VALUATION OF EQUITY-LINKED INSURANCE AND ANNUITY PRODUCTS WITH BINOMIAL MODELS

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ABSTRACT

In this paper we develop a valuation method for equity-linked insurance products. We assume that the premium information of term life insurances, pure endowment insurances, and endowment insurances at all maturities is obtainable within a company or from the insurance market. Using a method similar to that of Jarrow and Turnbull (1995), we derive three martingale probability measures associated with these basic insurance products. These measures are age-dependent, include an adjustment for the mortality risk, and reproduce the premiums of the respective insurance products. We then extend the martingale measures to include the financial market information using copulas and use them to evaluate equity-linked insurance contracts and equity-indexed annuities in particular. This is different from the traditional approach under which diversification of mortality risk is assumed. A detailed numerical analysis is performed for various existing equity-indexed annuities in the North American market.

1. INTRODUCTION

An equity-linked insurance or annuity is an insurance product with benefits linked to the performance of an equity market. Among others, equity-linked products include equity-indexed annuities (EIAs), variable annuities (VAs), universal life insurances (ULs), and variable universal life insurances (VULs). An equity-indexed annuity provides a limited participation in the performance of an equity index (e.g., S&P 500) while guaranteeing a minimum rate of return. Introduced by Keyport Life Insurance Co. in 1995, EIAs have been the most innovative annuity product over the last 10 years. They have become increasingly popular since their debut, and the sales of EIAs have broken the $10 billion barrier ($11.8 billion) in 2002 and reached $14.4 billion in 2003, $23.1 billion in 2004, and $27.3 billion in 2005. See the 2006 *Annuity Fact Book* (Table 7-8) from the National Association for Variable Annuities (NAVA). A variable annuity is similar to an equity-indexed annuity except that its premiums are invested in the policyholder choice of underlying stock and/or bond funds (mutual funds). It is wrapped with death and accumulation guarantees such as the guaranteed minimum death benefit (GMDB), the guaranteed minimum accumulation benefit (GMAB), the guaranteed minimum income benefit (GMIB), and/or the guaranteed minimum withdrawal benefit (GMWB). According to the 2006 *Annuity Fact Book,* the total variable annuity sales in 2005 had reached $133.4 billion, and the total assets were $1,202.8 billion (Charts 6-1 and 6-2). A universal life insurance is a life insurance that provides death benefits whose level is linked to the investment return of its issuer's general account with a minimum interest guarantee, while a variable universal life allows for its holder to invest its premiums into mutual funds similar to a variable annuity. See the monograph by Hardy (2003) for comprehensive discussions on these products.

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The traditional insurance and annuity pricing method calculates the net premium of a product as the expected present value of its benefits with respect to a mortality law. The (gross) premium, net of commissions and other nonmortality-related charges, is determined as the net premium plus a loading that is based on certain premium principle (see Bowers et al. 1997). Unfortunately, this traditional actuarial pricing approach often produces premiums inconsistent with the insurance market. Furthermore, it is difficult to extend to the valuation of equity-linked products since these products are embedded with various types of financial guarantees. Many attempts have been made to evaluate equity-linked products using option-pricing techniques. For instance, Møller (1998, 2001) employs the risk-minimization method developed in Schweizer (1995) to evaluate equity-linked life insurances. Bacinello (2003a,b) investigates a guaranteed life insurance participating contract with a surrender option. Under the Black-Scholes framework, Tiong (2000) and Lee (2002, 2003) use the Esscher transform method developed in Gerber and Shiu (1994) to obtain closed-form formulas for several equity-indexed annuities without surrender options. Lin and Tan (2003) consider a more general model for equity-indexed annuities, in which the external equity index and the interest rate are general stochastic differential equations. More recently, Coleman, Li, and Patron (2006) study certain hedging problems for variable annuities. It is generally assumed that insurance companies can diversify the mortality risk in these studies.

In this paper we propose a market consistent valuation method for equity-linked insurance and annuity products. Similar to Jarrow and Turnbull (1995), we derive martingale probability measures associated with basic insurance products and specifically the term life insurances, pure endowment insurances, and endowment insurances. We assume that the premium[1] information of term life insurances, pure endowment insurances, and endowment insurances at all maturities is obtainable. Three martingale measures, each of which serves a different purpose, are derived using the premium information from each of the three categories. Consequently, these measures are age-dependent, include an adjustment for the mortality risk, and reproduce the premiums of the respective insurance products. We extend the martingale measures to include the financial market information and use these measures to evaluate equity-linked insurance contracts and equity-indexed annuities in particular. The values of these contracts are the expected discounted payoffs of the contracts and hence linear with respect to the payoffs. Similar to Bacinello (2003a,b), we also evaluate equity-linked contracts that include a surrender option. It results in recursive algorithms that are similar to the valuation of American options (Hull 2000). We implement the approaches by considering several types of equity-indexed annuities. The assumption that the premium information of term life insurances, pure endowment insurances, and endowment insurances is available is what differs from many studies of the valuation of equity-linked products. We argue that this assumption is reasonable because an insurance company that issues equity-linked products often issues term life insurances, endowment insurances, and life annuities. The premium information and fee structures on these products are available within the company, and hence it may be possible, although it might not be easy, for a valuation actuary in the company to extract the aforementioned premium information by stripping the costs of any add-on conversion options and riders and by comparing the premiums of insurances or annuities with different maturities. Furthermore, because of the relative efficiency of the current insurance market and the standardization of these basic products, their premiums are often dictated by the market and hence vary insignificantly from company to company. As a result, the proposed method in this paper will provide market-consistent values for equity-linked products and will reduce subjectivity when determining loadings. In general, any market-consistent valuation method is less subjective, as discussed in Sheldon and Smith (2004).

