Regime-Switching Periodic Models for Claim Counts

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Abstract

We study a Cox risk model that accounts for both seasonal variations and random fluctuations in the claims intensity. This occurs with natural phenomena that evolve in a seasonal environment and affect insurance claims, such as hurricanes.

More precisely, we define an intensity process governed by a periodic function with a random peak level. The periodic intensity function follows a deterministic pattern in each short-term period and is illustrated by a beta-type function. A Markov chain with *m* states, corresponding to different risk levels, is chosen for the level process, yielding a so-called regime-switching process.

The properties of the corresponding claim-counting process are discussed in detail. By properly defining a Lundberg-type coefficient, we derive upper bounds for finite time ruin probabilities in a two-state case.

1. Introduction

Consider the risk process

$$U(t) = u + ct - \sum_{j=1}^{N(t)} X_j, \qquad t \geq 0, \tag{1.1}$$

where u is the initial value, c is the (constant) premium rate, $N = \{N(t)\}_{t\geq 0}$ is a point process counting the number of claims arriving in the time interval $[0, t)$, and X_j is the jth claim size. When N is a Poisson process with (constant) intensity λ and claim sizes $\{X_j\}_{j\geq 1}$ are i.i.d. and independent of N, then (1.1) is known as the classical (homogeneous Poisson) risk model, which has been investigated extensively in the actuarial literature.

The classical risk model is not realistic in some practical situations. Two main modifications are made here. First, a nonhomogeneous Poisson (NHP) process is used to model ''size fluctuations'' in the claim intensity of a risk subject to seasonality. Then a Cox process, also called a doubly stochastic Poisson process and a natural extension of the NHP process, is used to characterize the underlying ''risk fluctuations'' in the claims intensity (see Grandell 1991).

The risk theory literature gives fewer results when the claim-counting process is a NHP process. Dassios and Embrechts (1989) defines a risk model with periodic claim intensity and considers the corresponding ruin problems using a martingale approach. Similar models are also considered by Asmussen and Rolski (1994) and Rolski et al. (1999). Two-sided bounds and asymptotic formulas for ruin probabilities are derived by using an average arrival rate risk model. Berg and Haberman (1994) uses a nonhomogeneous Markov birth process, of which the NHP process is a special case, to predict trends in life insurance claim occurrences. Dimitrov, Chukova, and Garrido (2000) exploits some properties in a NHP risk model with a (single) periodic claim intensity. Morales (2004) chooses a Gaussian shape

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for the periodic claim intensity function. By contrast, Garrido and Lu (2004) and Lu and Garrido (2005) consider a more general doubly periodic intensity rate. Possible forms of intensity functions, like the double-beta and the sine-beta, are proposed.

An early reference to Cox risk models is Ammeter (1948). In his model the intensity λ_k over time intervals $[(k-1)\Lambda, k\Lambda)$ of (fixed) length Λ, for $k \in \mathbb{N}^+$, forms an i.i.d. sequence $\{\lambda_k\}_{k\geq 0}$. This model is generalized by Björk and Grandell (1988), who consider the intensity as $\lambda(t) = L_i$ if $\Sigma_{i-1} \leq t < \Sigma_i$, where $\Sigma_i = \sigma_1 + \cdots + \sigma_i$, with $\Sigma_0 = 0$ and (L_i, σ_i) a sequence of i.i.d. random vectors. Ammeter's model is revisited by Grandell (1995), and more properties of the model are discovered.

Asmussen (1989) proposes a Cox risk model, called a Markov-modulated Poisson process, whose intensity process $\{\lambda(t)\}_{t\geq 0}$ is given by $\lambda(t) = \lambda_{J(t)}$. Here the process $\{J(t)\}_{t\geq 0}$ models the random environment of an insurance business and is assumed to be an irreducible continuous-time Markov chain with a finite state space. Furthermore, a Cox risk process with a piecewise constant intensity is considered by Schmidli (1996), where the sequence $\{L_i\}_{i\geq 1}$ of successive levels of the intensity forms a Markov chain.

Ruin probabilities have been studied in these Cox models with a piecewise constant intensity. Lundberg inequalities hold, provided some assumptions are fulfilled. These may not be practical because of the difficulty in estimating the Lundberg coefficient and evaluating some constants within the inequalities. Other papers studying this topic are Embrechts, Grandell, and Schmidli (1993) and Schmidli (1997).

Most results in the risk theory literature regarding Cox processes are for piecewise constant intensities. Recently Schmidli (2003) considered a NHP model with doubly stochastic occurrences for the PCS catastrophes index, based on individual indices for PCS options, where the intensity is of the form $\Lambda\lambda(t)$, with Λ is stochastic and $\lambda(t)$ is a deterministic function.

Some natural phenomena evolve in a seasonal environment subject to random fluctuations that, in turn, affect insurance claims. For example, tropical storms and hurricanes periodically affect the coastal U.S. states along the Atlantic and the Gulf of Mexico. The claim intensity then forms a specific pattern for each year that can be modeled by a periodic function. Speculation exists regarding the significance and potential effects of the El Niño phenomenon on hurricane frequency and the strength attained by tropical cyclones during alternating El Niño–La Niña years. These are random effects that, in some sense, affect the risk propensity or the peak level of the seasonal intensity. They can be modeled by a stochastic process.

In this paper we propose a Cox model that accounts for both the seasonal variations and the random fluctuations in the claims intensity. Beard, Pentikäinen, and Pesonen (1984) and Daykin, Pentikäinen, and Pesonen (1994) suggest an intensity process λ as a composition of some factors, such as the normal trend, deviations from it and the short-term variations in risk propensity. Here we consider an intensity process with the following structure:

$$\lambda(t) = \lambda_\kappa\, \beta_S(t), \qquad t \geq 0, \tag{1.2}$$

where $\beta_S(t)$ is the short-term intensity function and λ_κ is a level process driven by a stochastic process κ. The periodicity of the short-term intensity function is also considered, taking into account those insurance claims affected by a periodic environment, like hurricanes or seasonal storms. A Markov chain with m states, corresponding to different (risk) levels, is chosen for the level process, yielding a so-called regime-switching process. Under this intensity process, properties of the claim-counting process and its corresponding risk process are studied in detail. By properly choosing a Lundberg-type coefficient, we derive upper bounds for finite time ruin probabilities in a two-state case. This two-state Cox process model with periodicity mirrors the underlying risk on the claim under, say, "normal" or "extraordinary" conditions and is akin to the regime-switching models used in finance.

The paper is organized as follows. The model is defined in Section 2. Section 3 discusses the properties of the claim-counting process. This gives a precise description of the model characteristics, such as the probabilities of recording k claims during the time interval $[0, t)$, for $t \geq 0$ and $k \in \mathbb{N}$, and the expectation of the integrated intensities in (1.2). In Section 4 we derive Lundberg-type upper bounds for finite time ruin probabilities and illustrate the results by some examples.

2. A Cox Model with a Regime-Switching Periodic Intensity

Consider an intensity process $\lambda = \{\lambda(t)\}_{t\geq 0}$ governed by a deterministic pattern in each short-term period, say, a year, and a random effect on its peak level, that is, the amplitude of the pattern. This fixed-intensity pattern can be seen as the short-term periodicity, like in the NHP process. Assume we have m different risk levels, $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$, which represent the risk under different risk conditions as "low season," "median season," "high season," etc. In practice, such conditions can be slippery roads, foggy days, stormy weather, years affected by the El Niño phenomenon, and so on.

