"On Optimal Dividend Strategies in the Compound Poisson Model" by Hans U. Gerber and Elias S. W. Shiu, April 2006

HANSJÖRG ALBRECHER* AND STEFAN THONHAUSER†

This elegant paper of Professors Gerber and Shiu provides a justification of the consideration of threshold models in connection with optimal dividend payout schemes in the classical risk model. In Section 9 the authors show that, in the case of bounded dividend intensity, for exponential claim sizes the optimal dividend strategy is a threshold strategy. In this discussion we would like to show that for Erlang(2) claim sizes one can find parameter choices for which the threshold strategy is optimal and others for which it is not.

For that purpose, let us first derive the solution of the integro-differential equations (5.1) and (5.2) of the paper for $p(y) = \eta^2 y e^{-\eta y}$ ($y \geq 0$, $\eta > 0$). Multiplying equation (5.1) by $e^{\eta x}$ and calculating the second derivative with respect to x, one obtains

$$-\delta\eta^2 V(x, b) + \eta(c\eta - 2(\delta + \lambda))V'(x, b) + (2c\eta - (\delta + \lambda))V''(x, b) + cV'''(x, b) = 0.$$

The solution of this homogeneous differential equation is

$$V_l(x, b) = A_1(b)\, e^{R_1 x} + A_2(b)\, e^{R_2 x} + A_3(b)\, e^{R_3 x},$$

where R_1, R_2, R_3 are the three (real-valued) roots of the polynomial equation

$$c\, R^3 + (2c\eta - (\delta + \lambda))\, R^2 + \eta(c\eta - 2(\delta + \lambda))\, R - \delta\eta^2 = 0,$$

two of which are negative and one of which is positive. Note that the coefficients $A_i(b)$ ($i = 1, 2, 3$) depend on the level b. A substitution of $V_l(x, b)$ back into equation (5.1) gives

$$\sum_{i=1}^{3} \frac{A_i(b)}{\eta + R_i} = 0,$$

$$\sum_{i=1}^{3} \frac{A_i(b)}{(\eta + R_i)^2} = 0.$$

The same procedure applied to equation (5.2) gives, for $x > b$,

$$-\delta\eta^2 V(x, b) + \eta((c - \alpha)\eta - 2(\delta + \lambda))V'(x, b) + (2(c - \alpha)\eta - (\delta + \lambda))V''(c, b) + (c - \alpha)V'''(x, b) = 0,$$

with solution

$$V_r(x, b) = \frac{\alpha}{\delta} + B_1(b)\, e^{S_1 x} + B_2(b)\, e^{S_2 x}.$$

Here S_1, S_2 are the two negative roots of the polynomial

$$Q(S) = -\delta\eta^2 + \eta((c - \alpha)\eta - 2(\delta + \lambda))\, S + (2(c - \alpha)\eta - (\delta + \lambda))\, S^2 + (c - \alpha)\, S^3$$

(whose existences are guaranteed by $\alpha \leq c$). Using the identity

* Hansjörg Albrecher, PhD, is an Associate Professor at the Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria, and Group Leader for Financial Mathematics at the Radon Institute, Austrian Academy of Sciences, Linz, albrecher@TUGraz.at.

† Stefan Thonhauser is a PhD candidate at the Radon Institute, Austrian Academy of Sciences, Linz, and supported by the Austrian Science Fund Project P18392, stefan.thonhauser@oeaw.ac.at.

$$\int_0^x V(x-y, b)\, p(y)\, dy = \int_0^{x-b} V_r(x-y, b)\, p(y)\, dy + \int_{x-b}^x V_l(x-y, b)\, p(y)\, dy$$

for $x \geq b$, a substitution of $V_r(x, b)$ back into equation (5.2) leads to the two equations

$$\sum_{i=1}^{3} \frac{\eta^2 A_i(b)}{(\eta + R_i)^2} e^{R_i b} - \sum_{i=1}^{2} \frac{\eta^2 B_i(b)}{(\eta + S_i)^2} e^{S_i b} = \frac{\alpha}{\delta},$$

$$\sum_{i=1}^{3} \frac{\eta A_i(b)}{\eta + R_i} e^{R_i b} - \sum_{i=1}^{2} \frac{\eta B_i(b)}{\eta + S_i} e^{S_i b} = \frac{\alpha}{\delta}.$$

Finally, assuming continuity in $x = b$, one has

$$A_1(b)\, e^{R_1 b} + A_2(b)\, e^{R_2 b} + A_3(b)\, e^{R_3 b} - B_1(b)\, e^{S_1 b} - B_2(b)\, e^{S_2 b} = \frac{\alpha}{\delta},$$

which uniquely determines the five coefficients $\{A_1(b), A_2(b), A_3(b), B_1(b), B_2(b)\}$, leading to the explicit solution

$$V_l(x, b) = \frac{\alpha\, S_1\, S_2\, (e^{R_1 x}(\eta + R_1)^2(R_2 - R_3) - e^{R_2 x}(\eta + R_2)^2(R_1 - R_3) + e^{R_3 x}(\eta + R_3)^2(R_1 - R_2))}{h(b)} \tag{D.1}$$

with

$$h(b) = \delta\eta^2\, (e^{R_1 b}(R_2 - R_3)(R_1 - S_1)(R_1 - S_2) - e^{R_2 b}(R_1 - R_3)(R_2 - S_1)(R_2 - S_2) + e^{R_3 b}(R_1 - R_2)(R_3 - S_1)(R_3 - S_2)).$$

Note the similarity of the structure of the above solution to equation (6.14) of the paper for the exponential case.

ON THE OPTIMALITY OF THRESHOLD STRATEGIES FOR ERLANG(2) CLAIMS

As stated in Section 3 of the paper, it follows from the term $(1 - V'(x))\, r(x)$ in the HJB equation (3.2) that an optimal dividend policy has to fulfill

$$r(X_t) = \begin{cases} \alpha, & V'(X_{t-}) < 1 \\ 0, & V'(X_{t-}) \geq 1. \end{cases}$$

In turn, if a solution $V(x, b^*)$ of equations (5.1) and (5.2) for a threshold level b^* has the properties

$$\begin{aligned} V'(x, b^*) &> 1, \quad x < b^*, \\ V'(x, b^*) &< 1, \quad x > b^*, \end{aligned} \tag{D.2}$$

and is differentiable at $x = b^*$, then it solves equation (3.2) and hence, due to the verification arguments of Section 4, is an optimal solution to the dividend maximization problem under bounded dividend intensity. In particular, then

$$V'(b^*, b^*) = V'(b^*-, b^*) = V'(b^*+, b^*) = 1 \tag{D.3}$$

(see eqs. (8.4) and (8.5) of the paper).

The determination of the optimal threshold level b^* can be done by either minimizing the function $|h(b)|$ in equation (D.1) or using equation (D.3) directly; however, both equations are implicit in b. Let us therefore look at two numerical examples.

