Direct Derivation of Finite-Time Ruin Probabilities in the Discrete Risk Model with Exponential or Geometric Claims

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Abstract

Growing research interest has been shown in finite-time ruin probabilities for discrete risk processes, even though the literature is not as extensive as for continuous-time models. The general approach is through the so-called Gerber-Shiu discounted penalty function, obtained for large families of claim severities and discrete risk models. This paper proposes another approach to deriving recursive and explicit formulas for finite-time ruin probabilities with exponential or geometric claim severities. The proposed method, as compared to the general Gerber-Shiu approach, is able to provide simpler derivation and straightforward expressions for these two special families of claims.

1. Introduction

We consider the discrete-time risk process

$$U(n) = u + cn - \sum_{i=1}^{n} X_i, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where u is the initial surplus, $c > 0$ is the constant premium rate, and $\{X_i, i \geq 1\}$ are independent and identically distributed (i.i.d.) claim size random variables.

In the discrete-time model (1.1), the ruin can occur only at time instants $n = 1, 2, \ldots$. The general approach for studying ruin probabilities in the discrete-time model is through the so-called Gerber-Shiu discounted penalty function; see, for instance, Pavlova and Willmot (2004), Dickson (2005, Chapter 6), and Li (2005a,b). In this paper we propose another approach to deriving recursive and explicit formulas for finite-time ruin probabilities with exponential or geometric claim severities. The derivation for the case of exponential claims relies on the special structure of the exponential distribution and a recursive formula for the finite-time ruin probability given by De Vylder and Goovaerts (1988). Numerical evaluation of the resulting ruin function can be performed easily. Moments of the time to ruin given that the ruin occurs can be obtained using the method proposed by Drekic and Willmot (2003). The proposed method, as compared to the general Gerber-Shiu approach, is able to provide simpler derivation and straightforward expressions for these two special families of claims.

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2. MAIN RESULTS

2.1 Exponential Claims

We first consider the finite-time ruin probability of the discrete-time risk process in equation (1.1) with exponential claim size distribution. The exponential density function is given by

$$f(x) = \alpha e^{-\alpha x}. \tag{2.1}$$

In the following exposition, it is implicitly assumed that the premium loading factor θ, defined by $c = (1 + \theta)E(X) = (1 + \theta)/\alpha$, is positive. Equivalently the range of α is $\alpha > 1/c$. For $n = 1, 2, \ldots,$ let

$$\phi_n(u) = \Pr\{U(1) \geq 0, U(2) \geq 0, \ldots, U(n) \geq 0 | U(0) = u\}$$

denote the survival probability at instants $1, 2, \ldots, n$ with the initial surplus $u \geq 0$. The probability of ruin at one of the instants $1, 2, \ldots, n$ is denoted by

$$\psi_n(u) = 1 - \phi_n(u). \tag{2.2}$$

Alternatively, $\phi_n(u)$ can be expressed in terms of $\{X_i\}_{i=1}^n$ and parameters u, c as

$$\phi_n(u) = \Pr\{X_1 \leq u + c, X_1 + X_2 \leq u + 2c, \ldots, X_1 + X_2 + \cdots + X_n \leq u + nc\}.$$

Obviously,

$$\phi_1(u) = \Pr\{X_1 \leq u + c\} = F(u + c), \tag{2.3}$$

where $F(x) = 1 - \exp(-\alpha x)$ is the distribution function of X_1. Conditioning on $X_1 = x$, we have the recursive relation

$$\phi_n(u) = \int_0^{u+c} \phi_{n-1}(u + c - x) f(x)\, dx. \tag{2.4}$$

Combining equation (2.2) with equations (2.3) and (2.4), we obtain the following relations for the ruin probabilities:

$$\psi_1(u) = \Pr(X_1 > u + c) = 1 - F(u + c) \tag{2.5}$$

and

$$\begin{aligned} \psi_n(u) &= 1 - F(u + c) + \int_0^{u+c} \psi_{n-1}(u + c - x) f(x)\, dx \\ &= 1 - F(u + c) + \int_0^{u+c} \psi_{n-1}(x) f(u + c - x)\, dx. \end{aligned} \tag{2.6}$$