Furthermore, assume that the intensity level is modulated by an irreducible discrete-time Markov process, $\kappa = \{\kappa_n\}_{n\geq 0}$, with finite state space $J = \{0, 1, \ldots, m - 1\}$ and the transition probability matrix $P = (p_{ij})_{i,j=1}^{m}$.

Without loss of generality, we assume that the short-term period is 1. Let β be a function defined on $[0, 1]$, such that $\beta(t^*) = 1$, where $t^* \in [0, 1]$ is the mode of the function. Consider the intensity process λ, given by

$$\lambda(t) = \lambda_{\kappa_{\lfloor t \rfloor}}\, \beta(t - \lfloor t \rfloor), \qquad t \geq 0. \tag{2.1}$$

This gives $\lambda(n + t^*) = \lambda_{\kappa_n}\, \beta(t^*) = \lambda_{\kappa_n}$ for $n \in \mathbb{N}$, that is, the peak of the function $\lambda(t)$ within the $(n + 1)$-th year (i.e., $t \in [n, n + 1)$), is λ_{κ_n}, which changes according to the Markov chain κ. As such, we call $\{\lambda_{\kappa_n}\}_{n\geq 0}$ the intensity level process.

In the sequel, the common annual intensity pattern β is illustrated by a beta-type function with parameters $p \geq 1$ and $q \geq 1$, given by

$$\beta(t) = \alpha^* t^{p-1}(1 - t)^{q-1}, \qquad 0 \leq t \leq 1, \tag{2.2}$$

where α^* is a scale factor, given by

$$\alpha^* = \frac{1}{(t^*)^{p-1}(1 - t^*)^{q-1}}$$

and

$$t^* = \frac{p - 1}{p + q - 2} \tag{2.3}$$

is the mode of $\beta(t)$, $t \in [0, 1]$, so that at the mode $\beta(t^*) = 1$ is the peak level (see Fig. 1). Also denote the beta function in the usual way:

$$B(p, q) = \int_0^1 v^{p-1}(1 - v)^{q-1}\, dv = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}, \qquad p, q \geq 1,$$

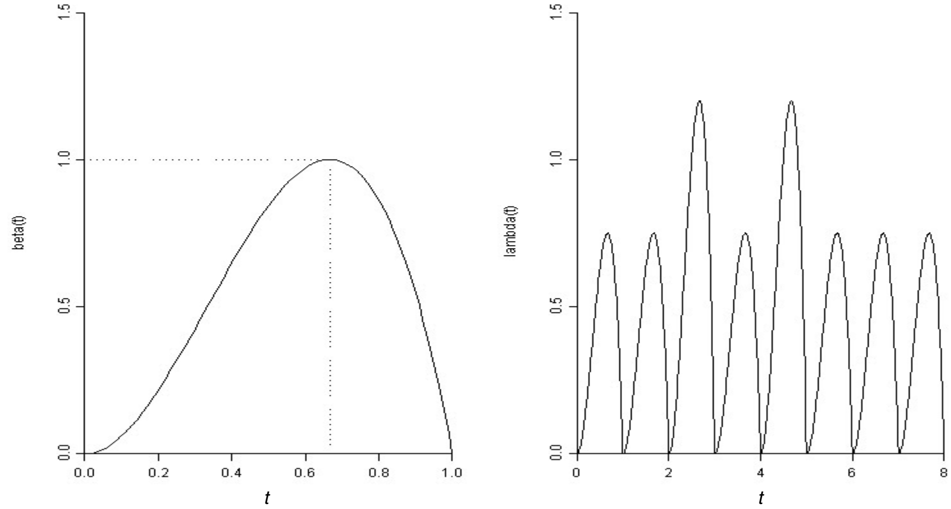
and the incomplete beta function at $p, q \geq 1$ as

$$B(p, q; t) = \int_0^t v^{p-1}(1 - v)^{q-1}\, dv, \qquad t \in (0, 1),$$

with $B(p, q; t) = 0$ if $t \leq 0$, while $B(p, q; t) = \lfloor t \rfloor B(p, q) + B(p, q; t - \lfloor t \rfloor)$, if $t \geq 1$. Figure 1 illustrates function $\beta(t)$, when $p = 3$ and $q = 2$, and the corresponding realization of the intensity process λ, when $\lambda_0 = 0.75$, $\lambda_1 = 1.2$, $p_{01} = 0.25$, and $p_{10} = 0.5$.

Consider the claim-counting process $N = \{N(t)\}_{t\geq 0}$ to be the special Cox process with intensity process as in (2.1). Because of the periodicity of the function $\beta(t - \lfloor t \rfloor)$, for $t \geq 0$, and the transitions, from year to year, within the m levels $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$, we call this risk model a regime-switching periodic nonhomogeneous Poisson (NHP) process.

Figure 1
$\beta(t)$ and One Realization of Intensity Process λ



Let $\{N_i(t)\}_{t\geq 0}$ for $i \in J$ (with $N_i(0) = 0$) denote a claim-counting NHP process with intensity function $\lambda_i\beta(t - \lfloor t \rfloor)$ over the time interval $[0, t)$: that is, $N_i(t)$ is Poisson distributed with mean $\lambda_i \int_0^t \beta(v - \lfloor v \rfloor)\, dv = \lambda_i\, \alpha^*\, B(p, q; t)$. Then the process N can be represented as

$$N(t) = \sum_{l=1}^{\lfloor t \rfloor} [N_{\kappa_{(l)}}(l) - N_{\kappa_{(l)}}(l - 1)] + N_{\kappa_{\lfloor t \rfloor}}(t) - N_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor)$$

$$= \sum_{i=0}^{m-1} \sum_{j=1}^{Y_i(\lfloor t \rfloor)} N_{i,j} + N_{\kappa_{\lfloor t \rfloor}}(t) - N_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor), \qquad t \geq 0, \tag{2.4}$$

where $Y_i(\lfloor t \rfloor) = \sum_{n=0}^{\lfloor t \rfloor - 1} \mathbb{I}(\kappa_n = i)$ denotes the number of years in $[0, \lfloor t \rfloor)$ that κ spends in state i, for $i \in J$, while $N_{i,j} = N_i(j) - N_i(j - 1)$ are i.i.d. random variables distributed as $N_i(1)$. This implies that the conditional expected number of claims in the time interval $[0, t)$, given the random environment, is

$$\mathbb{E}[N(t)|\kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}] = \sum_{n=0}^{\lfloor t \rfloor - 1} \int_0^1 \lambda_{\kappa_n}\beta(v)\, dv + \lambda_{\kappa_{\lfloor t \rfloor}} \int_0^{t - \lfloor t \rfloor} \beta(v)\, dv$$

$$= L(\lfloor t \rfloor)\alpha^* B(p, q) + \lambda_{\kappa_{\lfloor t \rfloor}}\, \alpha^* B(p, q; t - \lfloor t \rfloor), \qquad t \geq 0,$$