This paper is organized as follows. The next section presents binomial models for the equity index and for the term life insurances, pure endowment insurances, and endowment insurances. We then

[1] Hereafter, a premium is referred to a single premium that includes mortality charge but excludes commissions and other nonmortality-related charges.

derive martingale measures for those standard products in Section 3 and equity-linked products in Section 5. Each of those sections is followed by numerical examples (Sections 4 and 6) presenting their corresponding martingale probabilities. Section 7 focuses on recursive pricing formulas for equity-linked contracts with surrender options. Finally, we examine the implications of the proposed approaches on the EIAs by conducting a detailed numerical analysis in Section 8.

2. UNDERLYING BINOMIAL MODELS

Binomial models have widely been used to model stocks, stock indices, interest rates, and other financial securities because of their flexibility and tractability (see, e.g., Panjer et al. 1998). In this section we employ a modified CRR binomial model (Cox, Ross, and Rubinstein 1979) for a stock index. We then introduce three binomial insurance models: one for term life insurances, one for pure endowment insurances, and the third for endowment insurances.

Let $r(k)$, $k = 0, 1, \ldots$, be annual interest rates, that is, $r(k)$ is the interest rate for the time period $[k, k + 1]$. It is assumed that they are deterministic. For each year, assume that there are N trading periods, each with the length of $\Delta = 1/N$. The (stock) index process is denoted as $S(t)$, $t = 0, \Delta, 2\Delta, \ldots$, where $S(0)$ is the initial level of the index. For the time period $[t - \Delta, t]$, $t = \Delta, 2\Delta, \ldots$, the index has two possible outcomes, $S(t - \Delta)d(t)$ and $S(t - \Delta)u(t)$, with $d(t) < u(t)$. Hence, the index process can move up from $S(t - \Delta)$ to $S(t - \Delta)u(t)$, or down to $S(t - \Delta)d(t)$. Because of the deterministic assumption of the interest rates, the time-t value $B(t)$, $B(0) = 1$, of the money market account is given by

$$B(t) = \prod_{i=\{0,\Delta,\ldots,t-\Delta\}} [1 + r(\lfloor i \rfloor)]^{\Delta}, \qquad (2.1)$$

for $t = 0, \Delta, 2\Delta, \ldots$. Here $\lfloor . \rfloor$ is the floor function.

For the index process $S(t)$, since the interest rates remain constant each year, it is natural to assume that the values of $d(t)$ and $u(t)$ are constant within each year, that is, we may write $d(t) = d(k)$ and $u(t) = u(k)$ for $t = k - 1 + \Delta, k - 1 + 2\Delta, \ldots, k$, and $k = 1, 2, \ldots$. Moreover, to maintain a recombining binomial model, it is required that

$$u(k)d(k + 1) = d(k)u(k + 1), \qquad (2.2)$$

for $k = 1, 2, \ldots$. The values of $d(k)$ and $u(k)$ will be obtained using the volatility structure of the index process; they will be specified later.

The martingale probability measure Q can be obtained easily. Let $\pi(t)$ be the probability that the index value goes up during the period $[t - \Delta, t]$:

$$Q[S(t) = S(t - \Delta)u(t) | S(t - \Delta)] = \pi(t), \qquad (2.3)$$

and

$$Q[S(t) = S(t - \Delta)d(t) | S(t - \Delta)] = 1 - \pi(t), \qquad (2.4)$$

for $t = \Delta, 2\Delta, \ldots$. That the discounted value process $\{S(t)/B(t)\}$ is a martingale implies that

$$\pi(t) = \frac{[1 + r(k - 1)]^{\Delta} - d(k)}{u(k) - d(k)}, \qquad (2.5)$$

for $t = k - 1 + \Delta, k - 1 + 2\Delta, \ldots, k$, and $k = 1, 2, \ldots$. It follows from (2.5) that $\pi(t)$ is constant over each year, and it depends only on the index's returns $u(k)$ and $d(k)$ and the annual rate $r(k)$. For this reason, we write $\pi(t) = \pi(k)$ for all $t = k - 1 + \Delta, k - 1 + 2\Delta, \ldots, k$. The no-arbitrage condition thus requires

$$d(k) < [1 + r(k - 1)]^{\Delta} < u(k).$$

Moreover, under this model, the ratio $S(k)/S(k - 1)$ takes $N + 1$ possible values

$$\gamma(k, i) = u(k)^i d(k)^{N-i}, \qquad i = 0, 1, \ldots, N, \tag{2.6}$$

with corresponding martingale probabilities

$$Q\left[\frac{S(k)}{S(k-1)} = \gamma(k, i)\right] = \binom{N}{i} \pi(k)^i [1 - \pi(k)]^{N-i}, \qquad i = 0, 1, \ldots, N. \tag{2.7}$$

Some remarks are now made. The model assumes the usual frictionless market: no tax, no transaction costs, etc. The filtration associated with the index process is the one generated by the process. Although we assume deterministic interest rates, it can be extended to stochastic interest rates, which is considered in Gaillardetz (2006). Furthermore, for practical implementation purposes, one may also use current forward rates for $r(k)$.