Now we attempt to simplify equations (2.5) and (2.6). First, denote

$$\begin{aligned} c_1(u) &= u + c, \\ c_2(u) &= c_1(u) + c = u + 2c, \\ &\vdots \qquad\quad \vdots \\ c_n(u) &= c_{n-1}(u) + c = u + nc. \end{aligned}$$

After some derivation steps, we obtain the following:

1. *For* $n = 1$:

$$\psi_1(u) = e^{-\alpha(u+c)} = e^{-\alpha c_1(u)}$$

2. *For* $n = 2$:

$$\begin{aligned}\psi_2(u) &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \psi_1(x) f(c_1(u) - x)\, dx \\ &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} e^{-\alpha(x+c)} \alpha e^{-\alpha(c_1(u)-x)}\, dx \\ &= \psi_1(u) + \alpha c_1(u) e^{-\alpha c_2(u)}\end{aligned}$$

3. *For* $n = 3$:

$$\begin{aligned}\psi_3(u) &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \psi_2(x) f(c_1(u) - x)\, dx \\ &= \psi_2(u) + \alpha^2 \int_0^{c_1(u)} e^{-\alpha c_3(u)} (x + c)\, dx \\ &= \psi_2(u) + \alpha^2 e^{-\alpha c_3(u)} \frac{c_1(u) c_3(u)}{2}\end{aligned}$$

4. *For* $n = 4$:

$$\begin{aligned}\psi_4(u) &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \psi_3(x) f(c_1(u) - x)\, dx \\ &= \psi_3(u) + \alpha^3 \int_0^{c_1(u)} e^{-\alpha c_4(u)} \frac{(x + c)(x + 3c)}{2}\, dx \\ &= \psi_3(u) + \alpha^3 e^{-\alpha c_4(u)} \frac{c_1(u) [c_4(u)]^2}{3!}\end{aligned}$$

5. *For* $n = 5$:

$$\begin{aligned}\psi_5(u) &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \psi_4(x) f(c_1(u) - x)\, dx \\ &= \psi_4(u) + \alpha^4 \int_0^{c_1(u)} e^{-\alpha c_5(u)} \frac{(x + c)(x + 4c)^2}{3!}\, dx \\ &= \psi_4(u) + \alpha^4 e^{-\alpha c_5(u)} \frac{c_1(u) [c_5(u)]^3}{4!}.\end{aligned}$$

From the above recursive results, we observe a pattern:

$$\begin{aligned}\psi_n(u) &= \psi_{n-1}(u) + \frac{[\alpha c_n(u)]^{n-1}}{(n-1)!} e^{-\alpha c_n(u)} \frac{c_1(u)}{c_n(u)} \\ &= \sum_{k=1}^{n} \frac{[\alpha c_k(u)]^{k-1}}{(k-1)!} e^{-\alpha c_k(u)} \frac{c_1(u)}{c_k(u)},\end{aligned} \tag{2.7}$$

for $n = 1, 2, 3, \ldots$. A proof of the expression in equation (2.7) by induction is given in Appendix A.

Our main result in this section is the expression (2.7). Next, we define $p_n(u)$ as

$$p_n(u) = \psi_n(u) - \psi_{n-1}(u), \qquad n \geq 1,$$

with $\psi_0(u) = 0$. The notation $p_n(u)$ can be interpreted as the probability that the ruin occurs at the instant time n. Notice that

$$p_n(u) = \frac{[\alpha c_n(u)]^{n-1}}{(n - 1)!} e^{-\alpha c_n(u)} \frac{c_1(u)}{c_n(u)}, \tag{2.8}$$

and $\psi_n(u)$ can be rewritten as

$$\psi_n(u) = \sum_{k=1}^{n} p_k(u). \tag{2.9}$$

It should be noted that it might cause overflow problems if we directly compute the $p_n(u)$ probability using equation (2.8). Instead of direct calculation, we can make use of the ratio

$$\frac{p_{n+1}(u)}{p_n(u)} = \frac{\alpha}{n}\left(1 + \frac{c}{c_n(u)}\right)^{n-2} c_{n+1}(u)e^{-\alpha c}, \tag{2.10}$$

from which we can see that as $n \to \infty$,

$$\frac{p_{n+1}(u)}{p_n(u)} \to \alpha c e^{-(\alpha c-1)} = (1 + \theta)e^{-\theta}. \tag{2.11}$$