First, consider $\lambda = 2$, $c = 2.3$, $\eta = 2$, $\delta = 0.03$, $\alpha = 0.9$. In this case we obtain the optimal level $b^* = 5.33826$. Figure 1 shows $V'(x, b^*)$ as a function of x. Since $B_1(b^*), B_2(b^*) < 0$ in this case, $V'(x, b^*) < 1$ for all $x > b^*$ so that both equations (D.3) and (D.2) are fulfilled. Hence the threshold strategy with $b^* = 5.33826$ is indeed the optimal dividend strategy. Figure 2 illustrates the value of $V(x, b)$ for

Figure 1
Function $V'(x,b^*)$, Optimal Level $b^* = 5.33826$ ($\alpha = 0.9$)

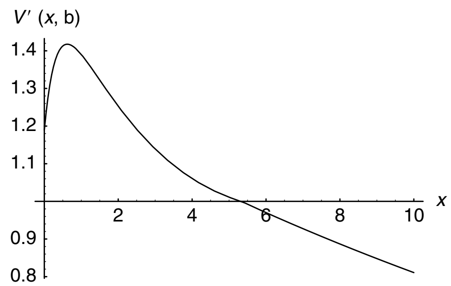


Figure 2
Value Functions, Optimal Level $b^* = 5.33826$ ($\alpha = 0.9$)

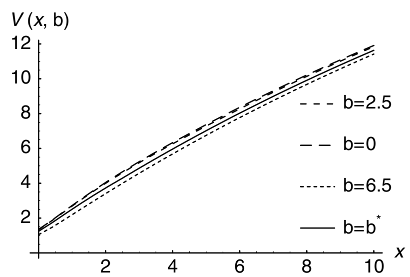


Figure 3
Function $V'(x,b^*)$, Optimal Level $b^* = 0$ ($\alpha = 0.1$)

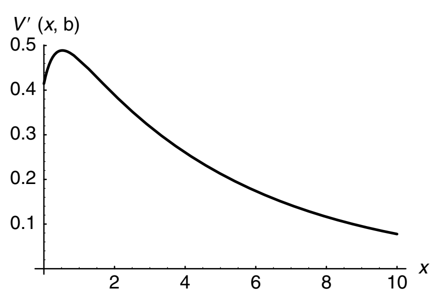


Figure 4
Value Functions, Optimal Level $b^* = 0$ ($\alpha = 0.1$)

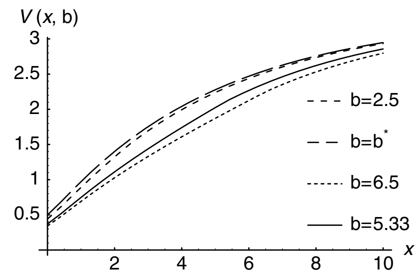


Figure 5
Function $V'(x,0)$ for Example 2

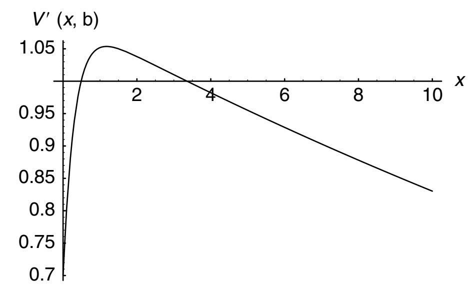
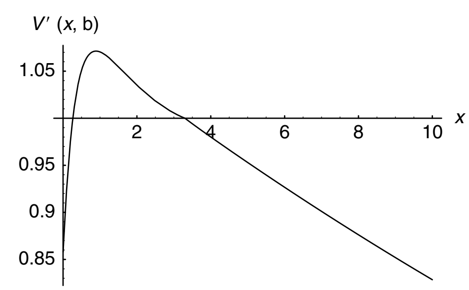


Figure 6
Function $V'(x,b^*)$ for Example 2



different thresholds b. If the maximal dividend payment intensity is instead bounded by $\alpha = 0.1$, then again a threshold strategy is optimal, this time, however, $b^* = 0$ (see Figs. 3 and 4).

As a second example, consider $\lambda = 4$, $c = 4.3$, $\eta = 2$, $\delta = 0.05$, $\alpha = 2$. One observes that $V'(0, 0) < 1$, but $V'(x, 0) < 1$ does not hold for all $x > 0$ (see Fig. 5), and therefore a threshold strategy with $b = 0$ cannot be optimal. Alternatively, condition (D.3) gives $b^* = 3.30058$, but $V'(x, b^*)$ does not fulfill conditions (D.2) (see Fig. 6; note also that the second derivative of $V(x, b^*)$ at $x = b^*$ does not exist), and hence the threshold strategy cannot be optimal. This observation is somewhat to be expected in view of recent results of Azcue and Muler (2005), who, as a by-product of their results, constructed an example with gamma-distributed claim amounts, where in the case of unbounded dividend intensity (i.e., $\alpha = \infty$) a *band* strategy with several bands is optimal (whereas the limit $\alpha \to \infty$ of a threshold strategy corresponds to a barrier strategy). Yet the first example above shows that, beyond exponential claim distributions, there are situations in which the threshold strategy is indeed also optimal for Erlang(2) claims.

Finally, we would like to point out that properties of the discounted penalty function in the threshold model have recently been studied by Lin and Pavlova (2006); for extensions to Sparre Andersen models see Albrecher et al. (2006).

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BANGWON KO*

Professors Gerber and Shiu have obtained many interesting results for optimal dividend strategies, especially when the individual claim amount distribution is an exponential or a mixture of exponentials. In the first part of this discussion, I use the method of Laplace transforms, instead of differential operators, to solve an integro-differential equation analogous to equation (10.2) in the paper and then present some interesting identities. In the second part, I illustrate how the optimal barrier b^* behaves as an increasing function of the dividend-rate ceiling α in the exponential claim amount case.

The function $L(x;\infty)$ can be generalized as

$$\phi(x) = \mathrm{E}[e^{-\delta T} w(|U(T)|)|X(0) = x], \qquad x \geq 0, \tag{D.1}$$

where $w(y)$ is a penalty function. Analogous to equation (10.2),

$$c\phi'(x) - (\lambda + \delta)\phi(x) + \lambda(\phi * p)(x) + \lambda\chi(x) = 0, \qquad x > 0, \tag{D.2}$$

where

$$\chi(x) := \int_x^\infty w(y - x)p(y)\, dy. \tag{D.3}$$

In Gerber and Shiu (1998, 2005), $\chi(x)$ is denoted as $\omega(x)$. The Laplace transform of equation (D.2) is

$$c[\xi\hat{\phi}(\xi) - \phi(0)] - (\lambda + \delta)\hat{\phi}(\xi) + \lambda\hat{\phi}(\xi)\hat{p}(\xi) + \lambda\hat{\chi}(\xi) = 0, \qquad \mathrm{Re}\ \xi \geq 0.$$

Thus,

$$\hat{\phi}(\xi) = \frac{c\phi(0) - \lambda\hat{\chi}(\xi)}{\mathcal{L}(\xi)}, \qquad \mathrm{Re}\ \xi \geq 0, \tag{D.4}$$

where $\mathcal{L}(\xi) := c\xi - (\lambda + \delta) + \lambda\hat{p}(\xi)$. Note that

$$\mathcal{L}(\xi) = 0 \tag{D.5}$$

* Bangwon Ko is a doctoral student in the Department of Statistics and Actuarial Science, University of Iowa, Iowa City, Iowa 52242-1409, bangwon-ko@uiowa.edu.

is *Lundberg's fundamental equation*. As in the paper, let ρ_0 be the positive solution of equation (D.5). Because $\hat{\phi}(\xi)$ is finite for Re $\xi \geq 0$ and because $\rho_0 > 0$, the numerator of formula (D.4) vanishes at $\xi = \rho_0$. Hence, ρ_0 is a removable singularity and

$$\phi(0) = \frac{\lambda}{c}\hat{\chi}(\rho_0). \tag{D.6}$$

These formulas are valid for an arbitrary individual claim amount density $p(y)$.

Now let us assume that the individual claim amount is a mixture of exponential distributions. Under formula (A.1) of the paper and with equation (D.3), we obtain

$$\hat{\chi}(\xi) = \sum_{i=1}^{n} \frac{A_i\beta_i\hat{w}(\beta_i)}{\beta_i + \xi} \tag{D.7}$$

We now see from formula (D.4) that $\hat{\phi}(\xi)$ is a rational function with singularities at $\xi = \rho_0, \rho_1, \ldots, \rho_n, -\beta_1, \ldots, -\beta_n$. It turns out that $-\beta_1, \ldots, -\beta_n$ are also removable singularities.