We next introduce binomial models for three standard insurance products. We will use the standard actuarial notation that can be found in Bowers et al. (1997). Let $T(x)$ be the future lifetime of insured (x) at time $t = 0$ and the curtate-future-lifetime

$$K(x) = \lfloor T(x) \rfloor, \tag{2.8}$$

the number of future whole years completed by the insured (x) prior to death.

For integers t and n, $0 \le t \le n$, let $V^{(1)}(x, t, n)$ denote the premium at time t for an n-year term life insurance that pays one monetary unit at $K(x) + 1$, given that (x) is alive at time t. Similarly, let $V^{(2)}(x, t, n)$ denote the premium at time t for an n-year pure endowment insurance that pays one monetary unit at the end of the n years, contingent on the survival of (x) at time n, and $V^{(3)}(x, t, n)$ the premium at time t for an n-year endowment insurance that pays one monetary unit at time $K(x) + 1$ if (x) dies before time n or at time n if (x) survives at time n. We note that the values $V^{(j)}(x, 0, n)$, $j = 1, 2, 3$, differ from the net premiums given in Bowers et al. (1997) and can be viewed as mortality-risk-adjusted single premiums before commissions and other nonmortality-related charges. These values may hence be obtained exogenously. The values $V^{(j)}(x, t, n)$, $t > 0$, $j = 1, 2, 3$, will be determined in a "no-arbitrage" way. The value process $W^{(1)}(x, t, n)$ of the n-year term life insurance is defined by

$$W^{(1)}(x, t, n) = \begin{cases} \dfrac{B(t)}{B(K(x) + 1)}, & K(x) < t, \\ V^{(1)}(x, t, n), & K(x) \ge t. \end{cases} \tag{2.9}$$

Thus, $W^{(1)}(x, t, n)$ is a binomial process that represents the intrinsic value of the insurance. Similarly, define $W^{(2)}(x, t, n)$ to be the value process of the n-year pure endowment insurance, and it is given by

$$W^{(2)}(x, t, n) = \begin{cases} 0, & K(x) < t, \\ V^{(2)}(x, t, n), & K(x) \ge t, \end{cases} \tag{2.10}$$

and $W^{(3)}(x, t, n)$ to be the value process generated by the n-year endowment insurance

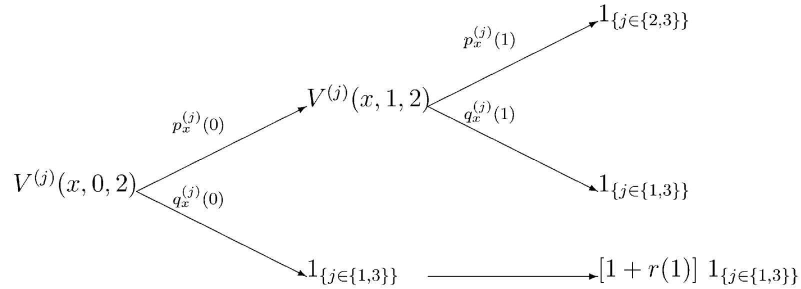
$$W^{(3)}(x, t, n) = \begin{cases} \dfrac{B(t)}{B(K(x) + 1)}, & K(x) < t, \\ V^{(3)}(x, t, n), & K(x) \ge t. \end{cases} \tag{2.11}$$

We note that the forms of (2.9) and (2.11) are identical except their terminal values, where $V^{(1)}(x, n, n) = 0$ and $V^{(3)}(x, n, n) = 1$ for all n. Figure 1 shows the dynamic of $W^{(j)}(x, t, 2)$ for $j = 1, 2, 3$, where $1_{\{A\}}$ is the indicator function with value 1 if A is satisfied and 0 otherwise.

3. MARTINGALE MEASURES FOR TERM INSURANCES AND ANNUITIES

In this section we employ a method similar to the approach of Jarrow and Turnbull (1995) to derive a martingale probability measure for each of the value processes introduced in the last section. Since

Figure 1
Probability Tree of Two-Year Insurance Payoffs $W^{(j)}(x, t, 2)$



we utilize exogenous premium information from the insurance and annuity markets, these martingale measures are age-dependent and include an adjustment to mortality risk, similar to the Commissioners' Extended Term (CET) Tables and the Annuity Tables in which a loading is added to or subtracted from each of the probabilities of death. However, the calculation of the mortality loadings in these tables often lack theoretical justifications. On the other hand, in finance the introduction of loadings or risk premiums of a tradable security via a risk-neutral measure or martingale measure is well justified using the no-arbitrage argument. We will apply a similar approach in this paper. Since insurance products are not tradable, the no-arbitrage argument is not totally valid. Nevertheless, we borrow the idea that the premium of a contingent payment is the expected present value of the payment under a martingale measure. Although we can no longer use the no-arbitrage argument, this approach provides us a consistent pricing framework. That is, if two products are identical, their premiums are the same, and if for two products, one always has a better payoff than the other, the premium of the former is higher than that of the latter.