It is easy to show that the constant on the right-hand side of equation (2.11) is less than 1. Therefore the (defective) probability function $\{p_n(u), n = 1, 2, 3, \ldots\}$ is asymptotically geometric, that is, it decays exponentially.

Finally, we provide numerical illustrations of the main results. We calculate the finite-time ruin probabilities of the discrete-time risk model using equations (2.7), (2.8), and (2.9). The chosen model parameter combinations are $\alpha = 1.0$; $\theta = 0.1$ and 0.25 (i.e., $c = 1.1$ and 1.25, respectively); and $u = 0$ and 10. Infinite-time ruin probabilities (i.e., $n \to \infty$) also are obtained using the equation in Dufresne (2001, p. 784) for comparison purposes.

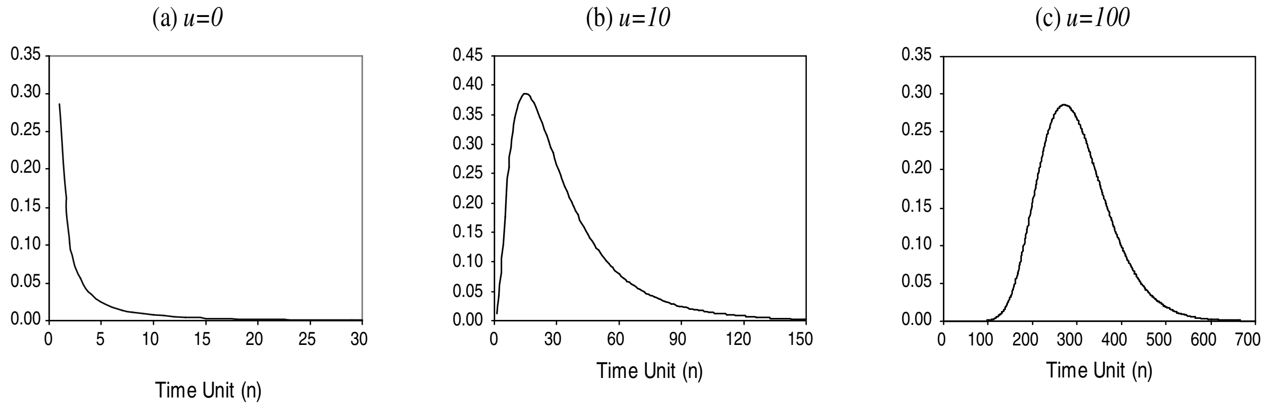
Table 1 shows values of $\psi_n(u)$ to five decimal places for various model parameters. We observe that, as n increases, values of $\psi_n(u)$ converge to the ultimate ruin probabilities as expected. Additionally, for a given value of u, convergence to the ultimate ruin probability is faster for larger value of θ. The expected number of time units given ruin occurs ($\varepsilon[n]$) in each model also is given in Table 1. It is clear that if ruin occurs, it is likely to occur sooner when $\theta = 0.25$ than when $\theta = 0.1$. Similar findings have been reported in Dickson, Hughes, and Zhang (2005).

Table 1

Ruin Probabilities $\psi_n(u)$ in the Discrete-Time Risk Model with Exponential Claims ($\alpha = 1$)

	$u = 0$		$u = 10$	
n	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$
5	0.59644	0.50387	0.00073	0.00044
10	0.67771	0.56432	0.00403	0.00189
20	0.73673	0.60193	0.01650	0.00565
30	0.76231	0.61490	0.03116	0.00875
40	0.77700	0.62084	0.04490	0.01090
50	0.78661	0.62396	0.05696	0.01235
100	0.80791	0.62808	0.09584	0.01487
200	0.81885	0.62861	0.12492	0.01532
300	0.82185	0.62863	0.13446	0.01533
400	0.82296	0.62863	0.13826	0.01533
500	0.82343	0.62863	0.13994	0.01533
1,000	0.82385	0.62863	0.14148	0.01533
∞	0.82386	0.62863	0.14148	0.01533
$\varepsilon[n]$	10.7	4.7	98.5	34.0

