"Lundberg-Type Bounds for the Joint Distribution of Surplus Immediately before and at Ruin under the Sparre Andersen Model," Andrew C. Y. Ng and Hailiang Yang, April 2005

David C. M. Dickson,[*] Steve Drekic,[†] David A. Stanford,[‡] and Gordon E. Willmot[†]

1. Discussion

The paper by Ng and Yang (2005) has two goals. The primary one pertains to the development of Lundberg-type bounds, and we welcome these developments, which add to the body of material in the field. We remark that the approach taken to the derivation of the Lundberg bounds in the paper involves the use of martingale arguments. It is instructive to note, however, that Gerber-Shiu discounted penalty functions satisfy defective renewal equations, and this includes the special cases examined in the paper. Consequently, very general Lundberg bounds may easily be derived using the approach of Willmot et al. (2001). Moreover, such bounds apply quite generally to the solution of defective renewal equations, and in particular apply to Gerber-Shiu discounted penalty functions, including the particular functions considered in the paper.

However, it is our opinion that the second goal of the paper stated in the abstract to "rederive the closed-form expression for the distribution of the severity of ruin" does not accurately assess what their paper and ours (Drekic et al. 2004) have achieved in terms of novelty. It is our view that the claim on page 94 of Ng and Yang of a "more direct approach" for their Proposition 1 that conditions "on the phase state of the claim when ruin occurs" is based on expressions that have been completely described in our earlier paper. Consequently, we feel their proof, while very quick in getting to the conclusion, reveals no new elements that were not already explicitly contained in Drekic et al. (2004).

We briefly restate our own approach in Drekic et al. (2004) and then compare it to the proof in Ng and Yang (2005). As with many papers on phase-type distributions, our Step 1 established the structural form of the distribution of the deficit at ruin, showing it possessed a phase-type representation. In short, we considered the question "Can you write it as a product of some initial probability vector and a matrix exponential?"

We answered this affirmatively in equation (5) of Drekic et al. (2004, p. 110), which establishes that only the claim causing ruin exerts direct influence on the actual size y of the deficit, via the matrix exponential term $\exp(yS)$. The original surplus u influences only the relative likelihood of the states for our initial probability vector γ_u in this formulation.

Step 2 of our proof was to interpret fully the initial probability vector γ_u as the vector of likelihoods for the phases J_u when ruin occurs, as per equation (7) of Drekic et al. (2004, p. 111), which we restate below:

$$\begin{aligned}\gamma_u &= \Pr(L > u, J_u = j; j = 1, 2, \ldots, m)\\ &= \alpha_+ \exp(uB).\end{aligned}$$

It is this expression that *explicitly* interprets the vector of probabilities γ_u as indicating which state we are in precisely at the moment of ruin.

Turning to Ng and Yang (2005), the authors work directly with the idea of a terminating renewal process in Asmussen (2000), and look at its state precisely at the moment of ruin. The first two lines of their proof establish that

$$\psi(u, j) = \Pr(T < \infty, m_u = j) = \alpha_+ e^{uQ} e,$$

where e denotes a column vector of ones. This is the same as γ_u times e in Drekic et al. (2004). The authors then proceed to show that since ruin has occurred, it must be the current claim that is causing it. This approach promptly arrives at

$$\Pr(T < \infty, |U(T)| > y) = \alpha_+ e^{uQ} e^{yT} e,$$

[*] Centre for Actuarial Studies, University of Melbourne, Australia.
[†] Department of Statistics and Actuarial Science, University of Waterloo, Canada.
[‡] Department of Statistical and Actuarial Sciences, University of Western Ontario, Canada.

which in turn is identical, except for notation, to the corollary in Drekic et al. (2004, p. 111).

Hence, we conclude that while the proof in Ng and Yang (2005) is much shorter, their ''new'' argument does not establish anything about the distribution of the deficit at ruin for phase-type claims that had not already been identified explicitly in Drekic et al. (2004).

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AUTHORS' REPLY

We thank Professors Dickson, Drekic, Stanford, and Willmot for their discussion of our paper. The first part of the discussion pointed out a possible alternative to obtaining more general and/or better results including the main result in our paper by using renewal theory. We look forward to seeing this development. Renewal theory and martingale method are two popular approaches used in ruin theory, as we know. For us, which method should be used is just a matter of taste. In our paper, as stated in the introduction, ''the purpose of this paper is twofold: we provide new results on upper bounds for the joint distributions and demonstrate the power of the martingale approach.'' The martingale method used in our paper can be applied to risk processes that can be expressed as a Markov additive process because a martingale can be constructed and a change of measure then gives the bounds desired. See Asmussen (2000), Rolski et al. (1999), and Ng and Yang (2005a,b).

The main part of the discussion is on our ''short'' rederivation of the result in their paper (Drekic et al. 2004). As we stated in the paper, the closed form expression for the distribution of the severity of ruin is first obtained in Drekic et al. (2004). We did not intend to improve their result. When we submitted the first version of this paper, we had not known of their paper, and it was a referee who pointed out the paper to us. Our idea and method are motivated from our work on a Markov-modulated model (see Ng and Yang 2005a,b). We felt that our derivation is different from theirs, and ours is shorter, so we kept this part. However, we clearly state that the result is theirs. When we say ''a more direct approach,'' we mean the same as in the discussion: ''the authors work directly with the idea of a terminating renewal process in Asmussen (2000) . . . . This approach promptly arrives at . . .''

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