On the Decomposition of the Ruin Probability for a Jump-Diffusion Surplus Process Compounded by a Geometric Brownian Motion

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Abstract

If one assumes that the surplus of an insurer follows a jump-diffusion process and the insurer would invest its surplus in a risky asset, whose prices are modeled by a geometric Brownian motion, the resulting surplus for the insurer is called a jump-diffusion surplus process compounded by a geometric Brownian motion. In this resulting surplus process, ruin may be caused by a claim or oscillation. We decompose the ruin probability in the resulting surplus process into the sum of two ruin probabilities: the probability that ruin is caused by a claim, and the probability that ruin is caused by oscillation. Integro-differential equations for these ruin probabilities are derived. When claim sizes are exponentially distributed, asymptotical formulas of the ruin probabilities are derived from the integro-differential equations, and it is shown that all three ruin probabilities are asymptotical power functions with the same orders and that the orders of the power functions are determined by the drift and volatility parameters of the geometric Brownian motion. It is known that the ruin probability for a jump-diffusion surplus process is an asymptotical exponential function when claim sizes are exponentially distributed. The results of this paper further confirm that risky investments for an insurer are dangerous in the sense that either ruin is certain or the ruin probabilities are asymptotical power functions, not asymptotical exponential functions, when claim sizes are exponentially distributed.

1. Introduction

Consider the surplus of an insurer follows the jump-diffusion process

$$U_t = u + ct - \sum_{k=1}^{N(t)} Z_k + \sigma W_t, \qquad t \geq 0, \tag{1.1}$$

which is also called a compound Poisson surplus process perturbed by a diffusion. Here $u \geq 0$ is the initial surplus, $c > 0$ is the rate of premiums, $\{Z_k, k = 1, 2, \ldots\}$ is a sequence of independent and identically distributed (i.i.d.) nonnegative random variables, denoting claim sizes, $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, representing the number of claims up to time t, $\{W_t, t \geq 0\}$ is a standard Brownian motion, and $\sigma > 0$ is a constant, representing the diffusion volatility. In addition, $\{Z_k, k = 1, 2, \ldots\}$, $\{N(t), t \geq 0\}$, and $\{W_t, t \geq 0\}$ are independent. Ruin in this perturbed compound Poisson surplus process has been studied extensively in the literature: see, for example, Dufresne and Gerber (1991), Schlegel (1998), and Yang and Zhang (2001) and references therein.

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Now, suppose that the insurer would invest its surplus in a risky asset, whose prices are given by a geometric Brownian motion $\{e^{\Delta t} = e^{\delta t + rB_t}, t \geq 0\}$, where $\delta > 0$ and r are constants, and $\{B_t, t \geq 0\}$ is a standard Brownian motion independent of $\{U_t, t \geq 0\}$. Thus, one unit invested at time 0 will accumulate to $e^{\Delta t}$ at time t. Recall that a classical model for stock prices or equity mutual funds is the geometric Brownian motion model.

Let X_t denote the surplus of the insurer at time t under this investment assumption. Thus,

$$X_t = e^{\Delta t}\left(u + \int_0^t e^{-\Delta s}\, dU_s\right), \qquad t \geq 0, X_0 = u. \tag{1.2}$$

This resulting surplus process $\{X_t\}$ is called a jump-diffusion surplus process compounded by a geometric Brownian motion.

In this surplus process (1.2), ruin may occur in two different situations: when ruin is caused by a claim, and when ruin is caused by oscillation. We denote the ruin time of the surplus process (1.2) by T, that is, $T = \inf\{t : X_t < 0\}$ and $= \infty$ if $X_t \geq 0$ for all $t \geq 0$. We denote the ruin probability with the initial surplus $u \geq 0$ by $\psi(u)$, namely, $\psi(u) = \Pr\{T < \infty | X_0 = u\} = \Pr\{X_t < 0 \text{ for some } t \geq 0 | X_0 = u\}$.

Furthermore, we denote $T_s = \inf\{t : X_t < 0, X_h > 0, 0 < h < t\}$ and ∞ if $X_t \geq 0$ for all $t > 0$; namely, T_s is the ruin time at which ruin is caused by a claim. We also denote $T_d = \inf\{t : X_t = 0, X_h > 0, 0 < h < t\}$ and ∞ if $X_t \geq 0$ for all $t \geq 0$; namely, T_d is the ruin time at which ruin is caused by oscillation. Then $T = \min\{T_s, T_d\}$. Moreover, we denote ruin probabilities in the two situations, respectively, by

$$\psi_s(u) = \Pr\{T_s < \infty | X_0 = u\}$$

and

$$\psi_d(u) = \Pr\{T_d < \infty | X_0 = u\}.$$

It is obvious that the ruin probability $\psi(u)$ can be decomposed as

$$\psi(u) = \psi_s(u) + \psi_d(u), \qquad u \geq 0. \tag{1.3}$$

In addition, it follows from the oscillating nature of the sample paths of $\{X_t\}$ that

$$\psi_d(0) = \psi(0) = 1 \quad \text{and} \quad \psi_s(0) = 0. \tag{1.4}$$

See Dufresne and Gerber (1991) for a detailed discussion of equations (1.3) and (1.4) in the perturbed compound Poisson surplus process (1.1) model.