where

$$L(\lfloor t \rfloor) = \sum_{i=0}^{m-1} Y_i(\lfloor t \rfloor)\lambda_i, \qquad t \geq 0 \tag{2.5}$$

denotes the sum of the corresponding $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ values that occurred in $[0, \lfloor t \rfloor)$. Hence, we have $\mathbb{E}[N(t)] \leq \max_{0 \leq i \leq m-1}\{\lambda_i\}\, \alpha^*\, B(p, q; t)$. The corresponding compound NHP process $S = \{S(t)\}_{t\geq 0}$ is given by

$$S(t) = \sum_{j=1}^{N(t)} X_j, \qquad t \geq 0, \tag{2.6}$$

where the X_j's are the claim sizes with distribution function F_X, expected claim size $\mu = \int_0^\infty v\, dF_X(v)$, and moment-generating function $\hat{m}_X(s) = \int_0^\infty e^{sv}\, dF_X(v)$, for some $s > 0$. These claim severities are assumed independent of the Markov environment process κ and hence of the claim-counting process N. As in (2.4) the process S can also be represented as

$$S(t) = \sum_{i=0}^{m-1} \sum_{j=1}^{Y_i(\lfloor t \rfloor)} S_{i,j} + S_{\kappa_{\lfloor t \rfloor}}(t) - S_{\kappa_{\lfloor t \rfloor}}(\lfloor t \rfloor), \qquad t \geq 0,$$

where $S_{i,j}$ are i.i.d. random variables distributed as $S_i(1) = \sum_{n=1}^{N_i(t)} X_{i,n}$ and $S_i(t) = \sum_{n=1}^{N_i(t)} X_{i,n}$.

Now consider the continuous-time surplus process $U = \{U(t)\}_{t\geq 0}$, given by

$$U(t) = u + ct - S(t), \qquad t \geq 0, \tag{2.7}$$

where u is the initial capital value and c is the constant premium rate. The aggregate claim process S is given in (2.6), and the claim-counting process N is the regime-switching periodic NHP process in (2.4).

Because the Markov environment process κ is assumed irreducible, it has a stationary initial distribution, denoted by $\pi = (\pi_0, \pi_1, \ldots, \pi_{m-1})$. Then by the law of large numbers for irreducible Markov processes, we have

$$\lim_{t\to\infty} \frac{U(t)}{t} = c - \mu \sum_{i=0}^{m-1} \pi_i \, \lambda_i \, \alpha^* B(p, q), \tag{2.8}$$

(see Rolski et al. 1999, ch. 12).

The limit result in (2.8) implies that ruin occurs almost surely if the process has a negative drift, that is, $c \leq \mu \sum_{i=0}^{m-1} \pi_i \, \lambda_i \, \alpha^* \, B(p, q)$. Therefore we assume that the net profit condition

$$c > \mu \sum_{i=0}^{m-1} \pi_i \lambda_i \alpha^* B(p, q) \tag{2.9}$$

holds in the sequel.

3. PROPERTIES OF THE REGIME-SWITCHING PERIODIC PROCESS

For the regime-switching periodic NHP process defined above, the random measure Λ in this Cox process, given the realization of the environment process κ up to time $\lfloor t \rfloor$, is

$$\Lambda(t) = \int_0^t \lambda(v)\, dv = L(\lfloor t \rfloor)\alpha^* B(p, q) + \lambda_{\kappa_{\lfloor t \rfloor}} \alpha^* B(p, q; t - \lfloor t \rfloor), \qquad t \geq 0, \tag{3.1}$$

where $L(\lfloor t \rfloor)$ is given in (2.5). Then the conditional probability that the number of claims be k in the time interval $[0, t)$ is obtained as

$$P\{N(t) = k | \kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\} = \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)}, \qquad k \in \mathbb{N},$$

where $\Lambda(t)$ is given in (3.1).

To calculate $P\{N(t) = k\}$, we need to know how many times the levels $\lambda_0, \lambda_1, \ldots, \lambda_{m-1}$ appear in the sequence $\{\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{\lfloor t \rfloor}}\}$. This is equivalent to finding how many times $0, 1, \ldots, m-1$ appear in the corresponding sequence $\{\kappa_0, \kappa_1, \ldots, \kappa_{\lfloor t \rfloor}\}$. Note that the Markov property, given that the sequence starts from $\kappa_0 = j$, $j \in J$, $Y_i(n) = \sum_{l=0}^{n-1} \mathbb{I}(\kappa_l = i)$ has the same distribution as the number of times that any successive n-length sequence of the Markov process κ, which starts from state $j \in J$, is in state i, for $i \in J$.

Let $p_i(n; y_0, y_1, \ldots, y_{m-1})$ denote the conditional probability of a total number y_j of visits to state j out of the following successive n-length sequence, for $j \in J$, respectively, given that the sequence starts from state i, $i \in J$. Note here $0 \leq y_j \leq n$, for $j \in J$ and $\sum_{j=0}^{m-1} y_j = n$. For convenience, define $p_i(n; y_0, y_1, \ldots, y_{m-1}) = 0$ for either $y_j < 0$, $n \geq 0$ and $i, j \in J$. It is also clear that $p_i(n; y_0, y_1, \ldots,$

$y_{m-1}) = 0$ when $y_0 + y_1 + \cdots + y_{m-1} > n$. We have the following lemma for the probabilities $p_i(n; y_0, y_1, \ldots, y_{m-1})$.

Lemma 1

The recursive formulas for $p_i(n; y_0, y_1, \ldots, y_{m-1})$ are given by

$$p_i(0; 0, \ldots, 0) = 1, \qquad i \in J,$$

$$p_i(n; y_0, y_1, \ldots, y_{m-1}) = \sum_{j=0}^{m-1} p_{ij}\, p_j(n-1; y_0, \ldots, y_j - 1, \ldots, y_{m-1}), \tag{3.2}$$

where $0 \le y_j \le n$, $n \ge 1$, $j \in J$ and $\Sigma_{j=0}^{m-1} y_j = n$.

Denote by $P(n; y_0, y_1, \ldots, y_{m-1})$, the probability of $Y_j(n) = y_j$, for $j \in J$ in an n-length sequence of the J-valued irreducible Markov chain κ. By the law of total probability, we obtain the formulas for the probabilities $P(n; y_0, y_1, \ldots, y_{m-1})$ given in the following lemma.

Lemma 2

Let $\kappa = \{\kappa_n\}_{n\ge 0}$ be a J-valued irreducible Markov chain with initial distribution $(\pi_0, \pi_1, \ldots, \pi_{m-1})$. Then the probabilities of $Y_j(n) = y_j$, for $j \in J$, in an n-length sequence of the Markov chain κ, for $n \ge 1$, are given by

$$P(0; 0, 0, \ldots, 0) = 1,$$

$$P(n; y_0, y_1, \ldots, y_{m-1}) = \sum_{j=0}^{m-1} \pi_j\, p_j(n-1; y_0, \ldots, y_j - 1, \ldots, y_{m-1}), \tag{3.3}$$

where $0 \le y_j \le n$, $j \in J$, and $\Sigma_{j=0}^{m-1} y_j = n$, and $p_j(n-1; y_0, \ldots, y_j - 1, \ldots, y_{m-1})$ can be calculated recursively from (3.2).