In the following, we derive martingale measures for three standard insurance products: the term insurance, the pure endowment insurance, and the endowment insurance. In other words, the martingale measures $Q_x^{(j)}$, $j = 1, 2, 3$, are such that $\{W^{(j)}(x, t, n)/B(t)\}$, $j = 1, 2, 3$, are martingales. As we mentioned earlier, we assume that the time-0 premiums of the term insurances, pure endowment insurances, and endowment insurances are given exogenously. These measures are calibrated to reproduce these premiums using an approach similar to that of Jarrow and Turnbull (1995). Let $q_x^{(j)}(t)$ be the probability under $Q_x^{(j)}$ that (x) dies before $x + t + 1$ given that the insured is alive at age $x + t$:

$$q_x^{(j)}(t) = Q_x^{(j)}[K(x) = t | K(x) \geq t], \tag{3.1}$$

for $t = 0, 1, \ldots$. Define the probability (under $Q_x^{(j)}$) that (x) survives to $x + t + 1$ given that the insured is alive at age $x + t$ by

$$p_x^{(j)}(t) = Q_x^{(j)}[K(x) > t | K(x) \geq t], \tag{3.2}$$

for $t = 0, 1, \ldots$. The relation between the two probabilities may be generalized as

$$p_x^{(j)}(t) = 1 - q_x^{(j)}(t), \tag{3.3}$$

and they are presented above each branch in Figure 1.

The main objective of this section is to extract $q_x^{(j)}$'s that will be used to evaluate equity-linked products in later sections.

3.1 Term Life Insurances

Proposition 3.1

For given $V^{(1)}(x, 0, n)$, $n = 1, 2, \ldots$, the age-dependent, mortality risk-adjusted martingale probabilities are

$$q_x^{(1)}(t) = \frac{[V^{(1)}(x, 0, t + 1) - V^{(1)}(x, 0, t)] \prod_{i=0}^{t} [1 + r(i)]}{\prod_{i=0}^{t-1} p_x^{(1)}(i)}, \tag{3.4}$$

with the premium at time t defined by

$$V^{(1)}(x, t, n) = \sum_{k=0}^{n-t-1} \left[\prod_{i=0}^{k-1} \frac{p_x^{(1)}(t + i)}{1 + r(t + i)}\right] \frac{q_x^{(1)}(t + k)}{1 + r(t + k)}. \tag{3.5}$$

PROOF

To prove Proposition 3.1, $V^{(1)}(x, 0, n)$ may be defined in terms of $V^{(1)}(x, 0, n + 1)$. First, a recursive formula for $V^{(1)}(x, t, n)$ is identified using the stochastic process $\{W^{(1)}(x, t, n)/B(t)\}$. By definition of $W^{(1)}(x, t, n)$ and the martingale property

$$\frac{V^{(1)}(x, t, n)}{B(t)} = E_x^{(1)} \left[\frac{W^{(1)}(x, t + 1, n)}{B(t + 1)} \middle| F^*(t)\right], \tag{3.6}$$

where $F^*(t)$ contains the information about the insured's life up to time t and $E_x^{(1)}[\cdot]$ represents the expectation with respect to $Q_x^{(1)}$. Considering that the insured died between 0 and t does not provide any information on the age-dependent, mortality-risk-adjusted martingale probabilities; however, the martingale property is preserved. Consequently, given that $K(x) \geq t$ and from (3.6),

$$V^{(1)}(x, t, n) = \frac{1}{1 + r(t)} \{q_x^{(1)}(t) + V^{(1)}(x, t + 1, n)p_x^{(1)}(t)\}. \tag{3.7}$$

$V^{(1)}(x, t, n)$ may be found only in terms of $V^{(1)}(x, t + 2, n)$ and the martingale probabilities. Using (3.7) for $V^{(1)}(x, t + 1, n)$ in (3.7) leads to

$$V^{(1)}(x, t, n) = \frac{q_x^{(1)}(t)}{1 + r(t)} + \left[\prod_{i=0}^{1} (1 + r(t + i))^{-1}\right]\left\{p_x^{(1)}(t)q_x^{(1)}(t + 1) + \left[\prod_{i=0}^{1} p_x^{(1)}(t + i)\right] V^{(1)}(x, t + 2, n)\right\}. \tag{3.8}$$

Applying this procedure from $V^{(1)}(x, t + 2, n)$ through $V^{(1)}(x, n - 1, n)$ leads to (3.5) since $V^{(1)}(x, n, n) = 0$. Setting $t = 0$ in (3.5),

$$V^{(1)}(x, 0, n) = V^{(1)}(x, 0, n - 1) + \left[\prod_{i=0}^{n-2} \frac{p_x^{(1)}(i)}{1 + r(i)}\right] \frac{q_x^{(1)}(n - 1)}{1 + r(n - 1)}, \tag{3.9}$$

which completes the proof. □

With the martingale structure identified, the n-year term life insurance premiums may be reproduced as the expected discounted payoff of the insurance

$$V^{(1)}(x, 0, n) = E_x^{(1)} \left[\frac{1_{\{K(x)<n\}}}{B(K(x) + 1)}\right]. \tag{3.10}$$

3.2 Pure Endowment Insurances

Proposition 3.2

For given $V^{(2)}(x, 0, n)$, $n = 1, 2, \ldots$, the age-dependent, mortality-risk-adjusted martingale probabilities are

$$p_x^{(2)}(t) = \frac{V^{(2)}(x, 0, t + 1)}{V^{(2)}(x, 0, t)} [1 + r(t)], \tag{3.11}$$

with the premium at time t defined by

$$V^{(2)}(x, t, n) = \prod_{i=0}^{n-t-1} \frac{p_x^{(2)}(t+i)}{1 + r(t+i)}. \tag{3.12}$$