The surplus process (1.2) can be viewed as a special case of the surplus process (2.4) of Paulsen and Gjessing (1997), who studied ruin probability for a jump-diffusion surplus process compounded by another jump-diffusion process. When $\sigma = 0$, the surplus process (1.2) is a compound Poisson surplus process compounded by a geometric Brownian motion, and Frolova, Kabanov, and Pergamenshchikoc (2002) showed that the ruin probability $\psi(u)$ in this case is an asymptotical power function. This surplus process also was discussed in Cai (2004), Wang and Wu (2001), and others. When $\sigma > 0$, the integro-differential equation for the ruin probability $\psi(u)$ can be found in Paulsen and Gjessing. However, they did not consider the ruin probabilities $\psi_s(u)$ and $\psi_d(u)$. Furthermore, the methods used in Paulsen and Gjessing (1997) do not apply to $\psi_s(u)$ and $\psi_d(u)$.

In this paper, we first derive integro-differential equations for the ruin probabilities $\psi_s(u)$ and $\psi_d(u)$ using a differential argument and then obtain the corresponding equation for $\psi(u)$ using equation (1.3). A lower bound for $\psi(u)$ is given based on the ruin probability in the pure diffusion risk model, that is, $\lambda = 0$ in equation (1.2). When claim sizes are exponentially distributed, we show that all the three ruin probabilities $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$ are asymptotical power functions with the same orders using the lower bound and Bellman's theorem for the second-order linear differential equation. Furthermore, the orders of the power functions are determined by the drift δ and volatility parameter r of the geometric Brownian motion and are the same as those in Frolova et al. (2002). These results further confirm that risky investments are dangerous for an insurer in the sense that the ruin probabilities are asymptotical power functions, not asymptotical exponential functions, when claim sizes are exponentially distributed.

2. INTEGRO-DIFFERENTIAL EQUATIONS FOR RUIN PROBABILITIES

In this section we first derive integro-differential equations for $\psi_s(u)$ and $\psi_d(u)$ using a differential argument, and then the integro-differential equation for $\psi(u)$ follows from equation (1.3). The differential argument is a common method in risk theory. See Grandell (1991) for the method used in the compound Poisson surplus process and Dufresne and Gerber (1991) for that used in the perturbed compound Poisson risk process.

Throughout this paper, we denote the distribution function of Z_1 by F with $F(0) = 0$ and the tail of a distribution function B by $\bar{B}(x) = 1 - B(x)$ and assume that F has a finite expectation. Furthermore, we define

$$a = \delta + \frac{r^2}{2} \tag{2.1}$$

and note that

$$2a > r^2. \tag{2.2}$$

The condition (2.2) guarantees that $\psi(u) < 1$ for $u \geq 0$; see, for example, Paulsen (1993).

Theorem 2.1

Assume that $\psi_s(u)$ is continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$. Then, for any $u > 0$, $\psi_s(u)$ satisfies the integro-differential equation

$$\tfrac{1}{2}(\sigma^2 + r^2u^2)\psi_s''(u) + (au + c)\psi_s'(u) + \lambda\bar{F}(u) = \lambda\psi_s(u) - \lambda \int_0^u \psi_s(u - y)\, dF(y) \tag{2.3}$$

and the following boundary conditions:

$$\begin{cases} \psi_s(+\infty) = 0, \\ \psi_s(0) = 0. \end{cases} \tag{2.4}$$

PROOF

Let

$$Y_t = ue^{\Delta t} + ce^{\Delta t}\int_0^t e^{-\Delta s}\, ds + \sigma e^{\Delta t}\int_0^t e^{-\Delta s}\, dW_s, \qquad Y_0 = u. \tag{2.5}$$

Consider the risk process $\{X_t\}$ defined by equation (1.2) in an infinitesimal time interval $(0, t]$. Note that $\{X_t\}$ is a strong Markov process. Furthermore, because $\{N(t)\}$ is a Poisson process, there are three possible cases in $(0, t]$:

1. There are no claims in $(0, t]$, thus $X_t = Y_t$.
2. There is exactly one claim in $(0, t]$:
   a. The amount of the claim $y < Y_t$, that is, ruin does not occur, and thus $X_t = Y_t - y$
   b. The amount of the claim $y > Y_t$, that is, ruin occurs due to the claim
   c. The amount of the claim $y = Y_t$, that is, ruin occurs due to oscillation (observe that the probability that this case occurs is zero).
3. There is more than one claim in $(0, t]$.