For example, in a three-length sequence, the probability that there are three visits to state 0 and no visits to states 2 and 3 is given by

$$P(3; 3, 0, 0) = \pi_0\, p_0(2; 2, 0, 0) = \pi_0\, p_{00}\, p_0(1; 1, 0, 0) = \pi_0 (p_{00})^2,$$

since it has to start first from state 0 and then stay in 1 for the next two steps, while the probability that there be a visit to state 0, two visits to state 1, and no visit to state 3 is

$$\begin{aligned} P(3; 1, 2, 0) &= \pi_0\, p_0(2; 0, 2, 0) + \pi_1\, p_1(2; 1, 1, 0) \\ &= \pi_0[p_{01}\, p_1(1; 0, 1, 0) + \pi_1[p_{10}\, p_0(1; 0, 1, 0) + p_{11}\, p_1(1; 1, 0, 0)] \\ &= \pi_0\, p_{01}\, p_{11} + \pi_1[p_{10}\, p_{01} + p_{11}\, p_{10}], \end{aligned}$$

since if the sequence starts at 0, it must go to 1 twice in the next two steps, and if the sequence starts at 1, it must go to 0 for the next step and go back to 1 for the last step, or else go to 1 for the next step and go back to 0 at the last step.

We introduce the following notation. Let $\Lambda(n; y_0, y_1, \ldots, y_{m-1})$ be the random measure under a realization of $y_0, \ldots, y_{m-1}$ counts, respectively, at levels $\lambda_0, \ldots, \lambda_{m-1}$ in the sequence $\lambda_{\kappa_0}, \lambda_{\kappa_1}, \ldots, \lambda_{\kappa_{n-1}}\}$, that is,

$$\Lambda(n; y_0, y_1, \ldots, y_{m-1}) = (y_0\, \lambda_0 + \cdots + y_{m-1}\, \lambda_{m-1})\alpha^* B(p, q), \qquad n \in \mathbb{N}, \tag{3.4}$$

where $0 \le y_j \le n$, $j \in J$, and $y_0 + y_1 + \cdots + y_{m-1} = n$. Then we have the following theorem for the probabilities $P\{N(t) = k\}$.

Theorem 1

Let $\kappa = \{\kappa_n\}_{n\geq 0}$ be a J-valued irreducible Markov chain with transition probabilities $P = (p_{ij})_{i,j=1}^{m}$ and initial distribution $(\pi_0, \pi_1, \ldots, \pi_{m-1})$. For the counting process N, given by (2.4), the probabilities that there be k claim occurrences during the time interval $[0, t)$, for $t \geq 0$ and $k \in \mathbb{N}$, are given by

$$\begin{aligned} P\{N(t) = k\} = {} & \frac{1}{k!} \sum_{j=0}^{m-1} \sum_{i=0}^{m-1} \pi_i\, p_{ij} \\ & \times \sum_{(y_0,\ldots,y_{m-1})\in S_i} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) e^{-[\Lambda(\lfloor t \rfloor; y_0,\ldots,y_{m-1}) + \lambda_j\, \alpha^* B(p,q;t-\lfloor t \rfloor)]} \\ & \times [\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) + \lambda_j\, \alpha^* B(p, q; t - \lfloor t \rfloor)]^k, \end{aligned} \tag{3.5}$$

where $S_i = \{(y_0, \ldots, y_{m-1}) | 0 \leq y_\nu \leq \lfloor t \rfloor - 1,\ \nu \neq i,\ 1 \leq y_i \leq \lfloor t \rfloor,\ \Sigma_{\nu=0}^{m-1} y_\nu = \lfloor t \rfloor\}$, $P(\lfloor t \rfloor; y_0, \ldots, y_{m-1})$, and $\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1})$ can be obtained from (3.3) and (3.4).

PROOF

See the Appendix. □

The random measure $\Lambda(t)$ of this special Cox process is given by (3.1). Taking expectations in (3.1) directly gives

$$\begin{aligned} \mathbb{E}[\Lambda(t)] &= \alpha^* B(p, q)\mathbb{E}[L(\lfloor t \rfloor)] + \mathbb{E}[\lambda_{\kappa_{\lfloor t \rfloor}}]\ \alpha^* B(p, q; t - \lfloor t \rfloor) \\ &= \alpha^* B(p, q) \sum_{i=0}^{m-1} \lambda_i\ \mathbb{E}[Y_i(\lfloor t \rfloor)] + \alpha^* B(p, q; t - \lfloor t \rfloor)\mathbb{E}[\lambda_{\kappa_{\lfloor t \rfloor}}], \end{aligned}$$

where $\mathbb{E}[Y_i(\lfloor t \rfloor)] = \Sigma_{y=0}^{\lfloor t \rfloor} yP\{Y_i(\lfloor t \rfloor) = y\}$, and $\mathbb{E}[\lambda_{\kappa_{\lfloor t \rfloor}}] = \Sigma_{i=0}^{m-1} \lambda_i\ \pi_i$. Let $S = \{(y_0, \ldots, y_{m-1}) | 0 \leq y_\nu \leq \lfloor t \rfloor,\ \Sigma_{\nu=0}^{m-1} y_\nu = \lfloor t \rfloor\}$, and $S_y = \{(y_0, \ldots, y_{m-1}) | 0 \leq y_\nu \leq \lfloor t \rfloor - y,\ \nu \neq i,\ \Sigma_{\nu\neq i}\, y_\nu = \lfloor t \rfloor - y,\ y_i = y\}$. Then for $0 \leq y \leq \lfloor t \rfloor$, we have

$$P\{Y_i(\lfloor t \rfloor) = y\} = \sum_{(y_0,\ldots,y_{m-1})\in S_y} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}).$$

By the equation

$$\sum_{(y_0\ldots,y_{m-1})\in S} y_i\, P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) = \sum_{y=0}^{\lfloor t \rfloor} \sum_{(y_0,\ldots,y_{m-1})\in S_y} yP(\lfloor t \rfloor; y_0, \ldots, y_{m-1}),$$

it follows immediately that

$$\begin{aligned} \mathbb{E}[\Lambda(t)] = {} & \alpha^* B(p, q) \sum_{(y_0,\ldots,y_{m-1})\in S} \left(\sum_{i=0}^{m-1} y_i\ \lambda_i \right) P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) \\ & + \alpha^* B(p, q; t - \lfloor t \rfloor) \sum_{i=0}^{m-1} \lambda_i \pi_i, \qquad t \geq 0. \end{aligned} \tag{3.6}$$

It is not difficult to see that (3.6) is equivalent to

$$\begin{aligned} \mathbb{E}[\Lambda(t)] = {} & \sum_{(y_0,\ldots,y_{m-1})\in S} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) \\ & \times \sum_{i=0}^{m-1} \pi_i[\Lambda(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) + \lambda_i \alpha^* B(p, q; t - \lfloor t \rfloor)], \end{aligned}$$

then when $t \geq 0$ and $s < a_\Lambda$, the moment-generating function of $\Lambda(t)$, $\hat{m}_{\Lambda(t)}(s) = \mathbb{E}[e^{s\Lambda(t)}]$, is given by

$$\hat{m}_{\Lambda(t)}(s) = \sum_{(y_0,\ldots,y_{m-1})\in S} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) \sum_{i=0}^{m-1} \pi_i\ e^{s[\Lambda(\lfloor t \rfloor; y_0,\ldots,y_{m-1}) + \lambda_i\, \alpha^* B(p,q;t-\lfloor t \rfloor)]}, \tag{3.7}$$

where a_Λ is a number such that $\lim_{s\uparrow a_\Lambda} \hat{m}_{\Lambda(t)}(s) = +\infty$, while $P(\lfloor t \rfloor; y_0, \ldots, y_{m-1})$ can be obtained from (3.3).