In fact,

$$\hat{\phi}(-\beta_i) = -\hat{w}(\beta_i), \qquad i = 1, 2, \ldots, n. \tag{D.8}$$

Thus,

$$\hat{\phi}(\xi) = \frac{\pi(\xi)}{\rho(\xi)} \tag{D.9}$$

where $\rho(\xi) := \prod_{k=1}^{n} (\xi - \rho_k)$ and $\pi(\xi)$ is a polynomial of degree $n - 1$ or less. It follows from equations (D.8) and (D.9) that

$$\pi(-\beta_i) = \hat{\phi}(-\beta_i)\rho(-\beta_i) = -\hat{w}(\beta_i)\rho(-\beta_i), \qquad i = 1, 2, \ldots, n. \tag{D.10}$$

Let $\beta(\xi) := \prod_{j=1}^{n} (\xi + \beta_j)$. By the Lagrange interpolation formula and equation (D.10),

$$\pi(\xi) = \beta(\xi) \sum_{j=1}^{n} \frac{\pi(-\beta_j)}{\beta'(-\beta_j)(\xi + \beta_j)} = \beta(\xi) \sum_{j=1}^{n} \frac{-\hat{w}(\beta_j)\rho(-\beta_j)}{\beta'(-\beta_j)(\xi + \beta_j)}$$

Applying the method of partial fractions to the right-hand side of equation (D.9), we have

$$\hat{\phi}(\xi) = \sum_{k=1}^{n} \frac{\pi(\rho_k)}{\rho'(\rho_k)(\xi - \rho_k)}.$$

Hence,

$$\phi(x) = \sum_{k=1}^{n} \frac{\pi(\rho_k)}{\rho'(\rho_k)} e^{\rho_k x} = \sum_{k=1}^{n}\sum_{j=1}^{n} \frac{-\hat{w}(\xi_j)\beta(\rho_k)\rho(-\beta_j)}{(\rho_k + \beta_j)\beta'(-\beta_j)\rho'(\rho_k)} e^{\rho_k x}. \tag{D.11}$$

Because $\beta'(-\beta_j) = \prod_{i=1,i\neq j}^{n} (\beta_i - \beta_j)$ and $\rho'(\rho_k) = \prod_{i=1,i\neq k}^{n} (\rho_k - \rho_i)$, formula (D.11) becomes

$$\phi(x) = \sum_{k=1}^{n}\sum_{j=1}^{n} \hat{w}(\beta_j)(\rho_k + \beta_j)\left(\prod_{i=1,i\neq j}^{n} \frac{\beta_i + \rho_k}{\beta_i - \beta_j}\right)\left(\prod_{i=1,i\neq k}^{n} \frac{\rho_i + \beta_j}{\rho_i - \rho_k}\right) e^{\rho_k x}, \qquad x \geq 0. \tag{D.12}$$

With $\hat{w}(\beta_j) = 1/\beta_j$, formula (D.12) is an explicit expression for $L(x;\infty)$.

Formula (D.12) can be simplified using the notation of *divided differences*. Let

$$\mathcal{E}_{x,j}(\xi) := e^{\xi x}\frac{\beta(\xi)}{\xi + \beta_j} = e^{\xi x}\prod_{i=1,i\neq j}^{n} (\xi + \beta_i). \tag{D.13}$$

Then the $(n - 1)$-th divided difference of $\mathscr{E}_{x,j}(\xi)$ with respect to the n collocation points, $\rho_1, \ldots, \rho_n$, is

$$\mathscr{E}_{x,j}[\rho_1, \rho_2, \ldots, \rho_n] = \sum_{k=1}^{n} e^{\rho_k x} \left(\prod_{i=1, i \neq j}^{n} (\beta_i + \rho_k) \right) \prod_{i=1, i \neq k}^{n} \frac{1}{\rho_i - \rho_k},$$

and hence formula (D.12) becomes

$$\phi(x) = \sum_{j=1}^{n} \hat{w}(\beta_j) \mathscr{E}_{x,j}[\rho_1, \rho_2, \ldots, \rho_n](\rho_j + \beta_j) \prod_{i=1, i \neq j}^{n} \frac{\rho_i + \beta_j}{\beta_i - \beta_j}, \qquad x \geq 0. \tag{D.14}$$

Next let us consider some interesting identities. For $j = 1, 2, \ldots, n$,

$$\mathscr{E}_{0,j}(\xi) = \frac{\beta(\xi)}{\xi + \beta_j} = \prod_{i=1, i \neq j}^{n} (\xi + \beta_i)$$

is a monic polynomial of degree $n - 1$. Thus, its $(n - 1)$-th divided difference is 1, and formula (D.14), with $x = 0$, is

$$\phi(0) = \sum_{j=1}^{n} \hat{w}(\beta_j)(\beta_j + \rho_j) \prod_{i=1, i \neq j}^{n} \frac{\beta_j + \rho_i}{\beta_j - \beta_i}, \tag{D.15}$$

which can be found in Gerber and Shiu (2005, eq. [5.29]) with different subscripts. On the other hand, from formulas (D.6) and (D.7), we have