PROOF

To prove Proposition 3.2, $V^{(2)}(x, 0, n)$ may be defined in terms of $V^{(2)}(x, 0, n + 1)$. First, a recursive formula for $V^{(2)}(x, t, n)$ is identified using the stochastic process $\{W^{(2)}(x, t, n)/B(t)\}$. As mentioned above, the fact that the policyholder died before time t does not provide any information about the martingale probabilities. For $K(x) \geq t$ and by definition of $W^{(2)}(x, t, n)$, we have

$$V^{(2)}(x, t, n) = \frac{1}{1 + r(t)} p_x^{(2)}(t) V^{(2)}(x, t+1, n). \tag{3.13}$$

$V^{(2)}(x, t, n)$ may be found only in terms of $V^{(2)}(x, t + 2, n)$ and the martingale probabilities. Using (3.13) for $V^{(2)}(x, t + 1, n)$ in (3.13) gives

$$V^{(2)}(x, t, n) = \prod_{i=0}^{1} \frac{p_x^{(2)}(t+i)}{1 + r(t+i)} V^{(2)}(x, t+2, n). \tag{3.14}$$

Applying this procedure from $V^{(2)}(x, t + 2, n)$ through $V^{(2)}(x, n - 1, n)$ leads to (3.12) since $V^{(2)}(x, n, n) = 1$. Replacing $V^{(2)}(x, 0, n + 1)$ and $V^{(2)}(x, 0, n)$ by (3.12) yields

$$\frac{V^{(2)}(x, 0, n+1)}{V^{(2)}(x, 0, n)} = \frac{p_x^{(2)}(n)}{1 + r(n)}, \tag{3.15}$$

which completes the proof. □

With the martingale structure identified, the n-year pure endowment insurance premiums may be reproduced as the expected discounted payoff of the insurance

$$V^{(2)}(x, 0, n) = E_x^{(2)}\left[\frac{1_{\{K(x) \geq n\}}}{B(n)}\right]. \tag{3.16}$$

3.3 Endowment Insurances

Proposition 3.3

For given $V^{(3)}(x, 0, n)$, $n = 1, 2, \ldots$, the age-dependent, mortality-risk-adjusted martingale probabilities are

$$p_x^{(3)}(t) = \frac{V^{(3)}(x, 0, t+1) - V^{(3)}(x, 0, t+2)}{\{\prod_{i=0}^{t} [1 + r(i)]^{-1} - \prod_{i=0}^{t+1} [1 + r(i)]^{-1}\} \prod_{i=0}^{t-1} p_x^{(3)}(i)}, \tag{3.17}$$

with the premium at time t defined by

$$V^{(3)}(x, t, n) = \sum_{k=0}^{n-t-2} \left[\prod_{i=0}^{k-1} \frac{p_x^{(3)}(t+i)}{1 + r(t+i)}\right] \frac{q_x^{(3)}(t+k)}{1 + r(t+k)} + \frac{1}{1 + r(n-1)} \prod_{i=0}^{n-t-2} \frac{p_x^{(3)}(t+i)}{1 + r(t+i)}. \tag{3.18}$$

PROOF

To prove Proposition 3.3, $V^{(3)}(x, 0, n)$ may be defined in terms of $V^{(3)}(x, 0, n + 1)$. First, a recursive formula for $V^{(3)}(x, t, n)$ is identified using the discounted value process $\{W^{(3)}(x, t, n)/B(t)\}$. For $K(x) \geq t$ and by definition of $W^{(3)}(x, t, n)$ it follows that

$$V^{(3)}(x, t, n) = \frac{1}{1 + r(t)} \{q_x^{(3)}(t) + p_x^{(3)}(t) V^{(3)}(x, t+1, n)\}. \tag{3.19}$$

$V^{(3)}(x, t, n)$ may be found only in terms of $V^{(3)}(x, t + 2, n)$ and the martingale probabilities. Using (3.19) for $V^{(3)}(x, t + 1, n)$ in (3.19), we obtain

$$V^{(3)}(x, t, n) = \frac{1}{1 + r(t)} \left\{ q_x^{(3)}(t) + \frac{p_x^{(3)}(t)}{1 + r(t + 1)} [q_x^{(3)}(t + 1) + p_x^{(3)}(t + 1)V^{(3)}(x, t + 2, n)] \right\}. \quad (3.20)$$

Applying this procedure from $V^{(3)}(x, t + 2, n)$ through $V^{(3)}(x, n - 1, n)$ results in (3.18) since $V^{(3)}(x, n, n) = 1$. Setting $t = 0$ in (3.18) leads to

$$V^{(3)}(x, 0, n + 1) = V^{(3)}(x, 0, n) + \left[\prod_{i=0}^{n} (1 + r(i))^{-1}\right]\left[\prod_{i=0}^{n-1} p_x^{(3)}(i)\right] q_x^{(3)}(n) + \prod_{i=0}^{n} \frac{p_x^{(3)}(i)}{1 + r(i)} - \prod_{i=0}^{n-1} \frac{p_x^{(3)}(i)}{1 + r(i)}. \quad (3.21)$$

It follows using (3.3) that

$$V^{(3)}(x, 0, n + 1) = V^{(3)}(x, 0, n) + \left[\prod_{i=0}^{n} (1 + r(i))^{-1}\right]\left[\prod_{i=0}^{n-1} p_x^{(3)}(i)\right] - \prod_{i=0}^{n-1} \frac{p_x^{(3)}(i)}{1 + r(i)}, \quad (3.22)$$

which completes the proof. □

With the martingale structure identified, the n-year endowment insurance premiums may be reproduced as the expected discounted payoff of the insurance

$$V^{(3)}(x, 0, n) = E_x^{(3)} \left[\frac{1_{\{K(x)<n-1\}}}{B(K(x) + 1)} + \frac{1_{\{K(x)\geq n-1\}}}{B(n)} \right]. \quad (3.23)$$