Thus, considering the three cases above and noticing that in case 2(b), $\psi_s(Y_t - y) = 1$ because $y > Y_t$, we have

$$\psi_s(u) = (1 - \lambda t)\mathrm{E}[\psi_s(Y_t)] + \lambda t \mathrm{E}\left[\int_0^{Y_t} \psi_s(Y_t - y)\, dF(y)\right] + \lambda t \mathrm{E}\left[\int_{Y_t}^{\infty} \psi_s(Y_t - y)\, dF(y)\right] + o(t)$$

$$= (1 - \lambda t)\mathrm{E}[\psi_s(Y_t)] + \lambda t \mathrm{E}\left[\int_0^{Y_t} \psi_s(Y_t - y)\, dF(y)\right] + \lambda t \mathrm{E}\,[\bar{F}(Y_t)] + o(t),$$

or equivalently,

$$\lambda t \mathrm{E}[\psi_s(Y_t)] = \mathrm{E}[\psi_s(Y_t)] - \psi_s(u) + \lambda t \mathrm{E}\left[\int_0^{Y_t} \psi_s(Y_t - y)\, dF(y)\right] + \lambda t \mathrm{E}[\bar{F}(Y_t)] + o(t). \tag{2.6}$$

Let A be the infinitesimal generator for Y_t. Similar to formula (2.6) of Paulsen and Gjessing (1997), we have

$$A\psi_s(u) = \lim_{t \to 0} \frac{\mathrm{E}[\psi_s(Y_t)] - \psi_s(u)}{t} = (c + au)\psi_s'(u) + \tfrac{1}{2}(\sigma^2 + r^2u^2)\psi_s''(u). \tag{2.7}$$

Therefore, by dividing t on both sides of equation (2.6), letting $t \to 0$, using equation (2.7), we obtain

$$\lambda\psi_s(u) = (au + c)\psi_s'(u) + \tfrac{1}{2}(\sigma^2 + r^2u^2)\psi_s''(u) + \lambda \int_0^u \psi_s(u - y)\, dF(y) + \lambda\bar{F}(u),$$

which implies equation (2.3).

In addition, the boundary condition $\psi_s(+\infty) = 0$ follows from $\psi_s(u) \le \psi(u)$ and $\psi(+\infty) = 0$; the boundary condition $\psi_s(0) = 0$ follows from equations (1.3) and (1.4). □

Theorem 2.2

Assume that $\psi_d(u)$ is continuous on $[0, \infty)$ and twice continuously differentiable on $(0, \infty)$. Then, for any $u > 0$, $\psi_d(u)$ satisfies the integro-differential equation

$$\tfrac{1}{2}(\sigma^2 + r^2u^2)\psi_d''(u) + (au + c)\psi_d'(u) = \lambda\psi_d(u) - \lambda \int_0^u \psi_d(u - y)\, dF(y) \tag{2.8}$$

and the following boundary conditions:

$$\begin{cases} \psi_d(+\infty) = 0, \\ \psi_d(0) = 1. \end{cases} \tag{2.9}$$

Proof

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and hence is omitted. □

Theorem 2.3

Under the assumptions of Theorems 2.1 and 2.2, for any $u > 0$, $\psi(u)$ satisfies the integro-differential equation

$$\tfrac{1}{2}(\sigma^2 + r^2u^2)\psi''(u) + (au + c)\psi'(u) + \lambda\bar{F}(u) = \lambda\psi(u) - \lambda \int_0^u \psi(u - y)\, dF(y) \tag{2.10}$$

and the following boundary conditions:

$$\begin{cases} \psi(+\infty) = 0, \\ \psi(0) = 1. \end{cases} \tag{2.11}$$

PROOF

It follows from equation (1.3) that $\psi'(u) = \psi_s'(u) + \psi_d'(u)$ and $\psi''(u) = \psi_s''(u) + \psi_d''(u)$. Hence, equation (2.10) follows from adding both sides of equation (2.3) to the corresponding sides of equation (2.8). □

We point out that Theorem 2.3 also can be obtained from Theorem 2.1 of Paulsen and Gjessing (1997). Furthermore, Cai and Yang (2005) derived the conditions under which $\psi(u)$ is twice continuously differentiable. However, as pointed out by Cai and Yang, their methods are not applicable for $\psi_s(u)$ and $\psi_d(u)$. We do not know under what conditions $\psi_s(u)$ and $\psi_d(u)$ are twice continuously differentiable. Also, Cai and Yang and Wang (2001) have considered the decomposition of the ruin probability in the jump-diffusion surplus process under a constant interest force.

3. RUIN WITH EXPONENTIAL CLAIM SIZES

In this section we derive asymptotical solutions of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$ when claim sizes are exponentially distributed. It is easy to derive the following corollary, which gives third-order differential equations satisfied by the ruin probabilities.

Corollary 3.1

Under the assumptions of Theorems 2.1–2.3, if F is an exponential distribution, with $F(x) = 1 - e^{-\beta x}$, $x > 0$, $\beta > 0$, then, for any $u > 0$, $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$ satisfy the same third-order differential equation:

$$h'''(u) + p(u)h''(u) + q(u)h'(u) = 0, \tag{3.1}$$

where $h(u)$ is any of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$,

$$p(u) = \beta + \frac{2(a + r^2)u}{r^2u^2 + \sigma^2} + \frac{2c}{r^2u^2 + \sigma^2},$$

and

$$q(u) = \frac{(2\beta a)u}{r^2u^2 + \sigma^2} + \frac{2(\beta c - \lambda + a)}{r^2u^2 + \sigma^2}.$$

The boundary conditions of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$ are given in Theorems 2.1–2.3 and the following equations:

$$\begin{cases} (\sigma^2/2)\psi_s''(0^+) + c\psi_s'(0^+) = -\lambda, \\ (\sigma^2/2)\psi_d''(0^+) + c\psi_d'(0^+) = \lambda, \\ (\sigma^2/2)\psi''(0^+) + c\psi'(0^+) = 0. \end{cases} \tag{3.2}$$

PROOF

The proof comes from Theorems 2.1–2.3 by setting $F(u) = 1 - e^{-\beta u}$. □

The third-order differential equations in Corollary 3.1, together with the boundary conditions, enable us to obtain asymptotical formulas for the ruin probabilities. To do so, we first give a lower bound for $\psi(u)$ and then recall some results about linear differential equations.

