"Optimal Dividends in an Ornstein-Uhlenbeck Type Model with Credit and Debit Interest," by Jun Cai, Hans U. Gerber, Hailang Yang, April 2006

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Professors Cai, Gerber, and Yang have written an interesting paper. Two of the main results in the paper are equations (2.8) and (5.5). In this discussion we will adapt a different approach to recover both of them. Our notation follows that in the paper.

To begin, we introduce a known result for diffusions. Let $\{X_t;\ t \geq 0\}$ be a continuous stochastic process satisfying the stochastic differential equation

$$dX_t = \alpha(X_t)dt + \sigma(X_t)dW_t, \qquad t \geq 0, \tag{D.1}$$

with $\sigma(\cdot) > 0$ and $\{W_t;\ t \geq 0\}$ a standard Wiener process. We further assume that functions $\alpha(\cdot)$ and $\sigma(\cdot)$, defined in $(-\infty, +\infty)$, satisfy that there exists a constant K such that for all $y, z \in (-\infty, +\infty)$,

$$|\alpha(y) - \alpha(z)| + |\sigma(y) - \sigma(z)| \leq K|y - z|$$

and

$$\alpha^2(y) + \sigma^2(y) \leq K^2(1 + y^2).$$

Under the above condition, equation (D.1) has a unique solution for each X_0 (see Gihman and Skorohod 1972, p. 40). Clearly, equation (2.1) in the paper satisfies such a condition.

Let functions $f_1(\cdot)$ and $f_2(\cdot)$ be any two independent solutions of the ordinary differential equation

$$\frac{1}{2}\sigma^2(y)f''(y) + \alpha(y)f'(y) = \delta f(y). \tag{D.2}$$

Denote

$$p(y, z) := f_1(y)f_2(z) - f_1(z)f_2(y),$$

and

$$\omega(y, z) := \partial p(y, z)/\partial z = f_1(y)f_2'(z) - f_1'(z)f_2(y).$$

Note that both functions $f_1(\cdot)$ and $f_2(\cdot)$ depend on δ.

The next lemma could be found in Chapter 16 of Breiman (1968). It gives the Laplace transform of the first exit time from a finite interval for a diffusion process. Write T_a for the time when process $\{X_t\}$ first reaches a.

Lemma 1

Under the above settings, for $a \leq x \leq b$, we have

$$\mathbb{E}[e^{-\delta T_a};\ T_a < T_b | X_0 = x] = \frac{p(x, b)}{p(a, b)},$$

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and

$$\mathbb{E}[e^{-\delta T_b}; T_b < T_a | X_0 = x] = \frac{p(a, x)}{p(a, b)}.$$

We first recover equation (2.8). Given $\varepsilon > 0$, the process $\{X_t\}$ starting at b first exits interval $[0, b + \varepsilon]$ either from below or from above. For the corresponding risk model with barrier strategy we have

$$V(b; b) \approx \varepsilon\mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b] + \mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b] V(b; b), \tag{D.3}$$

where the first and second terms on the right-hand side approximate, respectively, the expected present values of all the dividends until and after $\{X_t\}$ reaches level $b + \varepsilon$ before ruin. By "$\approx$" we mean the difference between its two sides is of the order $o(\varepsilon)$.

Solving (D.3) for $V(b; b)$ and letting $\varepsilon \to 0+$, by Lemma 1 we have

$$\begin{aligned} V(b; b) &= \lim_{\varepsilon \to 0+} \frac{\varepsilon\mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b]}{1 - \mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b]} \\ &= \lim_{\varepsilon \to 0+} \frac{\varepsilon p(0, b)/p(0, b + \varepsilon)}{1 - p(0, b)/p(0, b + \varepsilon)} \\ &= \frac{p(0, b)}{\omega(0, b)}. \end{aligned}$$

It then follows from the strong Markov property and Lemma 1 that for $0 < x < b$,

$$V(x; b) = \mathbb{E}[e^{-\delta T_b}; T_b < T_0 | X_0 = x] V(b; b) = \frac{p(0, x)}{p(0, b)} \frac{p(0, b)}{\omega(0, b)} = \frac{p(0, x)}{\omega(0, b)} = \frac{h(x)}{h'(b)},$$

where $h(y) := p(0, y)$ solves equation (D.2) with boundary condtion $h(0) = 0$. We have thus recovered equation (2.8).

Next, we proceed to prove equation (5.5). Again, we consider the exit time of process $\{X_t\}$ from the interval $[0, b + \varepsilon]$. For the corresponding risk process with dividend strategy, we have by the strong Markov property and Lemma 1,

$$L(b; b) \approx \mathbb{E}[e^{-\delta T_0}; T_0 < T_{b+\varepsilon} | X_0 = b] + \mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b] L(b; b).$$

It follows that

$$\begin{aligned} L(b; b) &= \lim_{\varepsilon \to 0+} \frac{\mathbb{E}[e^{-\delta T_0}; T_0 < T_{b+\varepsilon} | X_0 = b]}{1 - \mathbb{E}[e^{-\delta T_{b+\varepsilon}}; T_{b+\varepsilon} < T_0 | X_0 = b]} \\ &= \lim_{\varepsilon \to 0+} \frac{p(b, b + \varepsilon)/p(0, b + \varepsilon)}{1 - p(0, b)/p(0, b + \varepsilon)} \\ &= \lim_{\varepsilon \to 0+} \frac{p(b, b + \varepsilon)/\varepsilon}{(p(0, b + \varepsilon) - p(0, b))/\varepsilon} \\ &= \frac{\omega(b, b)}{\omega(0, b)}, \end{aligned}$$

where for the last identity we need $p(b, b) = 0$. Furthermore,

$$
\begin{aligned}
L(x; b) &= \mathbb{E}[e^{-\delta T_0}; T_0 < T_b | X_0 = x] + \mathbb{E}[e^{-\delta T_b}; T_b < T_0 | X_0 = x] L(b; b) \\
&= \frac{p(x, b)}{p(0, b)} + \frac{p(0, x)}{p(0, b)} \cdot \frac{\omega(b, b)}{\omega(0, b)} \\
&= \frac{\omega(x, b)}{\omega(0, b)},
\end{aligned}
$$

where to get the last identity we need to verify $p(x, b)\omega(0, b) + p(0, x)\omega(b, b) = p(0, b)\omega(x, b)$ via a little algebra.

It is clear that $g(\cdot) := \omega(\cdot, b)$ also satisfies (D.2) with boundary condition $g'(b) = 0$. Therefore, equation (5.5) follows.

All of our arguments can be made more rigorous. See a similar discussion on Lévy risk models in Zhou (2006).

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