It is interesting to see that (3.7) can be rewritten as

$$\begin{aligned} \hat{m}_{\Lambda(t)}(s) &= \sum_{(y_0,\ldots,y_{m-1})\in S} P(\lfloor t \rfloor; y_0, \ldots, y_{m-1}) e^{s\Lambda(\lfloor t \rfloor; y_0,\ldots,y_{m-1})} \sum_{i=0}^{m-1} \pi_i e^{s\lambda_i \alpha^* B(p,q;t-\lfloor t \rfloor)} \\ &= \hat{m}_{\Lambda(\lfloor t \rfloor)}(s)\hat{m}_{\Lambda(t-\lfloor t \rfloor)}(s), \qquad s < a_\Lambda, \end{aligned}$$

showing that $\Lambda(t) = \Lambda(\lfloor t \rfloor) + \Lambda(t - \lfloor t \rfloor)$ and that these are independent.

Theorem 1 and the above results for $\Lambda(t)$ allow for the derivation of the moments of $N(t)$. By well-known formulas for general Cox models, first we can find the probability-generating function $\hat{g}_{N(t)}(s) = \hat{m}_\Lambda(s - 1)$, for $|s| < 1$, and then the successive factorial moments of $N(t)$: $\mathbb{E}[N(t)[N(t) - 1] \cdots [N(t) - r + 1]] = \hat{g}^{(r)}_{N(t)}(1) = \mathbb{E}[\Lambda(t)^r]$. In particular, we have that the expectation of the number of claim for this special Cox process is $\mathbb{E}[N(t)] = \mathbb{E}[\Lambda(t)]$, where $\mathbb{E}[\Lambda(t)]$ is given by (3.6), and the variance of $N(t)$ for this Cox process can be derived from $\mathbb{V}[N(t)] = \mathbb{V}[\Lambda(t)] + \mathbb{E}[\Lambda(t)]$. Finally, note that the index of dispersion of $N(t)$ is $\mathbb{I}_{N(t)} = \mathbb{V}_{N(t)}/\mathbb{E}[N(t)] = 1 + \mathbb{I}_{\Lambda(t)} > 1$, showing that $N(t)$ is overdispersed, by contrast to the classical Poisson process.

4. A LUNDBERG-TYPE UPPER BOUND FOR FINITE TIME RUIN PROBABILITIES

This last section discusses the ruin problem for Cox processes. First, consider the income process, over the time interval $[0, t)$, with initial value $R(0) = 0$ and a constant premium rate c, given as

$$R(t) = ct - S(t) = ct - \sum_{j=1}^{N(t)} X_j, \qquad t \geq 0, \tag{4.1}$$

where the claim-counting process $N = \{N(t)\}_{t\geq 0}$ is a Cox process with intensity process $\{\lambda(t)\}_{t\geq 0}$, and the random intensity measure Λ, given by $\Lambda(t) = \int_0^t \lambda(v)\, dv$, and $S(t)$ is as in (2.6). Further assume that the moment-generating function $\hat{m}_X(s) = \int_0^\infty e^{sx}\, dF_X(x)$ is defined on an interval $[0, a_X)$, where $a_X > 0$.

Denote the conditional Laplace-Stieltjes transform of $R(t)$, given $\Lambda(t)$, by $l(r; t|\Lambda) = \mathbb{E}[e^{-rR(t)}|\Lambda(t)]$. Assuming that it exists, it is given by

$$l(r; t|\Lambda) = e^{\Lambda(t)[\hat{m}_X(r)-1]-rct}, \quad \text{for some } r \in [0, a_X),\ t \geq 0. \tag{4.2}$$

Finally, the time to ruin is defined in the usual way as $T_u = \inf\{t \geq 0 | u + R(t) < 0\}$, for $u \geq 0$. The ultimate ruin probability is then given by $\Psi(u) = P\{T_u < \infty\}$. Using the martingale approach to Cox models discussed in Grandell (1991), we have the following result.

Theorem 2

A Lundberg-type upper bound holds for the finite time ruin probability in model (4.1):

$$P\{T_u \leq t_0\} \leq e^{-ru}\, \mathbb{E}\left[\sup_{0\leq t\leq t_0} l(r; t|\Lambda)\right], \qquad 0 \leq t_0 < \infty. \tag{4.3}$$

A tighter upper bound can also be obtained for $0 \leq t_0 < \infty$, as

$$P\{T_u \leq t_0\} \leq e^{-ru}\, \mathbb{E}\left[\sup_{0\leq t\leq t_0} l(r; t|\Lambda)\right] \sup_{y\geq 0} \left\{\frac{e^{ry}[1 - F_X(y)]}{\int_y^\infty e^{rx}\, dF_X(x)}\right\}, \tag{4.4}$$

where $r \in [0, a_X)$ is a real number.

PROOF

For details see the Appendix. □

The upper bound given in (4.4) is difficult to use in practice. Now consider our regime-switching periodic NHP model with two levels λ_0 and λ_1 and function β in the intensity process as given by (2.2). To derive a corresponding useful bound for this special Cox model, first for $i = 0, 1$, some $r \in [0, a_X)$, and $t \geq 0$, let

$$l_i(r; t) = \mathbb{E}[e^{-rR_i(t)}] = \mathbb{E}[e^{-r(ct-\Sigma_{j=1}^{N_i(t)} X_j)}] = e^{\lambda_i \alpha^* B(p,q;t)[\hat{m}_X(r)-1]-rct}. \tag{4.5}$$

Then define the average risk level $\bar{\lambda} = \pi_0 \lambda_0 + \pi_1 \lambda_1$, where λ_1, the peak intensity under "extraordinary risk" years, is assumed larger than that in "normal risk" years (i.e., $\lambda_0 < \lambda_1$). Consider, for $r \geq 0$, the Lundberg-type equation

$$\theta(r) = \bar{\lambda}\alpha^* B(p, q)[\hat{m}_X(r) - 1] - rc = 0. \tag{4.6}$$

The solution to (4.6), say, $\gamma > 0$, also called the adjustment coefficient for the average risk level $\bar{\lambda}$, satisfies

$$\bar{\lambda}\alpha^* B(p, q)[\hat{m}_X(\gamma) - 1] = \gamma c. \tag{4.7}$$

It follows from (4.7) that

$$\lambda_i \alpha^* B(p, q)[\hat{m}_X(\gamma) - 1] = \frac{\lambda_i}{\bar{\lambda}} \gamma c, \qquad i = 0, 1. \tag{4.8}$$

The existence and unicity of γ in $[0, a_X)$ is guaranteed because $\theta(0) = 0$ and $\theta'(0) = \bar{\lambda}\alpha^* B(p, q)\mu - c < 0$, provided that the net profit condition (2.9) holds, and hence the convexity of $\theta(r)$ ensures that $\theta'(\gamma) > 0$.