Let $\lambda = 0$, then the risk model (1.2) reduces to a pure diffusion risk model, in which the surplus follows a Brownian motion with drift and the asset price follows a geometric Brownian motion. Denote the ruin probability in this pure diffusion risk model by $\psi_0(u)$. Then it is obvious that

$$\psi(u) \geq \psi_0(u), \qquad u \geq 0. \tag{3.3}$$

However, equation (2.10) implies that $\psi_0(u)$ satisfies the second-order differential equation

$$\tfrac{1}{2}(\sigma^2 + r^2u^2)\psi_0''(u) + (au + c)\psi_0'(u) = 0 \tag{3.4}$$

and the following boundary conditions:

$$\begin{cases}\psi_0(+\infty) = 0,\\ \psi_0(0) = 1.\end{cases} \tag{3.5}$$

It is easy to solve the second-order differential equation (3.4) and obtain

$$\psi_0(u) = C_0 \int_u^\infty (r^2x^2 + \sigma^2)^{-a/r^2} \exp\left\{-\frac{2c}{r\sigma} \arctan\left(\frac{rx}{\sigma}\right)\right\} dx, \tag{3.6}$$

where

$$C_0^{-1} = \int_0^\infty (r^2x^2 + \sigma^2)^{-a/r^2} \exp\left\{-\frac{2c}{r\sigma} \arctan\left(\frac{rx}{\sigma}\right)\right\} dx. \tag{3.7}$$

It follows from L'Hospital's rule and $\exp\{-2c/(r\sigma)\arctan(rx/\sigma)\} \to \exp\{-c\pi/(r\sigma)\}$ as $x \to \infty$ that

$$\psi_0(u) \sim C_0 e^{-c\pi/(r\sigma)} \int_u^\infty (r^2x^2)^{-a/r^2}\, dx = K_0 u^{1-(2a/r^2)}, \qquad u \ge 0, \tag{3.8}$$

where

$$K_0 = \frac{C_0 e^{-c\pi/(r\sigma)} r^{-2a/r^2}}{(2a/r^2 - 1)} > 0$$

is a constant.

Equations (3.3) and (3.8) will be used to derive the asymptotical formulas of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$. For this purpose, let

$$w(u) = \psi'(u).$$

Then the third-order differential equation for $\psi(u)$ is reduced to the second-order differential equation

$$w'' + p(u)w' + q(u)w = 0. \tag{3.9}$$

Further, it follows from Lemma 4 of Bellman (1953, p. 109) that the substitutions

$$w(u) = e^{-(1/2)\int_0^u p(x)\,dx}\, y(u)$$

and

$$y(u) = z(x) \quad \text{with} \quad x = \frac{\beta}{2} u$$

reduce equation (3.9) to

$$z'' - (1 + f(x))z = 0, \tag{3.10}$$

where

$$f(x) = \frac{B_1 x}{4r^2x^2 + \sigma^2\beta^2} + \frac{B_2}{4r^2x^2 + \sigma^2\beta^2} + \frac{B_3 x}{(4r^2x^2 + \sigma^2\beta^2)^2} + \frac{B_4}{(4r^2x^2 + \sigma^2\beta^2)^2},$$

and

$$B_1 = 8(r^2 - a)$$

$$B_2 = 8\lambda - 4a - 4\beta c + 4a^2/r^2$$

$$B_3 = 16\beta a c$$

$$B_4 = 4\beta^2c^2 + 4\beta^2r^2\sigma^2 - 4\beta^2a^2\sigma^2/r^2.$$

By integrating $f(x)$, we obtain

$$\int_0^x f(t)\,dt = \ln\left(\frac{4r^2x^2 + \sigma^2\beta^2}{\sigma^2\beta^2}\right)^{1-a/r^2} + K\arctan\left(\frac{2xr}{\sigma\beta}\right) + \varepsilon(x) + L, \tag{3.11}$$

where K and L are constant and $\varepsilon(x) \to 0$ as $x \to \infty$.

It follows from $\int_0^\infty (f(x))^2\,dx < \infty$ and Bellman's theorem (Bellman 1953, Theorem 12) that there exist

$$\tilde{\varepsilon}_1(x) \to 0 \quad \text{and} \quad \tilde{\varepsilon}_2(x) \to 0$$

as $x \to \infty$ so that two fundamental solutions z_1 and z_2 for z in equation (3.10) are given by

$$z_1(x) = \exp\left\{x + \tfrac{1}{2}\int_0^x f(t)\,dt + \tilde{\varepsilon}_1(x)\right\}, \qquad x \geq 0,$$

and

$$z_2(x) = \exp\left\{-\left(x + \tfrac{1}{2}\int_0^x f(t)\,dt\right) + \tilde{\varepsilon}_2(x)\right\}, \qquad x \geq 0.$$

