ is a Poisson process of rate γ_i , independent of the iid random variables Y_k^i , the Brownian motion $\{W(t)\}$, and the compound Poisson processes in other states.

Consider the change of $X(t)$ in the time interval $[0, dt]$. The probability of no transition of state is $1 + \lambda_i dt + o(dt)$, and the probability of a single transition to state j is $\lambda_j dt + o(dt)$. Thus we have, for $0 < x < b$,

$$\begin{aligned} V_i(x; b)e^{\delta_i dt} &= E[V_i(X(dt); b)|X(0) = x] \\ &= (1 + \lambda_i dt)[V_i(x; b) + E(dV_i(X(t); b)|X(0) = x, J(\tau) = i \text{ for } 0 \leq \tau \leq dt)] \\ &\quad + \sum_{j \neq i} \lambda_j dt \cdot V_j(x; b) + o(dt). \end{aligned} \tag{D.4}$$

Given that the state of economy is i in $[0, dt]$, we have, by Itô's lemma for diffusions with jumps,

$$\begin{aligned} dV_i(X(t); b) &= V'_i(X(t); b)dX(t) + \frac{1}{2}V''_i(X(t); b)(dX(t))^2 + [V_i(X(t) + Y^i; b) - V_i(X(t); b)]dN_i(t) \\ &= V'_i(X(t); b)dX(t) + \frac{\sigma_i^2}{2}V''_i(X(t); b)dt + [V_i(X(t) + Y^i; b) - V_i(X(t); b)]dN_i(t). \end{aligned} \tag{D.5}$$

Because $\{N_i(t)\}$ is a Poisson process, the probabilities of its having one jump and more than one jump in $[0, dt]$ are $\gamma_i dt + o(dt)$ and $o(dt)$, respectively. Thus

$$E(dN_i(t)) = \gamma_i dt + o(dt).$$

By equation (D.5) and the independence of Y^i and $N_i(t)$, equation (D.4) can be simplified as

$$\begin{aligned} V_i(x; b)(1 + \delta_i dt) &= (1 + \lambda_i dt) \left[V_i(x; b) + (\mu_i + \rho_i x)V'_i(x; b) dt + \frac{\sigma_i^2}{2} V''_i(x; b) dt \right] \\ &\quad + \sum_{j \neq i} \lambda_j V_j(x; b) dt + \gamma_i E[V_i(x + Y^i; b) - V_i(x; b)] dt + o(dt), \end{aligned}$$

where the expectation is taken with respect to the jump size random variable Y^i . Canceling $V_i(x; b)$ on both sides, dividing through by dt , and taking limit, we obtain, for $i \in \mathcal{M}$ and $0 < x < b$,

$$\begin{aligned} \delta_i V_i(x; b) &= \underbrace{(\mu_i + \rho_i x)V'_i(x; b)}_{(I)} + \underbrace{\frac{\sigma_i^2}{2}(V''_i(x; b))}_{(II)} + \underbrace{\sum_{j=1}^n \lambda_j V_j(x; b)}_{(III)} \\ &\quad + \underbrace{\gamma_i \left[\int_{-\infty}^{\infty} V_i(x + y; b) dF_i(y) - V_i(x; b) \right]}_{(IV)}, \end{aligned} \tag{D.6}$$

where F_i is the distribution function of Y_i . The boundary conditions of the system of $2n$ equations above

are $V_i(0; b) = 0$ and $V_i(b; b) = 1$ for $i \in \mathcal{M}$. The third and fourth terms of equation (D.6) correspond to change of state and jumps, respectively.

The system of integro-differential equations (D.6) above generalizes equation (2.4) but is very hard to solve. In the case of no jumps in any state of economy, we can rewrite equation (D.6) as a system of first-order linear differential equations. Let $\{g_i(x)\}_{i \in \mathcal{M}}$ be a set of nontrivial solutions for the following linear system of equations:

$$\delta_i g_i(x) = (\mu_i + \rho_i x) g_i'(x) + \frac{\sigma_i^2}{2} g_i''(x) + \sum_{j=1}^n \lambda_{ij} g_j(x). \quad (\text{D.7})$$

Define $\mathbf{g}(x) = (g_1(x), g_2(x), \dots, g_n(x))^T$. Then equation (D.7) can be written as

$$\begin{bmatrix} \mathbf{g}(x) \\ \mathbf{g}'(x) \end{bmatrix}' = \mathbf{A}(x) \begin{bmatrix} \mathbf{g}(x) \\ \mathbf{g}'(x) \end{bmatrix} \quad (\text{D.8})$$

where the coefficient matrix $\mathbf{A}(x)$ is defined by the block matrix

$$\begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \\ \mathbf{C} & \mathbf{D}(x) \end{bmatrix}.$$