Figure 1
Optimal Barrier b^* with $\lambda = 5$, $\beta = 100$, $\delta = 0.06$

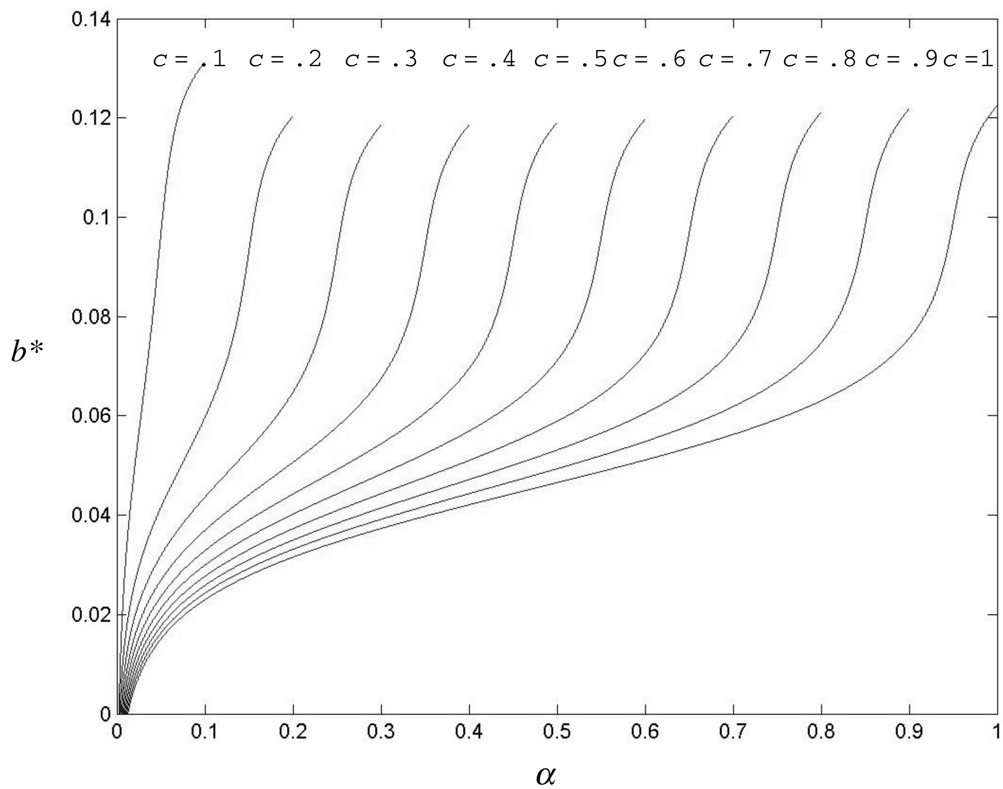
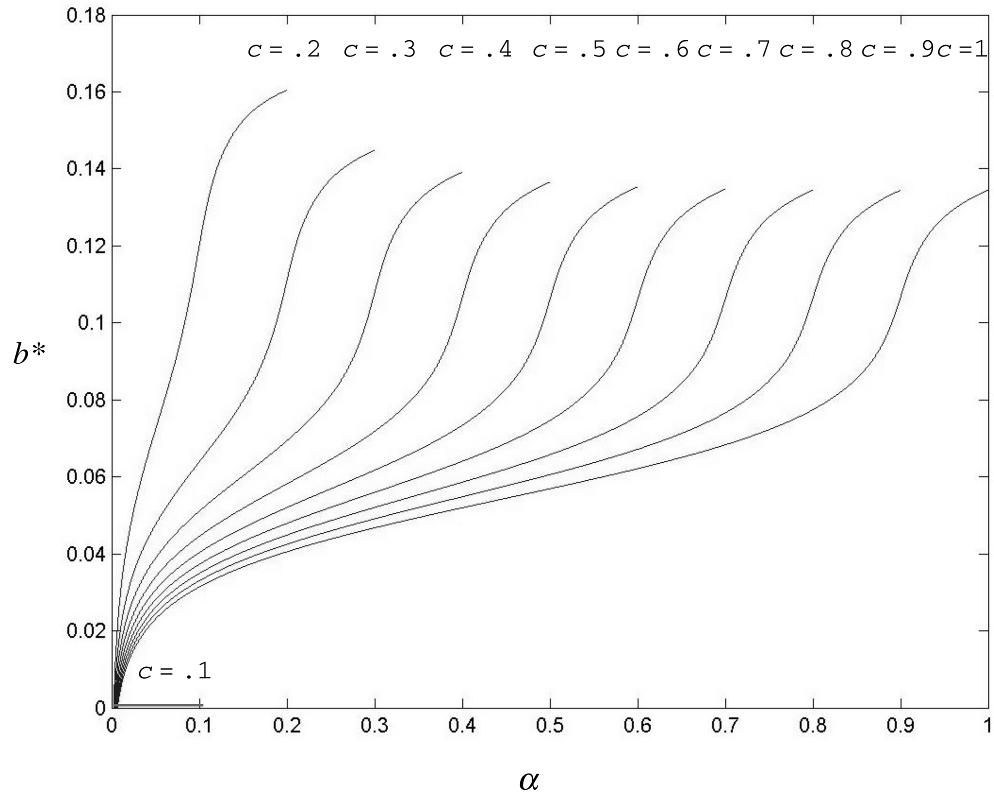


Figure 2
Optimal Barrier b^* with $\lambda = 10$, $\beta = 100$, $\delta = 0.06$



$$\phi(0) = \frac{\lambda}{c} \sum_{j=1}^{n} \frac{A_j \beta_j \hat{w}(\beta_j)}{\beta_j + \rho_0}. \tag{D.16}$$

Since formulas (D.15) and (D.16) hold for arbitrary functions $w(y)$, the coefficients of $\hat{w}(\beta_j)$ must be identical, and hence

$$\prod_{i=0}^{n} (\beta_j + \rho_i) = \frac{\lambda}{c} A_j \beta_j \prod_{k=1, k\neq j}^{n} (\beta_j - \beta_k), \qquad j = 1, 2, \ldots, n. \tag{D.17}$$

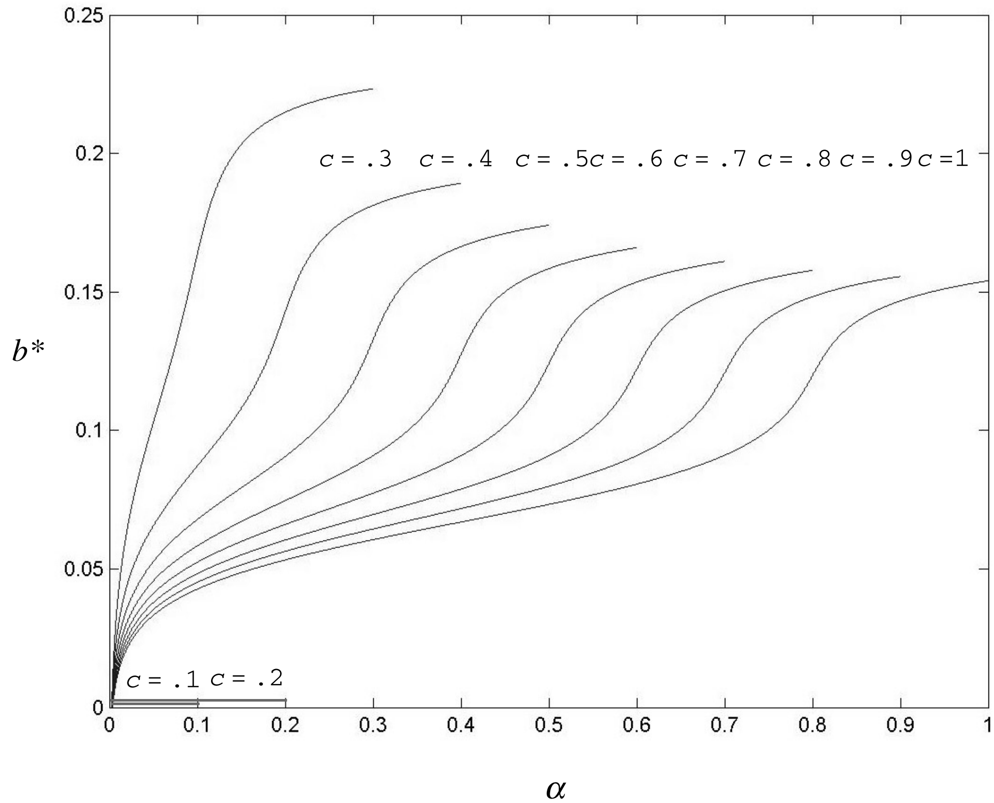
This is quite an interesting result, giving a set of n identities between the roots of *Lundberg's fundamental equation* and the constants defining it. Note that the right-hand side of identity (D.17) does not depend on δ, but the roots of *Lundberg's fundamental equation* are functions of δ; ρ_0 is an increasing function of δ, and $\rho_1, \rho_2, \ldots, \rho_n$ are decreasing functions. As $\delta \rightarrow \infty$, we have $\rho_0 \rightarrow \infty$ and $\rho_i \rightarrow -\beta_i$, $i = 1, 2, \ldots, n$, yielding another interesting result:

$$\lim_{\delta\rightarrow\infty} (\beta_j + \rho_0)(\beta_j + \rho_j) = \frac{\lambda}{c} A_j \beta_j, \qquad j = 1, 2, \ldots, n.$$

For $n = 1$, identity (D.17) is formula (4.34) in Gerber and Shiu (1998). Under a positive security loading assumption, we have $\rho_0 = 0$ if $\delta = 0$; then identity (D.17) can be found in Dufresne and Gerber (1991). Also, we can use inequalities (A.8) of the paper to see that both sides of identity (D.17) have the same sign.