Figure 1
Probability Function of the Time to Ruin in the Discrete-Time Risk Model with Exponential Claims (α = 1, θ = 0.25, and Various Values of u)



The probability functions of the time to ruin of the discrete-time risk model ($\alpha = 1$ and $\theta = 0.25$) with various values of initial surplus ($u = 0$, 10, and 100) are plotted in Figure 1. To compare the shapes of the probability functions, vertical axes of the graphs are rescaled linearly. If the initial surplus u is small, the probability function of the time to ruin has a very clear geometric distribution function feature. On the other hand, if the initial surplus u is large, the probability function of the time to ruin appears to be normal, in line with the classical result of Segeradhl (1955).

2.2 Geometric Claims

Following the framework of Rolski et al. (1999, p. 147), we consider the following discrete-time reserve process:

$$R(n) = u + n - \sum_{i=1}^{n} X_i, \tag{2.12}$$

where $\{X_i, i \geq 1\}$ are i.i.d. discrete claim variables taking values in $\mathcal{N} = \{0, 1, 2, \ldots\}$, $u \in \mathcal{N}$, and the common probability function for X is $\{p_k, k = 0, 1, 2, \ldots\}$, where $p_k = \Pr\{X = k\}$. It should be noted that we have taken the premium rate to 1 in model (2.12) for simplicity. Assume that the loading factor θ is positive, that is, $E(X) < 1$. The time of ruin for model (2.12) is defined by $\tau(u) = \min\{n \geq 1 : R(n) < 0\}$, and the ultimate ruin probability is defined by

$$\psi(u) = \Pr\{\tau(u) < \infty | R(0) = u\}. \tag{2.13}$$

In this section we investigate a special case of the model (2.12) in which claim size variables are assumed to follow a geometric distribution with probability function:

$$p_k = \Pr\{X = k\} = pq^k, \qquad k \geq 0,$$

where $p + q = 1$, $0 < p < 1$, and $E(X) = q/p$. It should be noted that the distribution of X_i's has a mass point at zero.

Using similar techniques as in Section 2.1, we obtain the following two equations, which correspond to the two expressions (2.5) and (2.6), respectively:

$$\psi_1(u) = \Pr(X_1 > u + 1) = 1 - P(u + 1) = q^{u+2}, \tag{2.14}$$

$$\psi_n(u) = 1 - P(u + 1) + \sum_{j=0}^{u+1} \psi_{n-1}(u + 1 - j)p_j$$
$$= q^{u+2} + \sum_{j=0}^{u+1} \psi_{n-1}(j)p_{u+1-j}. \quad (2.15)$$

We derive the finite-time ruin function for $n = 1, 2, \ldots, 5$ and study their structures:

$$\psi_1(u) = q^{u+2}$$
$$\psi_2(u) = \psi_1(u) + pq^{u+3}(u + 2)$$
$$\psi_3(u) = \psi_2(u) + p^2q^{u+4} \frac{(u + 2)(u + 5)}{2!}$$
$$\psi_4(u) = \psi_3(u) + p^3q^{u+5} \frac{(u + 2)(u + 6)(u + 7)}{3!}$$
$$\psi_5(u) = \psi_4(u) + p^4q^{u+6} \frac{(u + 2)(u + 7)(u + 8)(u + 9)}{4!}.$$

We observe a pattern that

$$\psi_{n+1}(u) = \psi_n(u) + h_{n+1}(u) \quad (2.16)$$

for $n \geq 2$, where $h_{n+1}(u)$ is a term defined by

$$h_{n+1}(u) = p^n q^{d(u)+n} \frac{d(u)[d(u) + (n + 1)][d(u) + (n + 2)] \cdots [d(u) + (2n - 1)]}{n!}, \quad (2.17)$$

with $d(u) = u + 2$.