The matrices $\mathbf{C} = (c_{ij})_{n \times n}$ and $\mathbf{D}(x) = (d_{ij}(x))_{n \times n}$ are defined by

$$\mathbf{C} = 2 \left(\text{diag} \left(\frac{\delta_i}{\sigma_i^2} \right) - \left(\frac{\lambda_{ij}}{\sigma_i^2} \right)_{n \times n} \right) \quad \text{and} \quad \mathbf{D}(x) = -2 \text{diag} \left(\frac{\mu_i + \rho_i x}{\sigma_i^2} \right).$$

If $\rho_i = 0$ for all $i \in \mathcal{M}$, then $\mathbf{A}(x) = \mathbf{A}$ does not depend on x . This special case generalizes the model discussed in Gerber and Shiu (2004) to a Markovian regime-switching setting; see Ng and Yang (2006) for an application of Markovian regime-switching model in ruin theory. The solution of equation (D.8) in this case is

$$\begin{bmatrix} \mathbf{g}(x) \\ \mathbf{g}'(x) \end{bmatrix} = e^{x\mathbf{A}} \begin{bmatrix} \mathbf{g}(0) \\ \mathbf{g}'(0) \end{bmatrix}.$$

Since the boundary conditions $V_i(0, b) = 0$ lead to $g_i(0) = 0$ for $i \in \mathcal{M}$, we have $\mathbf{g}(0) = \mathbf{0}$. It remains to determine $\mathbf{g}'(0)$. The boundary conditions $V_i(b; b) = 1$ lead to $g_i'(b) = 1$ for $i \in \mathcal{M}$, which give rise to a system of n linear equations for determining $\mathbf{g}'(0)$.

The method of obtaining equation (D.5) above can also be used to derive an integro-differential equation for the expected present value of a payment of 1 due at the time of ruin under the Markovian regime-switching setting. Let

$$L_i(x; b) = \mathbb{E} \left[\exp \left(- \int_0^T \delta_{J(t)} dt \right) \middle| X(0) = x, J(0) = i \right]$$

be defined similar to equation (5.1); then $L_i(x; b)$ satisfies

$$\delta_i L_i(x; b) = (\mu_i + \rho_i x) L_i'(x; b) + \frac{\sigma_i^2}{2} L_i''(x; b) + \sum_{j=1}^n \lambda_{ij} L_j(x; b) + \gamma_i \left[\int_{-\infty}^{\infty} L_i(x + y; b) dF_i(y) - L_i(x; b) \right]$$

for $0 < x < b$ in conjunction with the boundary conditions $L_i(0; b) = 1$ and $L'(b; b) = 0$ for all $i \in \mathcal{M}$. Again, if $\rho_i = 0$ for all $i \in \mathcal{M}$, the system of integro-differential equations can be solved analogously.

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JINXIA ZHU*

This paper is really interesting. In this discussion we are going to study the expected ruin time in the model where there is a barrier and the company is allowed to continue its business after its reserve falls below zero as long as it stays above the critical level. This discussion uses the same notation as in the paper. Here we assume that ruin occurs the first time the reserve process goes below the level y (the absorption point, $y \leq 0$). As stated in the paper, the reserve surplus process follows the following dynamics:

$$\begin{aligned} dX(t) &= (\mu + \rho X(t)) dt + \sigma dW(t), & 0 \leq X(t) \leq b, \\ dX(t) &= (\mu + \tau X(t)) dt + \sigma dW(t), & y \leq X(t) \leq 0, \end{aligned}$$

Define

$$E_x[\cdot] = E[\cdot | X(0) = x],$$

and the ruin time by

$$T(y) = \inf\{t : X(t) = y\}.$$

Let

$$L(x, y) = E_x[e^{-\delta T(y)}].$$

Using the same approach as in Gerber and Shiu (2004), it can be seen that $L(x, y)$ satisfies the following differential equations:

$$\sigma^2 \frac{\partial^2}{\partial x^2} L(x, y) + (\mu + \rho x) \frac{\partial}{\partial x} L(x, y) - \delta L(x, y) = 0, \quad 0 < x < b$$

and

$$\sigma^2 \frac{\partial^2}{\partial x^2} L(x, y) + (\mu + \tau x) \frac{\partial}{\partial x} L(x, y) - \delta L(x, y) = 0, \quad y < x < 0,$$

in combination with the boundary conditions

$$L(y, y) = 1,$$

$$\frac{\partial}{\partial x} L(x, y)|_{x=b} = 0,$$

$$L(0+, y) = L(0-, y),$$

and

$$\frac{\partial}{\partial x} L(x, y)|_{x=0+} = \frac{\partial}{\partial x} L(x, y)|_{x=0-}.$$