Finally, let us investigate how the optimal barrier b^* behaves as an increasing function of the dividend-rate ceiling α in the exponential claim amount case. Although a positive security-loading condition is not assumed, formula (9.20) implies that if we want a positive b^*, we need a stronger condition than

Figure 3
Optimal Barrier b^* with $\lambda = 20$, $\beta = 100$, $\delta = 0.06$



the positive security loading: that is, for a positive b^*, we should have $\beta c - \lambda - \delta > 0$, or equivalently $\theta > \delta/\lambda$ (expected to be small).

It seems reasonable that an insurance company without a positive security loading would be "inefficient." It also seems natural that if the company wants to pay dividends, it would charge more than the actuarially fair premium. If $\beta c - \lambda - \delta > 0$ is satisfied, we have one more condition set by formula (9.3) or, equivalently, by formula (9.20): b^* is 0 if the dividend-rate ceiling α is less than or equal to the expression on the right-hand side of formula (9.20). This tells us that if the company sets the ceiling close to zero, then the company will pay dividends at rate α until it is ruined, regardless of the surplus level. Although not clearly seen in the figures, b^* is also zero for α close to zero because of formula (9.20).

Figures 1–3 depict how the optimal barrier b^* varies according to the dividend-rate ceiling α. As shown in Section 9, it is an increasing function of α. While setting δ and β at 0.06 and 100, respectively, we change c in each figure and λ across the figures. In Figures 2 and 3, we observe some cases with $b^* = 0$ ($c = 0.1$ in Fig. 2 and $c = 0.1, 0.2$ in Fig. 3). There it happens that the condition $\beta c - \lambda - \delta > 0$ is not satisfied, and the company is "inefficient" due to the exogenously given parameters.

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NATHANIEL SMITH*

Reading this paper was quite a challenge and in the end a very positive experience. The purpose of this discussion is to point out a generalization of formula (6.14) to the case where the individual claim amount density is a mixture or a combination of n exponential densities as in Appendix A of the paper.

For $n = 2$, the formula is

$$V(x; b) = \frac{\alpha}{\delta} \frac{(-u_1)(-u_2)}{\beta_1\beta_2} \frac{\mathcal{N}(x)}{\mathcal{D}(b)}, \qquad 0 \le x \le b, \tag{1}$$

where the numerator and the denominator are

$$\begin{aligned} \mathcal{N}(x) &= e^{\rho_0 x}(\rho_0 + \beta_1)(\rho_0 + \beta_2)(\rho_2 - \rho_1) \\ &\quad - e^{\rho_1 x}(\rho_1 + \beta_1)(\rho_1 + \beta_2)(\rho_2 - \rho_0) \\ &\quad + e^{\rho_2 x}(\rho_2 + \beta_1)(\rho_2 + \beta_2)(\rho_1 - \rho_0) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathcal{D}(b) &= e^{\rho_0 b}(\rho_0 - u_1)(\rho_0 - u_2)(\rho_2 - \rho_1) \\ &\quad - e^{\rho_1 b}(\rho_1 - u_1)(\rho_1 - u_2)(\rho_2 - \rho_0) \\ &\quad + e^{\rho_2 b}(\rho_2 - u_1)(\rho_2 - u_2)(\rho_1 - \rho_0), \end{aligned} \tag{3}$$

respectively. Note that the expressions for $\mathcal{N}(x)$ and $\mathcal{D}(b)$ have a similar structure; by substituting $x \leftarrow b$, $\beta_1 \leftarrow -u_1$, $\beta_2 \leftarrow -u_2$ in $\mathcal{N}(x)$, we obtain $\mathcal{D}(b)$.

To verify (1)–(3), observe that

$$V(x; b) = \sum_{k=0}^{2} C_k(b) e^{\rho_k x}, \qquad 0 \le x \le b, \tag{4}$$

with $C_k(b) = \gamma(b)C_k$, $k = 0, 1, 2$. According to conditions (A.9), (A.15), and (A.16) of the paper, the coefficients $C_0(b)$, $C_1(b)$, $C_2(b)$, $D_1(b)$, $D_2(b)$ are obtained from the following matrix equation:

$$\begin{pmatrix} \frac{1}{\beta_1 + \rho_0} & \frac{1}{\beta_1 + \rho_1} & \frac{1}{\beta_1 + \rho_2} & 0 & 0 \\ \frac{1}{\beta_2 + \rho_0} & \frac{1}{\beta_2 + \rho_1} & \frac{1}{\beta_2 + \rho_2} & 0 & 0 \\ \frac{e^{\rho_0 b}}{\beta_1 + \rho_0} & \frac{e^{\rho_1 b}}{\beta_1 + \rho_1} & \frac{e^{\rho_2 b}}{\beta_1 + \rho_2} & -\frac{e^{u_1 b}}{\beta_1 + u_1} & -\frac{e^{u_2 b}}{\beta_1 + u_2} \\ \frac{e^{\rho_0 b}}{\beta_2 + \rho_0} & \frac{e^{\rho_1 b}}{\beta_2 + \rho_1} & \frac{e^{\rho_2 b}}{\beta_2 + \rho_2} & -\frac{e^{u_1 b}}{\beta_2 + u_1} & -\frac{e^{u_2 b}}{\beta_2 + u_2} \\ e^{\rho_0 b} & e^{\rho_1 b} & e^{\rho_2 b} & -e^{u_1 b} & -e^{u_2 b} \end{pmatrix} \begin{pmatrix} C_0(b) \\ C_1(b) \\ C_2(b) \\ D_1(b) \\ D_2(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\alpha}{\delta}\frac{1}{\beta_1} \\ \frac{\alpha}{\delta}\frac{1}{\beta_2} \\ \frac{\alpha}{\delta} \end{pmatrix}. \tag{5}$$

For the solution, Maple or Mathematica is of great help. Substitution in (4) followed by simplification yields (1)–(3).

* Nathaniel Smith is a doctoral student at the École des hautes études commerciales, University of Lausanne, CH-1015 Lausanne, Switzerland, nsmith@unil.ch.

For a general n, the formula is believed to be as follows:

$$V(x; b) = \frac{\alpha}{\delta} \frac{\prod_{i=1}^{n} (-u_i)}{\prod_{i=1}^{n} \beta_i} \frac{\mathcal{N}(x)}{\mathcal{D}(b)}, \qquad 0 \le x \le b, \tag{6}$$

with

$$\mathcal{N}(x) = \sum_{k=0}^{n} (-1)^{k+n} e^{\rho_k x} \left[\prod_{i=1}^{n} (\rho_k + \beta_i) \right] \prod_{\substack{j>l \\ j\neq k, l\neq k}} (\rho_j - \rho_l) \tag{7}$$

and

$$\mathcal{D}(b) = \sum_{k=0}^{n} (-1)^{k+n} e^{\rho_k b} \left[\prod_{i=1}^{n} (\rho_k - u_i) \right] \prod_{\substack{j>l \\ j\neq k, l\neq k}} (\rho_j - \rho_l). \tag{8}$$

Note that (7) is compatible with (A.10) of the paper. It would be interesting to see an analytical proof of (6)–(8).

By using the definition of $u_0, u_1, \ldots, u_n$ we can obtain an alternative expression for $V(x; b)$. The u_k's are the $n + 1$ solutions of equation (A.13) of the paper, which can be written as

$$(c - \alpha)u - \lambda \sum_{i=1}^{n} A_i \frac{u}{\beta_i + u} - \delta = 0. \tag{9}$$

By multiplying this equation with the product of all denominators, we obtain a polynomial equation of degree $n + 1$. Inspection of the coefficient of u^{n+1} and the constant term reveals that

$$(c - \alpha) \prod_{k=0}^{n} (-u_k) = -\delta \prod_{i=1}^{n} \beta_i. \tag{10}$$

With this, formula (6) is transformed into

$$V(x; b) = \frac{\alpha}{(\alpha - c)u_0} \frac{\mathcal{N}(x)}{\mathcal{D}(b)}, \qquad 0 \le x \le b. \tag{11}$$

It is interesting to compare this formula with formula (11.3) of the paper, which is valid for any claim amount distribution. From this comparison it follows that $\mathcal{N}(0) = \mathcal{D}(0)$.