A practical Lundberg-type upper bound for the finite-time ruin probability can be derived for a two-state regime-switching periodic NHP model by setting $r = \gamma$. For simplicity, assume that t_0 is an integer. Then with probability $P(t_0; t_0 - y, y)$ (here $Y_1(t_0) = y$ implies that $Y_0(t_0) = t_0 - y$), given by (3.3) when $m = 2$, $\Lambda(t_0)$ takes the following realization:

$$\Lambda(t_0; t_0 - y, y) = [(t_0 - y)\lambda_0 + y\lambda_1]\alpha^* B(p, q), \qquad 0 \leq y \leq t_0,\ t_0 \in \mathbb{N}.$$

When $0 \leq t \leq t_0$, we have two possibilities for $\Lambda(t)$, depending on the value of $\lambda_{\kappa_{\lfloor t \rfloor}}$. One is when $\lambda_{\kappa_{\lfloor t \rfloor}} = \lambda_0$:

$$\Lambda(t) = [(\lfloor t \rfloor - z)\lambda_0 + z\lambda_1]\ \alpha^* B(p, q) + \lambda_0 \alpha^* B(p, q; t - \lfloor t \rfloor), \qquad 0 \leq t \leq t_0, \tag{4.9}$$

where $0 \leq z \leq \min\{\lfloor t \rfloor, y\}$ and $\lfloor t \rfloor - z + 1 \leq t_0 - y$, or equivalently, $z \in C_1(t, y) = [\max\{0, \lfloor t \rfloor + 1 - (t_0 - y)\}, \min\{\lfloor t \rfloor, y\}]$. The other is when $\lambda_{\kappa_{\lfloor t \rfloor}} = \lambda_1$:

$$\Lambda(t) = [(\lfloor t \rfloor - z)\lambda_0 + z\lambda_1]\alpha^* B(p, q) + \lambda_1 \alpha^* B(p, q; t - \lfloor t \rfloor), \qquad 0 \leq t \leq t_0, \tag{4.10}$$

where similarly, $0 \leq z \leq \min\{\lfloor t \rfloor, y - 1\}$ and $\lfloor t \rfloor - z \leq t_0 - y$, or equivalently, $z \in C_2(t, y) = [\max\{0, \lfloor t \rfloor - (t_0 - y)\}, \min\{\lfloor t \rfloor, y - 1\}]$.

When $\Lambda(t)$ is given by (4.9), then (4.7) and (4.8) imply that

$$\begin{aligned}
\Lambda(t)[\hat{m}_X(\gamma) - 1] - \gamma ct &= (\lfloor t \rfloor - z)[\lambda_0\ \alpha^* B(p, q)(\hat{m}_X(\gamma) - 1) - \gamma c] \\
&\quad + z[\lambda_1 \alpha^* B(p, q)(\hat{m}_X(\gamma) - 1) - \gamma c] \\
&\quad + \lambda_0\ \alpha^* B(p, q; t - \lfloor t \rfloor)(\hat{m}_X(\gamma) - 1) - \gamma c(t - \lfloor t \rfloor) \\
&= -\lfloor t \rfloor \left(\frac{\bar{\lambda} - \lambda_0}{\bar{\lambda}}\right) \gamma c + z \left(\frac{\lambda_1 - \lambda_0}{\bar{\lambda}}\right) \gamma c \\
&\quad + \lambda_0\ \alpha^* B(p, q; t - \lfloor t \rfloor)[\hat{m}_X(\gamma) - 1] - \gamma c(t - \lfloor t \rfloor).
\end{aligned}$$

In turn,

$$\begin{aligned}\sup_{0\le t\le t_0} l(\gamma;\, t|\Lambda) &= \sup_{0\le t\le t_0} e^{\Lambda(t)[\hat{m}_X(\gamma)-1]} - \gamma ct \\ &\le \sup_{\substack{0\le t\le t_0\\ z\in C_1(t,y)}} e^{z((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c+\lambda_0\,\alpha^* B(p,q;t-\lfloor t\rfloor)[\hat{m}_X(\gamma)-1]-\gamma c(t-\lfloor t\rfloor)} \\ &= \max_{\substack{0\le t\le t_0\\ z\in C_1(t,y)}} e^{z(\lambda_1-\lambda_0/\lambda)\gamma c} \max_{0\le v\le 1} l_0(\gamma;\, v) \le e^{y((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c} \max_{0\le v<1} l_0(\gamma;\, v). \end{aligned} \tag{4.11}$$

Similarly, when $\Lambda(t)$ is given by (4.10), then

$$\sup_{0\le t\le t_0} l(\gamma;\, t|\Lambda) \le e^{(y-1)((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c} \max_{0\le v\le 1} l_1(\gamma;\, v) \le e^{y((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c} \max_{0\le v<1} l_1(\gamma;\, v),$$

which has a similar form as (4.11). Taking expectations gives

$$\mathbb{E}\left[\sup_{0\le t\le t_o} l(\gamma;\, t|\Lambda)\right] \le \left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,\, y)e^{y((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} l_i(\gamma;\, v).$$

Finally, a Lundberg-type upper bound for the finite time ruin probability in (4.4) is derived, that is,

$$P\{T_u \le t_0\} \le e^{-\gamma u}\left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,\, y)e^{y((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} l_i(\gamma;\, v) \sup_{y\ge 0}\left\{\frac{e^{\gamma y}\bar{F}_X(y)}{\int_y^\infty e^{\gamma x}\, dF_X(x)}\right\}, \qquad t_0 \in \mathbb{N}, \tag{4.12}$$

where γ satisfies (4.7) and $P(t_0;\, t_0 - y,\, y)$ is given in (3.3) when $m = 2$.

Obviously, the simpler bound for $P\{T_u \le t_0\}$ given by (4.3) can also be derived here:

$$P\{T_u \le t_0\} \le e^{-\gamma u}\left[\sum_{y=0}^{t_0} P(t_0;\, t_0-y,\, y)e^{y((\lambda_1-\lambda_0)/\bar{\lambda})\gamma c}\right] \max_{\substack{0\le v<1\\ i=0,1}} l_i(\gamma;\, v), \tag{4.13}$$

but (4.12) is tighter than (4.13), as shown in the following examples.