The recursive form of the ruin function $\psi_{n+1}(u) = \psi_n(u) + h_{n+1}(u)$ in equation (2.16) is the analogue of the result (2.7) in Section 2.1. A proof of the expression in equation (2.16) by induction is given in Appendix B.

Obviously, with equation (2.17), for $n \geq 2$, equation (2.16) has the equivalent expression

$$\psi_{n+1}(u) = \psi_2(u) + \sum_{k=2}^{n} h_{k+1}(u). \quad (2.18)$$

Notice that for $n \geq 3$, the sequence of $\{h_n(u), n = 3, 4, \ldots\}$ can be obtained recursively from the following ratio, with initial value $h_3(u)$:

$$\frac{h_{n+1}(u)}{h_n(u)} = pq \frac{[d(u) + (2n - 2)][d(u) + (2n - 1)]}{[d(u) + n]n}. \quad (2.19)$$

It can seen that as $n \to \infty$,

$$\frac{h_{n+1}(u)}{h_n(u)} \to 4pq,$$

and $4pq < 1$ for $q \neq p$; hence $\{h_n(u), n = 3, 4, \ldots\}$ decays exponentially.

Analogous to Table 1, we compute the ruin function (2.16) for various parameters u, θ, and different sizes of n. The corresponding ultimate ruin probabilities are calculated for comparison purposes. As in the case of exponential claims, the values of $\psi_n(u)$ for the case of geometric claims converge to the ultimate probability as expected. Hence, we do not report the details here.

3. CONCLUDING REMARKS

The main results in this paper are the two recursive expressions in equations (2.7) and (2.16). Although the analysis is not complex, it surprisingly provides such a straightforward answer to these two special cases.

APPENDIX A

PROOF OF EQUATION (2.7)

Before proving equation (2.7), we need to prove the following lemma.

Lemma A

Suppose that $u \geq 0$ and $c > 0$, then for $k \geq 1$, the following identity holds:

$$\frac{1}{k!}\int_0^{c_1(u)} (x + c)\{x + (k + 1)c\}^{k-1}\, dx = \frac{1}{(k + 1)!}\, c_1(u)\,[c_{k+2}(u)]^k. \tag{A.1}$$

PROOF

It is straightforward to prove the lemma by making the change of variable $y = x + (k + 1)c$ in the left-hand side of equation (A.1). □

Now, we proceed to show expression (2.7) by induction. Assume that

$$\psi_n(u) = \sum_{k=1}^{n} \frac{[\alpha c_k(u)]^{k-1}}{(k - 1)!}\, e^{-\alpha c_k(u)}\, \frac{c_1(u)}{c_k(u)}$$

holds; then we need to show that the formula is also true for the case of $(n + 1)$, that is,

$$\psi_{n+1}(u) = \sum_{k=1}^{n+1} \frac{[\alpha c_k(u)]^{k-1}}{(k -1)!}\, e^{-\alpha c_k(u)}\, \frac{c_1(u)}{c_k(u)}.$$

By equation (2.6) and the induction assumption that the case of n is true, we have

$$\begin{aligned}
\psi_{n+1}(u) &= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \psi_n(x) f(c_1(u) - x)\, dx \\
&= e^{-\alpha c_1(u)} + \int_0^{c_1(u)} \sum_{k=1}^{n} \frac{[\alpha c_k(x)]^{k-1}}{(k - 1)!}\, e^{-\alpha c_k(x)}\, \frac{c_1(x)}{c_k(x)}\, \alpha e^{-\alpha(c_1(u)-x)}\, dx \\
&= e^{-\alpha c_1(u)} + \alpha e^{-\alpha c_1(u)} \left[\int_0^{c_1(u)} \sum_{k=1}^{n} \frac{[\alpha c_k(x)]^{k-1}}{(k - 1)!}\, e^{-\alpha c_k(x)}\, \frac{c_1(x)}{c_k(x)}\, e^{\alpha x}\, dx\right] \\
&= e^{-\alpha c_1(u)} + \alpha e^{-\alpha c_1(u)} \sum_{k=1}^{n} \left[\int_0^{c_1(u)} \frac{[c_k(x)]^{k-2} c_1(x)}{(k - 1)!}\, e^{-\alpha ck} \alpha^{k-1}\, dx\right] \\
&= e^{-\alpha c_1(u)} + \sum_{k=1}^{n} \alpha^k e^{-c_{k+1}(u)}\, \frac{1}{(k - 1)!} \int_0^{c_1(u)} (x + kc)^{k-2}(x + c)\, dx.
\end{aligned}$$