An independent proof of $\mathcal{N}(0) = \mathcal{D}(0)$ is as follows. The key is that $\mathcal{N}(x)$ is the determinant of the matrix that is obtained if the last column of the Vandermonde matrix is replaced by the column with entries

$$e^{\rho_k x} \prod_{i=1}^{n} (\rho_k + \beta_i) \tag{12}$$

in row $k + 1$ ($k = 0, 1, \ldots, n$). For example, for $n = 2$, this matrix is

$$\begin{pmatrix} 1 & \rho_0 & e^{\rho_0 x}(\rho_0 + \beta_1)(\rho_0 + \beta_2) \\ 1 & \rho_1 & e^{\rho_1 x}(\rho_1 + \beta_1)(\rho_1 + \beta_2) \\ 1 & \rho_2 & e^{\rho_2 x}(\rho_2 + \beta_1)(\rho_2 + \beta_2) \end{pmatrix}. \tag{13}$$

To see that the determinant of this matrix is $\mathcal{N}(x)$, expand the determinant by the last column. For $x = 0$ a substantial simplification is possible: $\mathcal{N}(0)$ reduces to the determinant of the $(n + 1)$ by $(n + 1)$ Vandermonde matrix. To see this, note that the entries in the last column are now the values of a monic polynomial of degree n at $\rho_0, \rho_1, \ldots, \rho_n$. Thus, by subtracting from the last column an appropriate linear combination of the first n columns, we obtain the Vandermonde matrix. Evidently the same result holds for $\mathcal{D}(0)$. Hence,

$$\mathcal{N}(0) = \mathcal{D}(0) = \prod_{j>l} (\rho_j - \rho_l). \tag{14}$$

Finally, let us consider the case where the claim amount distribution is Erlang(n), that is, gamma with shape parameter n and scale parameter β. It can be considered as the limiting case $\beta_1 = \cdots = \beta_n = \beta$. Hence, the basic results for $V(x; b)$ carry over; it suffices to set $\beta_i = \beta$ and observe that now the ρ_k's and the u_k's are solutions of the equations

$$c\rho - (\lambda + \delta) + \lambda\left(\frac{\beta}{\beta + \rho}\right)^n = 0, \tag{15}$$

and

$$(c - \alpha)u - (\lambda + \delta) + \lambda\left(\frac{\beta}{\beta + u}\right)^n = 0, \tag{16}$$

respectively.

CHUANCUN YIN*

Professors Gerber and Shiu are to be congratulated for this interesting paper considering the optimal dividend strategies in the compound Poisson model. It is particularly surprising how many explicit formulae they were able to obtain. I would like to comment on the fact that the model of the paper can be extended to a more general framework.

Let $b_1, b_2, \ldots, b_m, c_1, \ldots, c_{m+1}$ be a finite number of nonnegative constants such that $0 < b_1 < b_2 < \ldots < b_m$. Instead of the model (2.2), consider the following compound Poisson risk model with multiple thresholds:

$$X(t) = x + \int_0^t \rho(X(s))\, ds - S(t), \qquad t \geq 0,$$

where x is the initial surplus, $S(t)$ is the aggregate claims up to time t, and

$$\rho(x) = \begin{cases} c_{m+1}, & x > b_m \\ c_m, & b_{m-1} < x \leq b_m \\ \vdots & \\ c_2, & b_1 < x \leq b_2 \\ c_1, & x \leq b_1. \end{cases}$$

The model contains the compound Poisson model under a constant barrier strategy (see Gerber 1979; Gerber and Shiu 1998; Lin et al. 2003), the compound Poisson model under a constant threshold strategy (see Gerber and Shiu's paper and Lin and Pavlova 2006) and the compound Poisson model with a two-step premium rate (see Asmussen 2000; Zhou 2004) as special cases.

From Rolski et al. (1999) we know that $\{X(t)\}$ is a piecewise deterministic Markov process (PDMP) taking values in $\mathcal{R}$ with extended generator $\mathcal{A}$ that satisfies

$$\mathcal{A}f(x) = \chi f(x) + \lambda \int_0^\infty (f(x - y) - f(x))p(y)\, dy,$$

where f belongs to the domain $\mathcal{D}(\mathcal{A})$ of the generator $\mathcal{A}$ of $\{X(t)\}$, and $\chi = \rho(x)\, d/dx$ is the vector field of the integral curves of the PDMP.

* Chuancun Yin, PhD, is a Professor in the Department of Mathematics, Qufu Normal University, Qufu 273165, China, ccyin@mail.qfnu.edu.cn.

Consider $L(x; b_1, \ldots, b_m)$ in formula (10.1) for the model above. As a function of x, L satisfies the following integro-differential equations:

$$\mathcal{A}L(x; b_1, \ldots, b_m) = \delta L(x; b_1, \ldots, b_m), \; 0 < x < \infty, \; x \neq b_1, b_2, \ldots, b_m,$$

which generalizes formulas (10.2) and (10.3) in the paper. Obviously $L(x; b_1, \ldots, b_m) = 1$ whenever x is negative, $\lim_{x\to\infty} L(x; b_1, \ldots, b_m) = 0$, and $L(x; b_1, \ldots, b_m)$ is continuous at $x = b_i$ $(1 \leq i \leq m)$. Furthermore,

$$c_i L'(b_i-; b_1, \ldots, b_m) = c_{i+1} L'(b_i+; b_1, \ldots, b_m), \; i = 1, 2, \ldots, m.$$

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XIAOWEN ZHOU*

Professors Gerber and Shiu have written another interesting paper, which attacks the challenging problem of optimal dividend strategies for risk models.

The threshold strategy is of particular interest in this paper since it often serves as the optimal dividend strategy. In this discussion I will derive expressions concerning the threshold strategy for the compound Poisson risk model. I will also obtain an expression for the expected present value of all dividends until ruin for the Lévy risk model with barrier.

First let us go over some preliminaries. As usual, we write $\{U(t)\}$ for the compound Poisson risk model with premium rate c, claim frequency λ, and claim amount density function $p(y)$, $y \geq 0$. Write W_δ for the *scale function* such that

$$\int_0^\infty e^{-tx} W_\delta(x)\, dx = \frac{1}{ct + \lambda(\hat{p}(t) - 1) - \delta}, \qquad t > \rho, \tag{D.1}$$

where $\hat{p}$ stands for the Laplace transform for p and $\rho := \rho(\delta)$ stands for the unique nonnegative solution to the *Lundberg fundamental equation*

$$c\xi + \lambda(\hat{p}(\xi) - 1) = \delta. \tag{D.2}$$

Denote by T^* the time when process $\{U(t)\}$, starting at $0 \leq u \leq b$, first exits from interval $(0, b)$. Put

$$J_b(\delta, u, x) := \frac{W_\delta(u) W_\delta(b - x)}{W_\delta(b)} - 1(u \geq x) W_\delta(u - x) \tag{D.3}$$

and

$$J_\infty(\delta, u, x) := W_\delta(u) e^{-x\rho} - 1(u \geq x) W_\delta(u - x). \tag{D.4}$$

Then it follows from equation (6) and Corollary 2 of Bertoin (1997) that

* Xiaowen Zhou, PhD, is an associate professor at Concordia University, 1455 de Maisonneuve Blvd. W. Montreal, Quebec H3G 1M8, Canada, zhou@alcor.concordia.ca.