Furthermore, by equation (3.11), $\arctan(2xr/(\sigma\beta)) \to \pi/2$, and $\varepsilon(x) \to 0$ as $x \to \infty$, we know that the two independent solutions z_1 and z_2, apart from constant factors, satisfy

$$z_1(x) = \exp\{x + \tilde{\varepsilon}_1(x)\}\left(\frac{4r^2x^2 + \sigma^2\beta^2}{\sigma^2\beta^2}\right)^{(1-a/r^2)/2}, \qquad x \geq 0,$$

and

$$z_2(x) = \exp\{-x + \tilde{\varepsilon}_2(x)\}\left(\frac{4r^2x^2 + \sigma^2\beta^2}{\sigma^2\beta^2}\right)^{-(1-a/r^2)/2}, \qquad x \geq 0.$$

In addition, we have

$$e^{-(1/2)\int_0^u p(t)\,dt} = e^{-\beta u/2}\left(\frac{r^2u^2 + \sigma^2}{\sigma^2}\right)^{-(1+a/r^2)/2} \exp\left\{-\left(\frac{c}{r\sigma}\right)\arctan\left(\frac{ru}{\sigma}\right)\right\}$$

and $\exp\{-(c/(r\sigma))\arctan(ru/\sigma)\} \to \exp\{-c\pi/(2r\sigma)\}$ as $u \to \infty$.

Thus, two fundamental solutions for w in equation (3.9), apart from some constant factors, satisfy

$$w_1(u) = e^{\varepsilon_1(u)}\left(\frac{r^2u^2 + \sigma^2}{\sigma^2}\right)^{-a/r^2}, \qquad u \geq 0,$$

and

$$w_2(u) = e^{\varepsilon_2(u)}\left(\frac{r^2u^2 + \sigma^2}{\sigma^2}\right)^{-1}, \qquad u \geq 0,$$

where

$$\varepsilon_i(u) = \tilde{\varepsilon}_i(\beta u/2), \qquad i = 1, 2,$$

and

$$\varepsilon_i(u) \to 0$$

as $u \to \infty$ for $i = 1, 2$.

Observe that $\psi(u)$, $\psi_s(u)$, $\psi_d(u)$ satisfy the same equation only with different boundary conditions. Hence, there exist constants C_1, C_2, C_1^s, C_2^s, C_1^d, and C_2^d so that the general solutions of $\psi(u)$, $\psi_s(u)$, $\psi_d(u)$ are given by

$$\psi(u) = C_1 \int_u^\infty w_1(x)\, dx + C_2 \int_u^\infty w_2(x)\, dx$$
$$= C_1 \int_u^\infty e^{\varepsilon_1(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-a/r^2} dx + C_2 \int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx,$$
$$\psi_s(u) = C_1^s \int_u^\infty e^{\varepsilon_1(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-a/r^2} dx + C_2^s \int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx,$$
$$\psi_d(u) = C_1^d \int_u^\infty e^{\varepsilon_1(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-a/r^2} dx + C_2^d \int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx.$$

It follows from L'Hospital's rule and $e^{\varepsilon_1(u)} \to 1$ as $u \to \infty$ that

$$\frac{\int_u^\infty e^{\varepsilon_1(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-a/r^2} dx}{\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx} \to 1,$$

and

$$\frac{\int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx}{\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx} \to 0.$$

Therefore,

$$\lim_{u\to\infty} \frac{\psi(u)}{\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx} = C_1 \geq 0, \tag{3.12}$$

$$\lim_{u\to\infty} \frac{\psi_s(u)}{\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx} = C_1^s \geq 0, \tag{3.13}$$

$$\lim_{u\to\infty} \frac{\psi_d(u)}{\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx} = C_1^d \geq 0. \tag{3.14}$$

Since $\psi(u) = \psi_s(u) + \psi_d(u)$, we have $C_1 = C_1^s + C_1^d$.

Now we show that

$$C_1 > 0, \qquad C_1^s > 0, \qquad C_1^d > 0. \tag{3.15}$$

First, if $C_1 = 0$, then $C_2 > 0$ and

$$\psi(u) = C_2 \int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx \leq Ce^{-\beta x}, \qquad u > u_0 \tag{3.16}$$

for some constant $C > 0$ and some suffcient large constant $u_0 > 0$. However, the inequality (3.16) does not hold because of equations (3.3) and (3.8). Hence, we conclude that $C_1 > 0$.

Second, if $C_1^d = 0$, then $C_1 = C_1^s > 0$. Thus, it follows from

$$\int_0^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx > 0, \tag{3.17}$$

$\psi(0) = 1$, $\psi_s(0) = 0$, and $C_1 = C_1^s$ that $C_2 = C_2^s$. Hence, $\psi(u) = \psi_s(u)$, which cannot be true, since the second and the third boundary conditions for $\psi(u)$ and $\psi_s(u)$ are different. Thus, we have $C_1^d > 0$.

Finally, if $C_1^s = 0$, then $C_1 = C_1^d > 0$ and

$$\psi_s(u) = C_2^s \int_u^\infty e^{-\beta x+\varepsilon_2(x)} \left(\frac{r^2x^2 + \sigma^2}{\sigma^2}\right)^{-1} dx, \qquad u \geq 0, \tag{3.18}$$

which, together with expression (3.17) and $\psi_s(0) = 0$ in equation (2.4), implies that $C_2^s = 0$. Thus, $\psi_s(u) = 0$ and hence $\psi(u) = \psi_d(u)$, which is not true. Therefore, we obtain $C_1^s > 0$.