By Lemma A, we have

$$\begin{aligned}\psi_{n+1}(u) &= e^{-\alpha c_1(u)} + \sum_{k=1}^{n} \alpha^k e^{-c_{k+1}(u)} \frac{1}{k!} c_1(u)[c_{k+1}(u)]^{k-1} \\ &= e^{-\alpha c_1(u)} + \sum_{k=1}^{n} \frac{[\alpha c_{k+1}(u)]^k}{k!} e^{-\alpha c_{k+1}(u)} \frac{c_1(u)}{c_{k+1}(u)} \\ &= \sum_{k=1}^{n+1} \frac{[\alpha c_k(u)]^{k-1}}{(k-1)!} e^{-\alpha c_k(u)} \frac{c_1(u)}{c_k(u)}.\end{aligned}$$

This completes the proof. □

APPENDIX B

PROOF OF EQUATION (2.16)

Before proving equation (2.16), we need to prove the following lemma.

Lemma B

The following equation holds for integers $k \geq 2$ and $N \geq 1$:

$$\begin{aligned}&\frac{1}{k!}\sum_{n=1}^{N} (n+1)(n+1+k+1)(n+1+k+2)\cdots(n+1+2k-1) \\ &= \frac{1}{(k+1)!} N(N+k+2)(N+k+3)\cdots(N+2k+1). \qquad \text{(B.1)}\end{aligned}$$

PROOF

It should be noted that equation (B.1) is the analogue of the formula (A.1) in Lemma A, where the integral is replaced by the summation. Obviously, equation (B.1) can be rewritten as

$$\begin{aligned}&(k+1)\sum_{n=1}^{N} (n+1)(n+k+2)(n+k+3)\cdots(n+2k) \\ &= N(N+k+2)(N+k+3)\cdots(N+2k+1),\end{aligned}$$

which can be proved by induction on N as follows:

1. When $N = 1$, the left-hand side is

$$(k+1)(2)(k+3)(k+4)\cdots(2k+1),$$

while the right-hand side is

$$1(k+3)(k+4)\cdots(2k+1)(2k+2).$$

Clearly the case of $N = 1$ is true.

2. Suppose that the case for N is true. Consider the case of $(N + 1)$. We have

$$\begin{aligned}
&(k+1)\sum_{n=1}^{N+1}(n+1)(n+k+2)(n+k+3)\cdots(n+2k)\\
&=(k+1)\sum_{n=1}^{N}(n+1)(n+k+2)(n+k+3)\cdots(n+2k)\\
&\quad+(k+1)(N+2)(N+k+3)(N+k+4)\cdots(N+2k+1)\\
&=N(N+k+2)(N+k+3)\cdots(N+2k+1)\\
&\quad+(k+1)(N+2)(N+k+3)(N+k+4)\cdots(N+2k+1)\\
&=\{N(N+k+2)+(k+1)(N+2)\}\\
&\quad\times(N+k+3)(N+k+4)\cdots(N+2k+1)\\
&=(N+1)(N+2k+2)(N+k+3)(N+k+4)\cdots(N+2k+1)\\
&=(N+1)(N+k+3)(N+k+4)\cdots(N+2k+1)(N+2k+2).
\end{aligned}$$

Therefore the case for $(N + 1)$ is true. This completes the proof of Lemma B. □

Now, we proceed to show the expression (2.16) by induction. Assume that

$$\psi_{n+1}(u) = \psi_n(u) + h_{n+1}(u)$$

holds, we need to show that

$$\psi_{n+2}(u) = \psi_{n+1}(u) + h_{n+2}(u)$$

also holds.