$$\mathrm{E}[e^{-\delta T^*};\ U(T^*) = b | U(0) = u] = \frac{W_\delta(u)}{W_\delta(b)} \tag{D.5}$$

and

$$\mathrm{E}[e^{-\delta T^*};\ U(T^*-) \in dx,\ -U(T^*) \in dy | U(0) = u] = J_b(\delta, u, x)\lambda p(x + y)\ dx\ dy \tag{D.6}$$

It follows from equation (1.11) of Zhou (2005) that

$$\mathrm{E}[e^{-\delta T};\ T < \infty,\ U(T-) \in dx,\ -U(T) \in dy | U(0) = u] = J_\infty(\delta, u, x)\lambda p(x + y)\ dx\ dy, \tag{D.7}$$

where T is the ruin time. Letting $\delta = 0$ in equation (D.5) we have

$$\mathrm{P}[U(T^*) = b | U(0) = u] = \frac{W_0(u)}{W_0(b)}. \tag{D.8}$$

Further, letting $b \to \infty$ we have

$$\mathrm{P}[T = \infty | U(0) = u] = \frac{W_0(u)}{W_0(\infty)}, \tag{D.9}$$

where $W_0(\infty) := \lim_{b\to\infty} W_0(b)$. Such results are used in Zhou (2005) for the study of risk models with barrier.

Write $\{X(t)\}$ for the surplus process net of dividend payments under a threshold strategy introduced in this paper. The time value of ruin under a threshold strategy is discussed there in Section 10. Here we will find an expression for $W_\delta(u; x, y)$ such that

$$\mathrm{E}[e^{-\delta T};\ X(T-) \in dx,\ X(T) \in dy | X(0) = u] = W_\delta(u; x, y)\ dx\ dy,$$

which would result in an expression for the Gerber-Shiu function.

With c replaced by $c - \alpha$, $0 < \alpha < c$, let W^*_δ and ρ^* be the scale function in equation (D.1) and the nonnegative solution to equation (D.2), respectively; let J^*_b and J^*_∞ be the functions defined in equations (D.3) and (D.4), respectively.

Observe that under the threshold strategy with threshold b, the process $\{X(t)\}$ evolves like a surplus process for risk model with premium rate either c or $c - \alpha$ depending on whether it takes values between 0 and b or greater than b, respectively. Starting at b and distinguishing between whether the first jump of $\{X(t)\}$ downcrossing level b causes ruin or not, by equations (D.5) and (D.7) together with the strong Markov property we have, for $x > b$,

$$W_\delta(b; x, y) = W^*_\delta(0)e^{-(x-b)\rho^*}\lambda p(x + y) + \int_0^b dy' W^*_\delta(0)e^{-(x-b)\rho^*}\lambda p(x - y')\frac{W_\delta(y')}{W_\delta(b)}W_\delta(b; x, y). \tag{D.10}$$

Notice from equation (D.1) that $W^*_\delta(0) = 1/(c - \alpha)$. Solving equation (D.10) for $W_\delta(b; x, y)$ we have

$$W_\delta(b; x, y) = \frac{\lambda W_\delta(b)p(x + y)e^{-(x-b)\rho^*}}{(c - \alpha)W_\delta(b) - \lambda e^{-(x-b)\rho^*}\int_0^b dy' p(x - y')W_\delta(y')}.$$

Similarly, for $X(T-) \in dx$, $0 < x < b$, to occur, the first jump of $\{X(t)\}$ downcrossing level b can not cause ruin. By equations (D.5), (D.6), and (D.7) we have

$$W_\delta(b; x, y) = \int_0^\infty dx' \int_0^b dy' W^*_\delta(0)e^{-x'\rho^*}\lambda p(x' + b - y')\left(\frac{W_\delta(y')}{W_\delta(b)}W_\delta(b; x, y) + J_b(y', x)\lambda p(x + y)\right).$$

Then for $0 < x < b$,

$$W_\delta(b; x, y) = \frac{\lambda^2 p(x + y)W_\delta(b)\int_0^\infty dx' \int_0^b dy' e^{-x'\rho^*}p(x' + b - y')J_b(y', x)}{(c - \alpha)W_\delta(b) - \lambda\int_0^\infty dx' \int_0^b dy' e^{-x'\rho^*}p(x' + b - y')W_\delta(y')}.$$

Moreover, for $0 \le u < b$, we have

$$W_\delta(u; x, y) = 1(0 < x < b)\left(J_b(\delta, u, x)\lambda p(x + y) + \frac{W_\delta(u)}{W_\delta(b)} W_\delta(b; x, y)\right)$$

$$+ 1(x > b)\frac{W_\delta(u)}{W_\delta(b)} W_\delta(b; x, y).$$

Finally, for $u > b$, we have

$$W_\delta(u; x, y) = 1(0 < x < b)\int_0^\infty dx' \int_0^b dy' J_\infty(\delta, u - b, x')\lambda p(x' + b - y')$$

$$\times \left(J_b(\delta, y', x)\lambda p(x + y) + \frac{W_\delta(y')}{W_\delta(b)} W_\delta(b; x, y)\right)$$

$$+ 1(x > b)\Bigg(J_\infty(\delta, u - b, x - b)\lambda p(x + y) + \int_0^\infty dx' \int_0^b dy' J_\infty(\delta, u - b, x')$$

$$\times \lambda p(x' + b - y')\frac{W_\delta(y')}{W_\delta(b)} W_\delta(b; x, y)\Bigg).$$

Next, we want to derive similar expressions for

$$V(u; b) := \alpha \mathrm{E}\left[\int_0^T 1(X(t) > b)e^{-\delta t}\, dt \,\middle|\, X(0) = u\right],$$

the expected present value of all dividends until ruin under the threshold strategy.

For $u = b$, by equations (D.5), (D.7), (D.8), (D.9) and the strong Markov property, we have

$$V(b; b) = \frac{\alpha W_0^*(0)}{\delta W_0^*(\infty)} + \frac{\alpha}{\delta}\int_0^\infty dx J_\infty^*(\delta, 0, x)\lambda(1 - P(x)) \tag{D.11}$$

$$+ \int_0^\infty dx J_\infty^*(\delta, 0, x)\int_0^b dy \lambda p(x + b - y)\frac{W_\delta(y)}{W_\delta(b)} V(b; b), \tag{D.12}$$

where $P(z) := \int_0^z p(x)\, dx$. Notice that the first term on the right-hand side of equation (D.11) is the expected present value of all dividends when the process $\{X(t)\}$ stays above level b forever. The second term on the right-hand side of equation (D.11) is the expected present value of all dividends up to the first downcrossing time of level b when the downcrossing time is finite. The term in (D.12) is the expected present value of all dividends after the downcrossing time. Therefore, $V(b; b)$ can be obtained by solving the above equation.

Similarly, for $0 < u < b$,

$$V(u; b) = \frac{W_\delta(u)}{W_\delta(b)} V(b; b),$$

and for $u > b$,

$$V(u; b) = \frac{\alpha W_0^*(u - b)}{\delta W_0^*(\infty)} + \frac{\alpha}{\delta}\int_0^\infty dx J_\infty^*(\delta, u - b, x)\lambda(1 - P(x))$$

$$+ \int_0^\infty dx J_\infty^*(\delta, u - b, x)\int_0^b dy \lambda p(x + b - y)\frac{W_\delta(y)}{W_\delta(b)} V(b; b).$$

Similar results were obtained in Lin and Pavlova (2006) via a different approach.

The remainder of this discussion will give a different derivation of expression (5.9) in the paper for evaluating the expected present value of all dividends until ruin in the case $\alpha = c$, that is, in the case where the threshold is a barrier, under a more general setting.