As seen, if the time-0 premiums are given exogenously, then it would be possible to extract the stochastic structure of each insurance products using Propositions 3.1, 3.2, and 3.3. However, there exist constraints on those insurance premiums since the martingale probabilities should be strictly positive. It follows that

$$V^{(1)}(x, 0, t + 1) > V^{(1)}(x, 0, t),$$

$$V^{(2)}(x, 0, t + 1) < \frac{V^{(2)}(x, 0, t)}{1 + r(t)},$$

and

$$V^{(3)}(x, 0, t + 1) > V^{(3)}(x, 0, t) + \left[\prod_{i=0}^{t-2} p_x^{(3)}(i)\right]\left\{\prod_{i=0}^{t} \frac{1}{1 + r(i)} - \prod_{i=0}^{t-1} \frac{1}{1 + r(i)}\right\},$$

since $q_x^{(j)}(t) > 0$ for $j = 1, 2, 3$, and $t = 0, 1, \ldots$. Since $p_x^{(j)}(t) > 0$ for $j = 1, 2, 3$, we obtain

$$V^{(1)}(x, 0, t + 1) < \left[\prod_{i=0}^{t} \frac{1}{1 + r(i)}\right]\left\{1 - \sum_{k=1}^{t} V^{(1)}(x, 0, k)\left(\prod_{i=0}^{k-1} [1 + r(i)] - \prod_{i=0}^{k} [1 + r(i)]\right)\right\},$$

$$V^{(2)}(x, 0, t + 1) > 0,$$

and

$$V^{(3)}(x, 0, t + 2) < V^{(3)}(x, 0, t + 1),$$

for $t = 0, 1, \ldots$.

Theoretically, there exists a natural hedging between the insurance and annuity products. However, it may not occur in practice because of certain regulatory and accounting constraints and issues such as moral harzard and antiselection. Hence, it is reasonable to evaluate insurances and annuities separately, as shown in this section.

4. LOADED MARTINGALE PROBABILITIES: NUMERICAL EXAMPLES

In this section we compare numerically the loadings obtained from a standard insured mortality table (1980 U.S. CET), annuity table (1983 U.S. GAM) with the martingale probabilities. All mortality tables

Table 1

Loadings Obtained from Standard Deviation Principle 5%

t	CSO	CET		GAM		Martingale Probabilities					
	q_{55+t}	q_{55+t}	L	q_{55+t}	L	$q_{55}^{(1)}(t)$	L	$q_{55}^{(2)}(t)$	L	$q_{55}^{(3)}(t)$	L
0	10.47	13.61	3.14	6.81	−3.66	15.56	5.09	5.38	−5.09	15.56	5.09
1	11.46	14.90	3.44	7.35	−4.11	13.69	2.23	9.18	−2.28	12.75	1.29
2	12.49	16.24	3.75	7.93	−4.56	14.29	1.80	10.61	−1.88	14.03	1.54
3	13.59	17.67	4.08	8.58	−5.01	15.18	1.59	11.89	−1.70	14.98	1.39
4	14.77	19.20	4.43	9.32	−5.46	16.24	1.47	13.17	−1.60	16.06	1.29
5	16.08	20.90	4.82	10.18	−5.91	17.47	1.39	14.52	−1.56	17.31	1.23
6	17.54	22.80	5.26	11.18	−6.36	18.89	1.35	16.00	−1.54	18.73	1.19
7	19.19	24.95	5.76	12.37	−6.82	20.51	1.32	17.64	−1.55	20.36	1.17
8	21.06	27.38	6.32	13.77	−7.29	22.37	1.31	19.48	−1.58	22.23	1.17
9	23.14	30.08	6.94	15.41	−7.73	24.45	1.31	21.52	−1.62	—	—

used in this section may be downloaded from www.soa.org. We conduct the numerical examples on a male aged 55 at time 0. The values, $V^{(j)}(55, 0, n)$ (j = 1, 2, 3, and n = 1, . . . , 10), are determined using the standard deviation premium principle and with either a constant factor (5% and 10%) or an increasing factor (1% and 2%) (see Bowers et al. 1997), where the mortality is assumed to follow the 1980 U.S. CSO table, which is used as the reference table. We also assume that the short-term rate $r(t)$ is constant over time and equal to 5% for simplicity. Note that the interest rate level has no impact on the loadings using those premium principles. The loadings (L) in the tables represent the difference between the relevant "q_x" and the one from the CSO table. All numbers presented in the different tables are multiplied by 1,000.

Tables 1 and 2 present the mortality probabilities given in the CSO, CET, and GAM tables. The martingale probabilities presented in those tables are obtained from equations (3.4), (3.11), and (3.17). More precisely, the insurance products are determined using the standard deviation principle with constant loading factors of 5% in Table 1 and 10% in Table 2.