EXAMPLE 1

Consider exponentially distributed claim sizes with mean μ. Their moment-generating function is $\hat{m}_X(s) = 1/(1 - \mu s)$, for $s < a_X = 1/\mu$. The adjustment coefficient for parameter $\bar{\lambda}$ is then given by

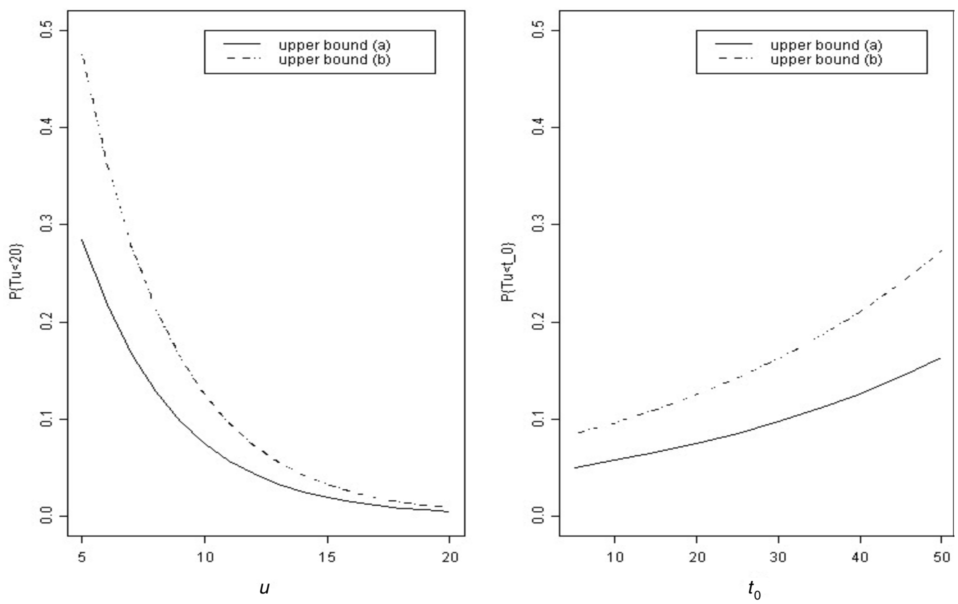
$$\gamma = \frac{c - \bar{\lambda}\alpha^* B(p,\, q)\mu}{c\mu} = \frac{1}{\mu} - \frac{\bar{\lambda}\alpha^* B(p,q)}{c}, \tag{4.14}$$

which is the positive solution to (4.7). The corresponding $l_i(\gamma;\, v)$, given in (4.5), takes the form

$$l_i(\gamma;\, v) = e^{(\lambda_i/\bar{\lambda})(B(p,q;v)/B(p,q)) - v)\gamma c}, \qquad 0 \le v < 1,\ i = 0,\, 1. \tag{4.15}$$

Figure 2 illustrates the upper bounds in this exponential case, as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen to be $\lambda_0 = 1$, $\lambda_1 = 1.2$, $p = 3$, $q = 2$, $p_{01} = 0.25$, $p_{10} = 0.5$, $c = 1.5$, $\mu = 1.5$, and $\gamma = 0.267$, which is obtained from (4.14). Clearly, the upper bounds (a), given by (4.12), are sharper than those in (b), given by (4.13).

Figure 2
Upper Bounds for Exponential Claims vs. u (t_0 = 20) and t_0 (u = 10)



EXAMPLE 2

Consider the case of inverse Gaussian distributed claims, with mean μ, variance $\mu\beta$, and density function

$$f_X(x) = \frac{\mu}{\sqrt{2\pi\beta x^3}} e^{-(x-\mu)^2/2\beta x}, \qquad x > 0.$$

Their moment-generating function $\hat{m}_X(s) = e^{(\mu/\beta)(1-\sqrt{1-2\beta s})}$ exists for $s < 1/2\beta$. The adjustment coefficient γ with respect to parameter $\bar{\lambda}$ is the positive solution to the equation

$$\bar{\lambda}\alpha^* B(p, q)[e^{(\mu/\beta)(1-\sqrt{1-2\beta\gamma})} - 1] = \gamma c, \tag{4.16}$$

and $l_i(v; \gamma)$, for $i = 0, 1$, is of the same form as in (4.15).

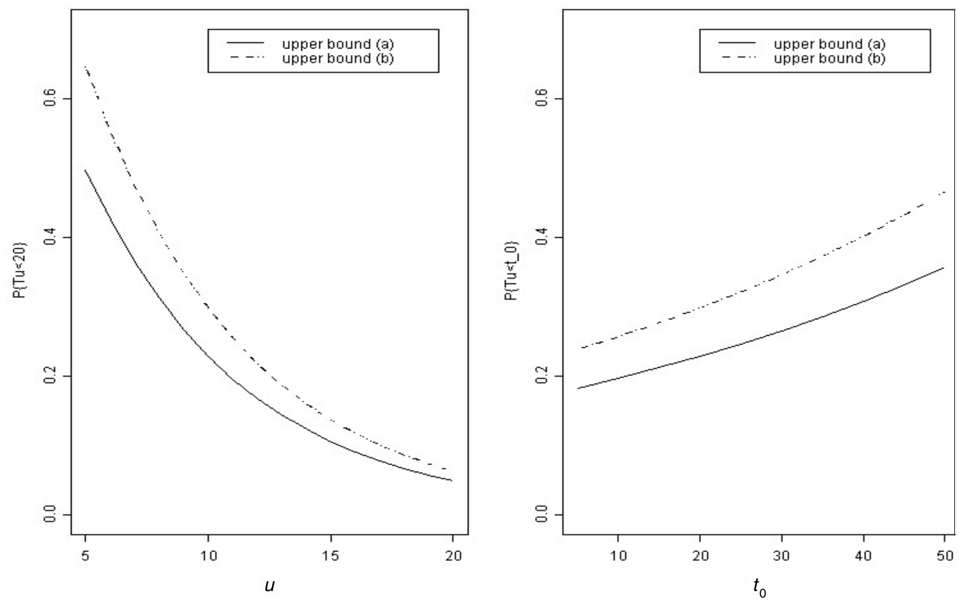
Figure 3 illustrates the upper bounds in this inverse Gaussian case, again as a function of u (left graph), when $t_0 = 20$, and as a function of t_0 (right graph), when $u = 10$. The other parameters are chosen as for Figure 2 and $\beta = \frac{8}{3}$, which gives a variance of 4. Here $\gamma = 0.155$ is obtained from (4.16). Again the upper bounds in (a), given by (4.12), are sharper than those in (b), given by (4.13).

5. CONCLUSIONS

Regime-switching periodic NHP processes can be useful in modeling risk processes under periodic and random environments. A beta-type short-term intensity function is proposed with an m-state Markov process to model the peak level in the intensity of this Cox risk process. This generalizes the periodic NHP model. Also, it can provide a more realistic description than Cox models with piecewise constant intensities.

The flexible shape of the beta function and the explicit results, in terms of the complete beta and incomplete beta functions, obtained for the Cox risk model should make these regime-switching period NHP models more practical than Cox processes with piecewise constant intensities, or than the usual NHP process. However, this work can be extended to other reasonable short-term intensity functions.

Figure 3
Upper Bounds for Inverse Gaussian Claims vs. t_0 (u = 10) and u (t_0 = 20)



Furthermore, statistical methods to estimate the beta and level parameters of the model from real data are readily available and shall be illustrated in subsequent work.