Let us write $\{U(t)\}$ for a Lévy process with no positive jumps; see Chapter VII in Bertoin (1996) for an introduction to such processes. Then there exists a σ-finite *Lévy measure* Π on $(-\infty, 0)$ such that

$$\int_{-\infty}^{0} (1 \wedge x^2)\Pi(dx) < \infty$$

and

$$\mathrm{E}[e^{\lambda(U(t)-U(0))}] = e^{t\psi(\lambda)}, \qquad \lambda \geq 0,$$

where the *Laplace exponent* ψ is of the form

$$\psi(\lambda) = m\lambda + \sigma^2\lambda^2/2 + \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x 1(x > -1))\Pi(dx).$$

Its scale function W_δ has Laplace transform $(\psi(t) - \delta)^{-1}$ for $t > \rho$, where ρ is the unique nonnegative solution to $\psi(t) = \delta$.

For $0 \leq U(0) \leq b$ put

$$D(t) := \sup_{0 \leq s \leq t} (U(s) - b) \vee 0.$$

Then $\{U(t)\}$ corresponds to the surplus process for risk model without barrier; $\{U(t) - D(t)\}$ corresponds to the surplus process for risk model with barrier at b; $\{D(t)\}$ corresponds to the all dividends until time t. Moreover, exit time

$$T := \inf\{t \geq 0 : U(t) - D(t) < 0\}$$

corresponds to the ruin time for a risk model with barrier.

Abusing notation slightly, denote

$$V(u; b) = \mathrm{E}\left[\int_0^T e^{-\delta t}\, dD(t) \,\middle|\, U(0) = u\right].$$

We will proceed to find an expression of $V(u; b)$ in terms of W_δ.

Start with the case $U(0) = b$. For any positive integer n, write T_n for the first exit time from interval $(1/n, b + 1/n)$ for the process $\{U(t)\}$, which is also the exit time of $\{U(t) - U(0)\}$ from interval $(-b + 1/n, 1/n)$. By the strong Markov property and equation (D.5) we have

$$\begin{aligned}
V(b; b) &= \mathrm{E}\left[\int_0^T e^{-\delta t}\, dD(t);\, U(T_n) \leq \frac{1}{n}\right] + \mathrm{E}\left[\int_0^T e^{-\delta t}\, dD(t);\, U(T_n) = b + \frac{1}{n}\right] \\
&\geq \mathrm{E}\left[\int_0^{T_n} e^{-\delta t}\, dD(t);\, U(T_n) \leq \frac{1}{n}\right] + \mathrm{E}\left[\int_0^{T_n} e^{-\delta t}\, dD(t);\, U(T_n) = b + \frac{1}{n}\right] \\
&\quad + \mathrm{E}\left[e^{-\delta T_n};\, U(T_n) = b + \frac{1}{n}\right] V(b; b) \\
&= \mathrm{E}\left[\int_0^{T_n} e^{-\delta t}\, dD(t);\, U(T_n) \leq \frac{1}{n}\right] + \frac{1}{n}\mathrm{E}\left[e^{-\delta T_n};\, U(T_n) = b + \frac{1}{n}\right] \qquad \text{(D.13)} \\
&\quad + \delta \mathrm{E}\left[\int_0^{T_n} e^{-\delta t}\, D(t)\, dt;\, U(T_n) = b + \frac{1}{n}\right] + \frac{W_\delta(b - 1/n)}{W_\delta(b)} V(b; b) \\
&\geq \frac{W_\delta(b - 1/n)}{nW_\delta(b)} + \frac{W_\delta(b - 1/n)}{W_\delta(b)} V(b; b),
\end{aligned}$$

where to obtain equation (D.13) we have used the integration by parts

$$\int_0^{T_n} e^{-\delta t}\, dD(t) = e^{-\delta T_n} D(T_n) + \delta \int_0^{T_n} e^{-\delta t} D(t)\, dt,$$

the fact that $T_n \leq T$, and the fact that $D(T_n) = 1/n$ and $T_n < T$ when $U(T_n) = b + 1/n$.

Similarly, write T'_n for the exit time of $\{U(t)\}$ from interval $(0, b + 1/n)$, and we have

$$\begin{aligned} V(b; b) &= \mathrm{E}\left[\int_0^T e^{-\delta t}\, dD(t);\, U(T'_n) \leq 0\right] + \mathrm{E}\left[\int_0^T e^{-\delta t}\, dD(t);\, U(T'_n) = b + \frac{1}{n}\right] \\ &\leq \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t}\, dD(t);\, U(T'_n) \leq 0\right] + \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t}\, dD(t);\, U(T'_n) = b + \frac{1}{n}\right] \\ &\quad + \mathrm{E}\left[e^{-\delta T'_n};\, U(T'_n) = b + \frac{1}{n}\right] V(b; b) \\ &= \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t}\, dD(t);\, U(T'_n) \leq 0\right] + \frac{W_\delta(b)}{nW_\delta(b + 1/n)} \\ &\quad + \delta \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t} D(t)\, dt;\, U(T'_n) = b + \frac{1}{n}\right] + \frac{W_\delta(b)}{W_\delta(b + 1/n)} V(b; b). \end{aligned}$$

Notice that

$$\begin{aligned} \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t}\, dD(t);\, U(T'_n) \leq 0\right] &\leq \mathrm{E}[D(T'_n);\, U(T'_n) \leq 0] \\ &\leq \frac{1}{n} \mathrm{P}\{U(T'_n) \leq 0\} \\ &= \frac{1}{n}\left(1 - \frac{W_0(b)}{W_0(b + 1/n)}\right) \end{aligned}$$

and

$$\begin{aligned} \delta \mathrm{E}\left[\int_0^{T'_n} e^{-\delta t} D(t)\, dt;\, U(T'_n) = b + \frac{1}{n}\right] &\leq \frac{1}{n} \mathrm{E}\left[1 - e^{-\delta T'_n};\, U(T'_n) = b + \frac{1}{n}\right] \\ &= \frac{1}{n}\left(\frac{W_0(b)}{W_0(b + 1/n)} - \frac{W_\delta(b)}{W_\delta(b + 1/n)}\right). \end{aligned}$$

Therefore,

$$V(b; b) = \lim_{n\to\infty} \frac{\dfrac{W_\delta(b - 1/n)}{nW_\delta(b)}}{1 - \dfrac{W_\delta(b - 1/n)}{W_\delta(b)}} = \lim_{n\to\infty} \frac{W_\delta(b - 1/n)}{n(W_\delta(b) - W_\delta(b - 1/n))} = \frac{W_\delta(b)}{W'_\delta(b)}.$$

Now turning to the case $0 \leq U(0) = u < b$, noting that $\{D(t)\}$ will not increase until $\{U(t)\}$ reaches lever b before ruin, by equation (D.5) again we have

$$V(u; b) = \frac{W_\delta(u)}{W_\delta(b)} V(b; b) = \frac{W_\delta(u)}{W'_\delta(b)},$$

which generalizes equation (1.19) of Zhou (2005).

Such a result had been obtained in equation (8.9) of Gerber (1972) for a compound Poisson risk models perturbed by an independent diffusion. Evidently, $W_\delta(b)/W'_\delta(b)$ increases in b. Moreover,

$$\mathrm{E}[D(t)|U(0) = u] = \frac{W_0(u)}{W_0'(b)}.$$

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AUTHORS' REPLY

We are very grateful to have received five excellent discussions that add so much to our paper. Nearly 50 years ago, Bruno de Finetti (June 13, 1906–July 20, 1985) published his path-breaking paper on optimal dividend strategies. He would be proud to know that this topic has remained an active area of research. Finally, we take this opportunity to add a reference, which we did not know until recently.

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