The CET loadings to the probabilities of death are approximately 30% of the corresponding values in the CSO table. This means that the loading increases with age. For example, with the above assumptions, the value of five-year term life insurance is 0.0679 based on the CET table, which leads to a loaded premium of 7.4% under the standard deviation principle. The GAM loadings are negative and approximately 35%. For example, the value of five-year pure endowment insurance is 0.753 using the GAM table, which leads to a 9.1% loaded premium. The loadings at time t = 0 are heftier than the subsequent loadings for all martingale probabilities. They also slowly converge to zero as time passes and follow a similar pattern regardless the premium factor. As expected, they are increasing in the

Table 2

Loadings Obtained from Standard Deviation Principle 10%

t	CSO	CET		GAM		Martingale Probabilities					
	q_{55+t}	q_{55+t}	L	q_{55+t}	L	$q_{55}^{(1)}(t)$	L	$q_{55}^{(2)}(t)$	L	$q_{55}^{(3)}(t)$	L
0	10.47	13.61	3.14	6.81	−3.66	20.65	10.18	0.29	−10.18	20.65	10.18
1	11.46	14.90	3.44	7.35	−4.11	15.94	4.48	6.91	−4.55	14.06	2.60
2	12.49	16.24	3.75	7.93	−4.56	16.11	3.62	8.76	−3.73	15.59	3.10
3	13.59	17.67	4.08	8.58	−5.01	16.80	3.21	10.22	−3.37	16.38	2.79
4	14.77	19.20	4.43	9.32	−5.46	17.74	2.97	11.60	−3.17	17.38	2.61
5	16.08	20.90	4.82	10.18	−5.91	18.90	2.82	13.00	−3.08	18.57	2.49
6	17.54	22.80	5.26	11.18	−6.36	20.27	2.73	14.50	−3.04	19.96	2.42
7	19.19	24.95	5.76	12.37	−6.82	21.87	2.68	16.13	−3.06	21.57	2.38
8	21.06	27.38	6.32	13.77	−7.29	23.72	2.66	17.95	−3.11	23.43	2.37
9	23.14	30.08	6.94	15.41	−7.73	25.81	2.67	19.95	−3.19	—	—

Table 3

Loadings Obtained from Standard Deviation Principle Increasing by 1%

	CSO	CET		GAM		Martingale Probabilities					
t	q_{55+t}	q_{55+t}	L	q_{55+t}	L	$q_{55}^{(1)}(t)$	L	$q_{55}^{(2)}(t)$	L	$q_{55}^{(3)}(t)$	L
0	10.47	13.61	3.14	6.81	−3.66	10.47	0.00	10.47	0.00	11.49	1.02
1	11.46	14.90	3.44	7.35	−4.11	12.97	1.51	9.98	−1.48	14.07	2.61
2	12.49	16.24	3.75	7.93	−4.56	14.80	2.31	10.26	−2.23	16.12	3.63
3	13.59	17.67	4.08	8.58	−5.01	16.59	3.00	10.72	−2.87	18.14	4.55
4	14.77	19.20	4.43	9.32	−5.46	18.44	3.67	11.30	−3.47	20.22	5.45
5	16.08	20.90	4.82	10.18	−5.91	20.42	4.34	12.03	−4.05	22.44	6.36
6	17.54	22.80	5.26	11.18	−6.36	22.58	5.04	12.90	−4.64	24.85	7.31
7	19.19	24.95	5.76	12.37	−6.82	24.97	5.78	13.95	−5.24	27.50	8.31
8	21.06	27.38	6.32	13.77	−7.29	27.64	6.58	15.19	−5.87	30.46	9.40
9	23.14	30.08	6.94	15.41	−7.73	30.58	7.44	16.61	−6.53	—	—

case of the term life insurance and endowment insurance as the premium factor increases. In the pure endowment case, the loadings (L) are decreasing as this factor increases.

The CET loadings may be compared to the ones obtained from the term life insurance products. However, they present different patterns. Similar arguments may be made for the GAM and the pure endowment insurance loadings. The endowment insurance loadings cannot be compared with loadings from other tables. These represent an average of the term life insurance and the pure endowment insurance loadings.

Tables 3 and 4 compare the mortality probabilities that are observable in the CSO, CET, and GAM tables with the martingale probabilities obtained from equations (3.4), (3.11), and (3.17) using increasing loading factors. More precisely in Table 3, the insurance products are determined using the standard deviation principle with increasing factors of 1%. This means that there is no loading for the one-year insurance products, a loading factor of 1% for the two-year insurance products, a loading factor of 2% for the three-year contracts, and so on. In Table 4 the loading factor is increased by 2%.

The loadings obtained from the martingale probabilities under the standard deviation principle with an increasing factor follow a similar pattern as the CET and GAM loadings. They increase in time. However, they are not constant percentage-wise. In the endowment case, the loadings are greater than the one obtained from the term life insurances because the martingale probability at time t is obtained from the value of the endowment insurance at time $t + 1$ and $t + 2$ combined with an increasing factor.

We may also use the expected premium principle (see Bowers at al. 1997) to evaluate the standard insurance premiums and obtain the martingale probabilities. However, the range for the loading factors is limited because the pricing method based on this principle failed to capture the volatility of the

Table 4

Loadings Obtained from Standard Deviation Principle Increasing by 2%

	CSO	CET		GAM		Martingale Probabilities					
t	q_{55+t}	q_{55+t}	L	q_{55+t}	L	$q_{55}^{(1)}(t)$	L	$q_{55}^{(2)}(t)$	L	$q_{55}^{(3)}(t)$	L
0	10.47	13.61	3.14	6.81	−3.66	10.47	0.00	10.47	0.00	12.51	2.04
1	11.46	14.90	3.44	7.35	−4.11	14.48	3.02	8.51	−2.95	16.69	5.23
2	12.49	16.24	3.75	7.93	−4.56	17.11	4.62	8.04	−4.45	19.77	7.28
3	13.59	17.67	4.08	8.58	−5.01	19.62	6.03	7.87	−5.72	22.76	9.17
4	14.77	19.20	4.43	9.32	−5.46	22.16	7.39	7.88	−6.89	25.80	11.03
5	16.08	20.90	4.82	10.18	−5.91	24.86	8.78	8.06	−8.02	29.03	12.95
6	17.54	22.80	5.26	11.18	−6.36	27.78	10.24	8.40	−9.14	32.52	14.98
7	19.19	24.95	5.76	12.37	−6.82	31.00	11.81	8.91	−10.28	36.37	17.18
8	21.06	27.38	6.32	13.77	−7.29	34.58	13.52	9.60	−11.46	40.66	19.60
9	23.14	30.08	6.94	15.41	−7.73	38.54	15.40	10.45	−12.69	—	—