APPENDIX

PROOF OF THEOREM 1

By the law of total probability, it is easily seen that

$$\begin{aligned} P\{N(t) = k\} &= \sum_{i=0}^{m-1} P\{N(t) = k | \kappa_{\lfloor t \rfloor - 1} = i\} P\{\kappa_{\lfloor t \rfloor - 1} = i\} \\ &= \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} P\{N(t) = k | \kappa_{\lfloor t \rfloor - 1} = i, \kappa_{\lfloor t \rfloor} = j\} p_{ij} \pi_i. \end{aligned}$$

Furthermore, since N is the Cox process corresponding to Λ, then $N(t)$ has independent increments and $N(t) - N(s)$ is Poisson distributed with mean $\Lambda(t) - \Lambda(s)$ (with respect to $\mathcal{F}_\infty^\Lambda$, see Grandell 1991), and so we have

$$P\{N(t) = k | \kappa_{\lfloor t \rfloor - 1} = i, \kappa_{\lfloor t \rfloor} = j\} = \sum_{l=0}^{k} P\{N(\lfloor t \rfloor) = l | \kappa_{\lfloor t \rfloor - 1} = i\} P\{N(t) - N(\lfloor t \rfloor) = k - l | \kappa_{\kappa_{\lfloor t \rfloor}} = j\}.$$

For $S_i = \{(y_0, \ldots, y_{m-1}) | 0 \le y_\nu \le \lfloor t \rfloor - 1, \nu \ne i, 1 \le y_i \le \lfloor t \rfloor, \Sigma_{\nu=0}^{m-1} y_\nu = \lfloor t \rfloor\}$

we have

$$
\begin{aligned}
P\{N(t) &= k|\kappa_{\lfloor t\rfloor-1} = i,\ \kappa_{\lfloor t\rfloor} = j\} \\
&= \sum_{l=0}^{k}\left(\sum_{(y_0,\ldots,y_{m-1})\in S_i} P(\lfloor t\rfloor; y_0, \ldots, y_{m-1}) \frac{[\Lambda(\lfloor t\rfloor; y_0, \ldots, y_{m-1})]^l}{l!} e^{-\Lambda(\lfloor t\rfloor; y_0,\ldots,y_{m-1})}\right) \\
&\quad \times \frac{[\lambda_j\, \alpha^* B(p, q; t - \lfloor t\rfloor)]^{k-l}}{(k-l)!} e^{-\lambda_j \alpha^* B(p,q;t-\lfloor t\rfloor)} \\
&= \sum_{(y_0,\ldots,y_{m-1})\in S_i} P(\lfloor t\rfloor; y_0, \ldots, y_{m-1})\, e^{-[\Lambda(\lfloor t\rfloor; y_0,\ldots,y_{m-1})+\lambda_j \alpha^* B(p,q;t-\lfloor t\rfloor)]} \\
&\quad \times \sum_{l=0}^{k} \frac{[\Lambda(\lfloor t\rfloor; y_0, \ldots, y_{m-1})]^l[\lambda_j\, \alpha^* B(p, q; t - \lfloor t\rfloor)]^{k-l}}{l!(k-l)!} \\
&= \sum_{(y_0,\ldots,y_{m-1})\in S_i} P(\lfloor t\rfloor; y_0, \ldots, y_{m-1})\, e^{-[\Lambda(\lfloor t\rfloor; y_0,\ldots,y_{m-1})+\lambda_j \alpha^* B(p,q;t-\lfloor t\rfloor)]} \\
&\quad \times \frac{[\Lambda(\lfloor t\rfloor; y_0, \ldots, y_{m-1}) + \lambda_j\, \alpha^* B(p, q; t - \lfloor t\rfloor)]^k}{k!}.
\end{aligned}
$$

Then (3.5) immediately follows. □

PROOF OF THEOREM 2

The proof of (4.3) can be found in Grandell (1991), where a martingale approach for a general Cox model is used. We briefly review the main steps for a better understanding of the proof of (4.4).

Let **F** be a suitable filtration, M be a positive **F**-martingale (or **F**-supermartingale), and T be an **F**-stopping time. Choose $t_0 < \infty$ and consider $t_0 \wedge T$, a bounded **F**-stopping time. By the Optional Stopping Theorem (see Rolski et al. 1999, Section 9.1.6), we have that

$$M(0) \geq \mathbb{E}^{\mathcal{F}_0}[M(t_0 \wedge T)] \geq \mathbb{E}^{\mathcal{F}_0}[M(T)|t \leq t_0]P^{\mathcal{F}_0}\{T \leq t_0\},$$

and therefore

$$P^{\mathcal{F}_0}\{T \leq t_0\} \leq \frac{M(0)}{\mathbb{E}^{\mathcal{F}_0}[M(T)|T \leq t_0]}, \qquad t_0 < \infty.$$

Now consider N to be a Cox process with intensity process $\{\lambda(t)\}_{t\geq 0}$ and random intensity measure Λ, given by $\Lambda(t) = \int_0^t \lambda(v)\, dv$. A suitable filtration **F** is defined as $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$ and thus $\mathcal{F}_0 = \mathcal{F}_\infty^\Lambda$. Consider the following choice of process M:

$$M(t) = \frac{e^{-r[u+R(t)]}}{e^{\Lambda(t)[\hat{m}_X(r)-1]-rct}}, \quad \text{for some } r \in [0, a_X),\ t \geq 0,$$

where $R(t)$ is given in (4.1).

It can be shown that M is an **F**-martingale for the filtration given by $\mathcal{F}_t = \mathcal{F}_\infty^\Lambda \vee \mathcal{F}_t^R$. A conditional lower bound is obtained when $0 \leq t_0 < \infty$ as

$$
\begin{aligned}
\mathbb{E}^{\mathcal{F}_0}[M(T_u)|T_u \leq t_0] &\geq \mathbb{E}^{\mathcal{F}_0}[e^{-\Lambda(T_u)[\hat{m}_X(r)-1]+rcT_u}|T_u \leq t_0] \\
&\geq \inf_{0\leq t\leq t_o} e^{-\Lambda(t)[\hat{m}_X(r)-1]+rct}. \qquad \text{(A.1)}
\end{aligned}
$$

More precisely, since the ruin-causing claim is greater than the surplus just before ruin, we have that

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_0}[M(T_u)|T_u \le t_0] &= \mathbb{E}^{\mathcal{F}_0}[e^{-r[u+R(T_u)]}e^{-\Lambda(T_u)[\hat{m}_X(r)-1]+rcT_u}|T_u \le t_0] \\ &\ge \inf_{0\le t\le t_o}\{e^{-\Lambda(t)[\hat{m}_X(r)-1]+rct}\}\mathbb{E}^{\mathcal{F}_0}[e^{-r[u+R(T_u)]}|T_u \le t_0] \\ &\ge \inf_{0\le t\le t_o}\{e^{-\Lambda(t)[\hat{m}_X(r)-1]+rct}\}\inf_{y\ge 0}\left\{\frac{\int_y^\infty e^{-r(y-x)}\,dF_X(x)}{1-F_X(y)}\right\}. \end{aligned} \tag{A.2}$$

Then we get, from (A.1), that the conditional ruin probability

$$P^{\mathcal{F}_0}\{T_u \le t_0\} \le \frac{M(0)}{\mathbb{E}^{\mathcal{F}_0}[M(T_u)|T_u \le t_0]} \le e^{-ru}\sup_{0\le t\le t_o} l(r;t|\Lambda). \tag{A.3}$$

Taking expectations proves (4.3). Using (A.2) in (A.3) yields (4.4). □

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