By equation (2.15) and the assumption that the case of $(n + 1)$ is true, we have

$$\begin{aligned}
\psi_{n+2}(u) &= q^{u+2} + \sum_{j=0}^{u+1}\psi_{n+1}(j)p_{u+1-j}\\
&= q^{u+2} + \sum_{j=0}^{u+1}(\psi_n(j) + h_{n+1}(j))p_{u+1-j}\\
&= q^{u+2} + \sum_{j=0}^{u+1}\left(q^{j+2} + pq^{j+3}(j+2) + \sum_{k=2}^{n}h_{k+1}(j)\right)p_{u+1-j}\\
&= q^{u+2} + \sum_{j=0}^{u+1}q^{j+2}p_{u+1-j} + \sum_{j=0}^{u+1}pq^{j+3}(j+2)p_{u+1-j} + \sum_{j=0}^{u+1}\sum_{k=2}^{n}h_{k+1}(j)p_{u+1-j}\\
&= q^{u+2} + \Delta_1 + \Delta_2 + \Delta_3.
\end{aligned}$$

It should be noted that $\psi_1(u) = q^{u+2}$. By the definition of geometric distribution, $p_{u+1-j} = pq^{u+1-j}$, we have

$$\begin{aligned}
\Delta_1 &= \sum_{j=0}^{u+1}q^{j+2}pq^{u+1-j}\\
&= pq^{u+3}\sum_{j=0}^{u+1}1 = pq^{u+3}(u+2).
\end{aligned}$$

Notice that $\psi_2(u) = \psi_1(u) + pq^{u+3}(u + 2) = q^{u+2} + \Delta_1$. Moreover,

$$\begin{aligned}\Delta_2 &= \sum_{j=0}^{u+1} pq^{j+3}(j + 2)pq^{u+1-j}\\ &= p^2q^{u+4} \sum_{j=0}^{u+1} (j + 2)\\ &= p^2q^{u+4} \frac{(u + 2)(u + 5)}{2}\\ &= h_3(u),\end{aligned}$$

and

$$\begin{aligned}\Delta_3 &= \sum_{j=0}^{u+1} \left(\sum_{k=2}^{n} p^kq^{d(j)+k} \frac{d(j)[d(j) + k + 1] \cdots [d(j) + 2k - 1]}{k!}\right) pq^{u+1-j}\\ &= \sum_{k=2}^{n} \left(\sum_{j=0}^{u+1} \frac{d(j)[d(j) + k + 1] \cdots [d(j) + 2k - 1]}{k!}\right) p^{k+1}q^{k+1}q^{u+2}\\ &= \sum_{k=2}^{n} \left(\sum_{j=1}^{u+2} \frac{(j + 1)(j + 1 + k + 1) \cdots (j + 1 + 2k - 1)}{k!}\right) p^{k+1}q^{k+1}q^{u+2}.\end{aligned}$$

By Lemma B, we have

$$\begin{aligned}\Delta_3 &= \sum_{k=2}^{n} \frac{1}{(k + 1)!} (u + 2)(u + 2 + k + 2) \cdots (u + 2 + 2k + 1)p^{k+1}q^{k+1}q^{u+2}\\ &= \sum_{k=2}^{n} \frac{1}{(k + 1)!} d(u)[d(u) + k + 2)] \cdots [d(u) + 2k + 1)]p^{k+1}q^{d(u)+k+1}\\ &= \sum_{k=2}^{n} h_{k+2}(u).\end{aligned}$$

Putting together the results, we get

$$\begin{aligned}\psi_{n+2}(u) &= \psi_2(u) + h_3(u) + \sum_{k=2}^{n} h_{k+2}(u)\\ &= \left(\psi_2(u) + \sum_{k=2}^{n} h_{k+1}(u)\right) + h_{n+2}(u).\end{aligned}$$

Finally, by equation (2.18), we have

$$\psi_{n+2}(u) = \psi_{n+1}(u) + h_{n+2}(u).$$

This completes the proof. □

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