Then, by summarizing the arguments above and using the formula

$$\int_u^\infty \left(\frac{r^2x^2}{\sigma^2}\right)^{-a/r^2} dx = \left(\frac{r}{\sigma}\right)^{-2a/r^2} \left(\frac{2a}{r^2} - 1\right) u^{1-2a/r^2},$$

we obtain the following theorem for the asymptotical behaviors of $\psi(u)$, $\psi_s(u)$, and $\psi_d(u)$.

Theorem 3.1

Under the assumptions of Corollary 3.1, there exist constants $C > 0$, $C_s > 0$ and $C_d > 0$ so that $C = C_s + C_d$ and as $u \to \infty$,

$$\psi(u) \sim Cu^{1-2a/r^2}, \tag{3.19}$$

$$\psi_s(u) \sim C_s u^{1-2a/r^2}, \tag{3.20}$$

$$\psi_d(u) \sim C_d u^{1-2a/r^2}. \tag{3.21}$$

Note that the condition (2.2) implies that the order $1 - 2a/r^2$ in functions (3.19)–(3.21) is negative. Indeed, if the condition (2.2) does not hold, the ruin probability $\psi(u) = 1$ for $u \geq 0$. We also point out that the orders of the power functions (3.19)–(3.21) are the same and are determined by the drift parameter a and volatility parameter r of the geometric Brownian motion. However, the drift/premium parameter c and volatility parameter σ, together with the parameters λ and β, in the surplus process influence the ruin probabilities only through the constants C, C_s and C_d. Furthermore, we point out that the order is the same as that in Frolova et al. (2002).

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REFERENCES

BELLMAN, RICHARD. 1953. *Stability Theory of Differential Equations*. New York: Dover.

CAI, JUN. 2004. Ruin Probabilities and Penalty Functions with Stochastic Rates of Interest. *Stochastic Processes and Their Applications* 112: 53–78.

CAI, JUN, AND HAILIANG YANG. 2005. Ruin in the Perturbed Compound Poisson Risk Process under Interest Force. *Advances in Applied Probability* 37: 819–35.

DUFRESNE, FRANÇOIS, AND HANS U. GERBER. 1991. Risk Theory for the Compound Poisson Process That Is Perturbed by Diffusion. *Insurance: Mathematics and Economics* 10: 51–59.

FROLOVA, ANNA, YURI KABANOV, AND SERGUEI PERGAMENSHCHIKOC. 2002. In the Insurance Business Risky Investments Are Dangerous. *Finance and Stochastics* 6: 227–35.

GRANDELL, JAN. 1991. *Aspects of Risk Theory.* New York: Springer.

PAULSEN, JOSTEIN. 1993. Risk Theory in a Stochastic Economic Environment. *Stochastic Processes and Their Applications* 46: 327–61.

PAULSEN, JOSTEIN, AND HÅKON K. GJESSING. 1997. Ruin Theory with Stochastic Return on Investments. *Advances in Applied Probability* 29: 965–85.

SCHLEGEL, SABINE. 1998. Ruin Probabilities in Perturbed Risk Models. *Insurance: Mathematics and Economics* 22: 93–104.

YANG, HAILIANG, AND LIANZENG ZHANG. 2001. Spectrally Negative Lévy Processes with Applications in Risk Theory. *Advances in Applied Probability* 33: 281–91.

WANG, GUOJING. 2001. A Decomposition of the Ruin Probability for the Risk Process Perturbed by Diffusion. *Insurance: Mathematics and Economics* 28: 49–59.

WANG, GUOJING, AND RONG WU. 2001. Distributions for the Risk Process with a Stochastic Return on Investments. *Stochastic Processes and Their Applications* 95: 329–41.

DISCUSSION

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Professors Cai and Xu are to be congratulated for this interesting paper on ruin theory. This paper presents an insurance risk model in which 100% of the surplus is invested in a risky asset. In practice it is more reasonable to assume that the surplus can be invested in both risky and risk-free assets. In this discussion, I present an alternative model.

Assume that the surplus process is the same as given in equation (1.1) of the paper, and the surplus can be invested in a risk-free bond or a risky asset. The price of risk-free bond follows

$$dR(t) = r_0 R(t) dt,$$

where $R(t)$ is the price of the risk-free bond at time t, r_0 is the risk-free interest rate ($r_0 \geq 0$). The price of the risky asset follows

$$dS(t) = \mu S(t) dt + \sigma_1 S(t) dB_t,$$

where B_t is a standard Brownian motion independent of W_t and the claim process. Note that this is the same as in the paper. I assume that W_t and B_t are correlated with $dW_t dB_t = \rho dt$.