We can calculate the expectation of the time of ruin by the formula

$$E_x[T(y)] = -\frac{\partial L(x, y)}{\partial \delta} \Big|_{\delta=0}.$$

Now, we will consider two cases. In the first case, the absorption point $y = 0$: that is, ruin occurs

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Table 1
Expected Ruin Time $E_x[T(y)]$ When $\mu = 1$, $\sigma = 3.0$, $\tau = 6\%$, $b = 10$, $y = 0$ ($y = \lambda = -16.67$)

ρ	x			
	0.2	1	6	10
0	1.63 (604.47)	7.27 (610.14)	24.58 (627.44)	27.03 (629.89)
0.01	1.70 (636.71)	7.71 (642.72)	25.97 (660.99)	28.49 (663.50)
0.02	1.81 (671.57)	8.17 (677.95)	27.47 (697.24)	30.06 (699.82)
0.04	2.04 (750.09)	9.25 (757.30)	30.82 (778.87)	33.56 (781.61)

whenever the reserve first reaches 0. In the second case, the absorption point is the critical value λ : that is, the company is allowed to continue when its reserve is negative as long as it stays above the critical level. In both cases we assume $\mu = 1$, $b = 10$, $\sigma = 3$, and $\tau = 6\%$. Mathematica has been used to calculate the expected ruin time $E_x[T(y)]$ for different values of credit rate ρ and initial surplus x . The results are displayed in Table 1, where the results for the second case are shown in parentheses.

Notice that the quantities in parentheses are remarkably larger than the corresponding ones outside the parentheses. This seems surprising considering that a higher debit interest rate is imposed on the company when its reserve is negative, which is adverse to the length of the ruin time. To find the underlying factors that may contribute to this, we are going to look at more cases.

We will consider another two cases, where in one case the absorption value is 0 and the barrier is raised to 20, and in the other the absorption value is lowered to -10 and the barrier is 10. The expected ruin times for different initial surplus in these cases are shown in Tables 2 and 3 (in parentheses), respectively.

In Table 2, we see that the quantities in the first row are larger than the corresponding ones in the second row except for the last column. This phenomenon is surprising since we expect that the barrier should have some effect on the ruin time. Before we explain this, look at Table 3, the results of which seem to be reasonable. Note that the distance between the initial surplus and the absorption point in each column is distinctly different for cases in Table 2, while it is the same for those in Table 3. From these, we guess that it is the distance between the initial surplus and the absorption point that causes the big difference in the expected ruin times for the two cases in Table 1.

But, if we further look at Table 2, we find that the last number of the second row is larger than the one in the first row, and the last two numbers of the third row are larger than the corresponding ones of the first row, respectively, despite that the distances between the initial surplus and the absorption point in the second and third rows are larger than that of the first row. This indicates that the impact of the barrier on the expected ruin time also could be great for a relatively large initial surplus. Here is the explanation: When the distance is very small, the process is very likely to go down to the absorption point before it goes up high enough to reach the barrier, giving little chance for the barrier to have some effect on it. When the distance is relatively large, the probability for the reserve process to reach the barrier before it stops is not so small. Therefore the barrier has a large chance of influencing the expected ruin time. This also accounts for the observation that the larger the fixed initial surplus, the greater impact the barrier has on the expected ruin time (see Table 4).

Table 2
Expected Ruin Time $E_x[T(y)]$ When $\mu = 1.0$, $\sigma = 3.0$, $\rho = 1\%$, $\tau = 6\%$

x	0.2	0.6	1	6
$y = -10, b = 10$	255.24	258.40	261.25	279.51
$y = 0, b = 20$	22.19	63.69	101.61	369.93
$y = 0, b = 25$	83.43	239.51	382.21	1397.85

Table 3
**Expected Ruin Time $E_x[T(y)]$ When $\mu = 1.0$, $\sigma = 3.0$, $\rho = 1\%$, $\tau = 6\%$, $b = 10$,
 $y = -10$ ($b = 20$, $y = 0$)**

x	-9.8(0.2)	-9.4(0.6)	-9(1)	-4(6)
$E[T(b, y)]$	9.14 (22.19)	26.88 (63.69)	43.92 (101.61)	197.35 (369.93)