Assume that the insurance company invests a fixed proportion α of its surplus in the risky asset. That is, the insurance company will rebalance its portfolio continuously to maintain the proportion. If W_t and B_t are independent, this model can be reduced to the model in the paper by considering a new risky asset that is a mixture of the old risky asset and the risk-free asset, always according the proportion α and $1 - \alpha$. The problem becomes difficult if we use some kind of an optimal strategy rather than a fixed proportion. Let $X(t)$ denote the surplus of the insurer under this investment strategy; then we have

$$dX(t) = \alpha X(t) \frac{dS(t)}{S(t)} + (1 - \alpha) X(t) \frac{dR(t)}{R(t)} + dU(t),$$

or, more clearly,

$$dX(t) = [\alpha\mu + (1 - \alpha) r_0] X(t) dt + c dt + \alpha\sigma_1 X(t) dB_t + \alpha dW_t - dL(t),$$
$$X(0) = u. \tag{D.1}$$

Here $L(t) = \sum_{k=1}^{N(t)} Z_k$ is the aggregate claim process, α is the proportion invested in the risky asset, $1 - \alpha$ is the proportion invested in the risk-free asset, and all other notation is the same as in the paper. Note that this model reduces to the model in the paper if $\alpha = 1$ (μ here is $\delta + r^2/2$ in the paper).

Now we can define the time of ruin as $T = \inf\{t : X_t < 0\}$ and $= \infty$ if $X_t > 0$ for all $t \geq 0$, and the ruin probability is defined as $\psi(u) = P\{T < \infty | X_0 = u\}$. We also can define the ruin probability caused

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by claim and the ruin probability caused by oscillation, respectively, in the same way as in the paper. In Frolova et al. (2002) and in this paper by Cai and Xu, if the risky asset is too risky as determined by whether $1 < 2\mu/\sigma_1^2$ or not, then ruin is certain. In the classical ruin model, if the net-profit condition is not satisfied, then ruin is certain. The condition $1 < 2\mu/\sigma_1^2$ is like the net-profit condition. This is not necessarily the case in our model, because the company does not need to invest all the funds in the risky asset.

To obtain the integro-differential equation satisfied by the ruin probability, we introduce an auxiliary problem. We assume that the insurance company pays dividends continuously at rate of $\delta(X_\delta(t))$ at time t. Then the dynamics of the surplus process $X_\delta(t)$ is given by

$$\begin{aligned} dX_\delta(t) &= \{[\alpha\mu + (1-\alpha)r_0]X_\delta(t) + c - \delta(X_\delta(t))\}dt + \alpha\sigma_1 X_\delta(t)dB_t + \sigma dW_t - dL(t), \\ X_\delta(0) &= u. \end{aligned} \tag{D.2}$$

As before, the ruin probability associated with the risk process (D.2) is defined as

$$\psi_\delta(u) = \Pr\{X_\delta(t) < 0 \quad \text{for some} \quad t \geq 0 | X_\delta(0) = u\}. \tag{D.3}$$

The objective of the insurance company is to minimize the ruin probability $\psi_\delta(u)$ by choosing the optimal dividend strategy. By the dynamic programming principle, the value function $\psi_\delta(u)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\sup_\delta\{\lambda E[\psi_\delta(u - Z_1) - \psi_\delta(u)] + [\alpha\mu u + (1-\alpha)r_0 u + c - \delta(u)]\psi'_\delta(u) + \tfrac{1}{2}(\sigma^2 + 2\rho\alpha\sigma\sigma_1 u + \alpha^2\sigma_1^2 u^2)\psi''_\delta(u)\} = 0. \tag{D.4}$$

Of course, the optimal dividend policy that minimizes the ruin probability is that the insurance company pays no dividends, that is, $\delta = 0$. Note that $\psi_0(u) = \psi(u)$. By the verification theorem, the HJB equation (D.4) becomes

$$\lambda E[\psi(u - Z_1) - \psi(u)] + [\alpha\mu u + (1-\alpha)r_0 u + c]\psi'(u) + \tfrac{1}{2}(\sigma^2 + 2\rho\alpha\sigma\sigma_1 u + \alpha^2\sigma_1^2 u^2)\psi''(u) = 0, \qquad u \geq 0, \tag{D.5}$$

with the boundary conditions

$$\psi(\infty) = 0,$$
$$\psi(0) = 1.$$

Equation (D.5) is the equation satisfied by the ruin probability, which reduces to equation (2.10) in the paper by Cai and Xu if $\alpha = 1$ and $\rho = 0$.

If $\lambda = 0$, that is, if the surplus is given by a Brownian motion with drift, then equation (D.5) becomes

$$[\alpha\mu u + (1-\alpha)r_0 u + c]\psi'(u) + \tfrac{1}{2}(\sigma^2 + 2\rho\alpha\sigma\sigma_1 u + \alpha^2\sigma_1^2 u^2)\psi''(u) = 0, \qquad u \geq 0. \tag{D.6}$$

The solution of this equation is given by

$$\psi(u) = 1 - \frac{g(u)}{g(\infty)},$$

where

$$g(y) = \int_0^y (\alpha^2\sigma_1^2 x^2 + 2\rho\alpha\sigma\sigma_1 x + \sigma^2)^{-\frac{\alpha\mu + (1-\alpha)r_0}{\alpha^2\sigma_1^2}} e^{-\frac{2c\alpha\sigma_1 - \rho(\alpha\mu + (1-\alpha)r_0)\sigma}{\alpha^2\sigma\sigma_1^2\sqrt{1-\rho^2}} \arctan\frac{\alpha\sigma_1 x + \rho\sigma}{\sigma\sqrt{1-\rho^2}}} dx.$$

This result reduces to equation (3.6) of the paper if $\alpha = 1$ and $\rho = 0$. Note that it is clear from equation (3.6) of the paper that $\psi(u) = 1$, for all u, if $2\mu/\sigma_1^2 < 1$. For our model, $\psi(u) = 1$ if $2[\alpha\mu + (1-\alpha)r_0]/(\alpha^2\sigma_1^2) < 1$ (for $r_0 < \mu$ and $0 < \alpha < 1$, this condition is more restrictive). That is, $\psi(u) < 1$, for $\alpha < \alpha_0$, where α_0 is the positive solution of

$$\sigma_1^2\alpha^2 - 2(\mu - r_0)\alpha - 2r_0 = 0.$$

For example, if $\mu = 3\%$, $r_0 = 1\%$ and $\sigma_1 = 30\%$, then $\psi(u) = 1$ if we always invest 100% in the risky asset. However, if we always invest less than 74% in the risky asset, then $\psi(u) < 1$.

AUTHORS' REPLY

We are very grateful to Professor Yang for his discussion, which provides a new model for further research in which a fixed proportion of the current surplus is invested in a risky asset and the rest in a risk-free asset. The main point of the model is to allow the two Brownian motions in the underlying surplus process and the risky asset to be correlated. Thus, the model can describe the dependence between insurance risks and finance risks.

Professor Yang points out that our model (1.2) "presents an insurance risk model in which 100% of the surplus is invested in a risky asset." Indeed, we interpret the model (1.2) by assuming that an insurer invests all its surplus in the stock market. However, we point out that the model (1.2) also can model the case in which an insurer earns interest at stochastic interest forces or makes its investments with stochastic returns. In fact, as shown in the discussion, when the two Brownian motions are independent, the model of the discussion is reduced to the model (1.2) of the paper. The model (1.2) and its analogs have attracted a lot of attention in the literature. See, for example, the references in the paper and references therein. These models and the corresponding results show the effects of risky investments or stochastic interest forces on ruin probabilities and reveal how dangerous it can be if an insurer invests in risky assets.

As pointed out by Professor Yang, the ruin probability in his model still can be decomposed into two ruin probabilities, caused by a claim and oscillation, respectively. However, the HJB arguments used in the discussion do not apply to the two ruin probabilities. We hope that by using the arguments of the paper it is possible to derive the integro-differential equations for the two ruin probabilities.

It is very interesting to consider the combination of a risky investment and a risk-free investment. The investment strategy proposed by Professor Yang is easy to implement in practice. When the two Brownian motions are independent, such a fixed proportion investment strategy has appeared in the literature. In general, such a strategy is not an optimal one that minimizes ruin probability. However, for a few special models and under some conditions, a fixed proportion strategy is asymptotically optimal.

For example, when the underlying surplus risk process is the classical surplus risk process or $\sigma = 0$ in equation (1.1) of the paper, the two Brownian motions are independent, and an insurer has large claim sizes, then the asymptotically optimal strategy is to hold a fixed proportion of the current surplus in the risky asset when the current surplus goes to infinity. See equation (28) of Gaier and Grandits (2004), which implies that when a company is rich and has larger claim sizes, it should invest more in the risky asset. This seems to be consistent with the analysis of Griffn and Boomgaardt (1999, p. 52) that "P&C insurers should use riskier assets than life companies because their liabilities are riskier." An interesting discussion of this statement was given in Shiu (2000). However, the story is different when an insurer has small claim sizes. In this case, the asymptotically optimal strategy is to hold a fixed amount in the risky asset when the current surplus goes to infinity. See equation (59) of Gaier, Grandits, and Schachermayer (2003) or Theorem 9 of Hipp and Schmidli (2004). It means that when a company is rich and has small claim sizes, the company should invest less in the risky asset.

Finally, we wish to emphasize that in the papers about optimal investment strategies such as Gaier and Grandits (2004), Gaier, Grandits, and Schachermayer (2003), and Hipp and Schmidli (2004), the investment portfolio is continuously rebalanced: that is, the amounts in risk and risk-free assets are adjusted at each moment of time.

REFERENCES

GAIER, JOHANNA, AND PETER GRANDITS. 2004. Ruin Probabilities and Investment under Interest Force in the Presence of Regularly Varying Tails. *Scandinavian Actuarial Journal:* 256–78.

GAIER, JOHANNA, PETER GRANDITS, AND WALTER SCHACHERMAYER. 2003. Asymptotic Ruin Probabilities and Optimal Investment. *Annals of Applied Probability* 13: 1054–76.

GRIFFIN, MARK, AND RICK BOOMGAARDT. 1999. Enterprise Risk and Return Management for Financial Institutions. *North American Actuarial Journal* 3(2): 48–56.

HIPP, CHRISTIAN, AND HANSPETER SCHMIDLI. 2004. Asymptotics of Ruin Probabilities for Controlled Risk Processes in the Small Claims Case. *Scandinavian Actuarial Journal:* 321–35.

SHIU, ELIAS S. W. 2000. Discussion on ''Enterprise Risk and Return Management for Financial Institutions.'' *North American Actuarial Journal* 4(1): 116–17.

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