Table 4
Expected Ruin Time $E_x[T(0)]$ When $\mu = 1.0$, $\sigma = 3.0$, $\rho = 1\%$

b	x						
	0.2	0.6	0.8	1	2	4	6
10	1.70	4.86	6.32	7.71	13.68	21.66	25.97
15	6.15	17.62	22.97	28.08	50.32	81.60	100.59
20	22.19	63.69	83.08	101.61	182.59	297.95	369.93
25	83.43	239.51	312.46	382.21	687.39	1123.67	1397.85

Table 5
Expected Ruin Time $E_x[T(y)]$ When $\mu = 1.0$, $\sigma = 3.0$, $b = 10$, $\rho = 1\%$, $\tau = 6\%$

x	y								
	0	-0.2	-0.4	-0.8	-2	-5	-8	-10	-15
0.2	1.70	3.49	5.37	9.40	23.96	80.16	171.87	255.24	531.57
0.4	3.32	5.11	6.98	11.02	25.58	81.78	173.49	256.86	533.19
1	7.71	9.50	11.38	15.41	29.97	86.17	177.88	261.25	537.58
2	13.68	15.46	17.34	21.37	35.93	92.14	183.85	267.21	543.55
4	21.66	23.44	25.32	29.35	43.91	100.12	191.83	275.19	551.53
8	27.96	29.75	31.63	35.66	50.22	106.42	198.13	281.50	557.83

Moreover, we note that for a fixed initial surplus, the higher the barrier, the greater impact it has on the expected ruin time. For instance, the change of barrier from 20 to 25 extends the expected ruin time more than that from 15 to 20. This is easy to understand considering that the instantaneous drift increases with the reserve.

Furthermore, from Table 5 we observe that the change of absorption point y causes the same amount of change of the expected ruin time for different x when $x > y$, and that the same amount of change of the initial surplus x gives rise to the same amount of increment to the expected ruin time for any y when $x > y$. These are easy to verify, noticing that for any $x > y$ and $h > 0$,

$$E_x[T(y - h)] = E_x[T(y)] + E_y[T(y - h)]$$

and

$$E_{x+h}[T(y)] = E_{x+h}[\tau(x)] + E_x[T(y)],$$

where $\tau(x) = \inf\{t \geq 0: X(t) = x\}$.

REFERENCE

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AUTHORS' REPLY

We are grateful to Nathaniel Smith, Andrew C. Y. Ng, and Jinxia Zhu for their discussions. Their contributions add new models and results and provide alternative methods and solutions. We feel that these three discussions truly enhance our paper.

Mr. Smith develops a clever idea. Because the diffusion process in our paper can be obtained as a limit of a family of shifted compound Poisson processes, he proposes to do the computations for the latter and then to obtain the results of our paper as limits. His observation that the convergence is proportional to the mean of the exponential claim amount distribution is most interesting. The content of this discussion could be a paper in itself.

In the first part of Mr. Ng's discussion, he develops an alternative and appealing set of formulas to calculate the expectation of the discounted dividends until ruin. For large σ such as $\sigma = 3, 5, 7,$ and 10 , using Mathematica, Ng's formula (D.2) yields the same results as formula (3.4) of the paper for $V(x, 10)$ in Table 1 of the paper. However, we found that for small values of σ such as $\sigma = 0.5$ and 0.9 and some values of ρ such as $\rho = 1\%, 2\%,$ and 3% , using Mathematica, Ng's formula (D.2) yields some negative values for $V(x, 10)$ in Table 1 of the paper. We guess that these unreasonable negative values are because Ng's formula (D.2) or Mathematica is very sensitive to the small values of σ . In any case, it would be worthwhile to make further comparisons between Ng's formula (D.2) and formula (3.4) of the paper. The second part of the discussion considers a Markovian regime-switching model and derives a version of formula (2.4) for this model. It is not easy to solve the equation in this case. If the goal is the search for an optimal dividend strategy, one might want to look at barrier strategies where the barrier parameter is a function of the state of the nature.

Ms. Zhu's discussion studies the expected time to ruin when the company is allowed to continue its business even if its surplus falls below zero, as long as it stays above the critical level. Because ruin is certain under a barrier strategy, it is indeed an excellent idea to study the expected time to ruin and, more generally, the expected time to fall to a certain level. Zhu makes a useful observation: because the sample paths of the surplus process are continuous, the expected travel times are additive.

Additional discussions on this paper can be submitted until October 1, 2006. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.