mortality risk. For example, assume now that $V^{(j)}(x, 0, n)$, $j = 1, 2, 3$, and $n = 1, 2, \ldots$, are determined using the expected premium principle with a constant factor θ applies to the 1980 U.S. CSO table. It follows from equation (3.4) that

$$q_x^{(1)}(t) = (1 + \theta)q_{x+t},$$

for $t = 0, 1, \ldots$, and from (3.11) and (3.17) we have

$$q_x^{(j)}(0) = 1 - (1 + \theta)(1 - q_x) \quad \text{and} \quad q_x^{(j)}(t) = q_{x+t},$$

for $j = 2, 3$, $t = 1, 2, \ldots$, and for all (x). For example, the probability that (55) dies in the next year q_{55} is equal to 0.01047. The loading factor θ should be smaller than or equal to 1.05% since

$$\begin{aligned} q_{55}^{(j)}(0) &= 1 - (1 + \theta)(1 - q_{55}) \\ &= 1 - (1 + \theta)(0.98953) \\ &> 0, \end{aligned}$$

for $j = 2, 3$. Solving for θ leads to $\theta < 1.058078\%$. Hence, values obtained with a constant factor greater than or equal to 1.06% should be rejected by the constraints presented in the previous section.

5. MARTINGALE MEASURES FOR EQUITY-LINKED PRODUCTS

Because of their unique designs, equity-linked products involve mortality and financial risks since these types of contracts provide both death and accumulation/survival benefits. Moreover, the level of these benefits are linked to the financial market performance and an equity index in particular. Hence, it is natural to assume that equity-linked products belong to a combined insurance and financial markets since they are simultaneously subject to the interest rate risk, equity risk, and mortality risk. Similarly to Section 3, we evaluate these type of products by valuing the death benefits and survival benefits separately since it is the common practice in insurance industry that the reserves for insurances are separated from those for annuities. Under this approach, two martingale measures again need to be generated: one for death benefits and another for survival benefits. Furthermore, these martingale measures should be such that they reproduce the index values in Section 2 and the premiums of insurance products in Section 3. In other words, the marginal probabilities derived in the previous sections should be preserved, and the martingale measures $Q_x^{(j)+}$, $j = 1, 2$, are such that $\{W^{(j)}(x, t, n)/B(t), t = 0, 1, \ldots\}$ and $\{S(k)/B(k), k = 0, \Delta, \ldots\}$ will remain martingales. Let $e_x^{(j)}(t, i))$ denote the probability under $Q_x^{(j)+}$ that

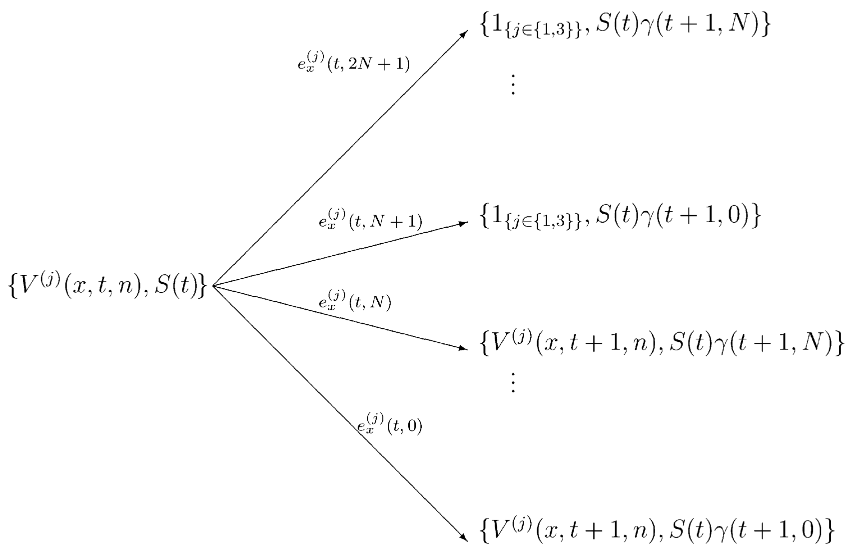
$$\begin{cases} (x) \text{ survives and } S(t + 1) = S(t)\gamma(t + 1, i), & \text{if } i = 0, \ldots, N, \\ (x) \text{ dies and } S(t + 1) = S(t)\gamma(t + 1, i - (N + 1)), & \text{if } i = N + 1, \ldots, 2N + 1, \end{cases} \tag{5.1}$$

between t and $t + 1$ given $S(t)$ and $K(x) \geq t$ as illustrated in Figure 2 for time period $[t, t + 1]$, where $\gamma(t, l)$ is given by $u(t)^l d(t)^{N-l}$ for $l = 0, 1, \ldots, N$.

What remains is to determine the probabilities $e_x^{(j)}(t, i)$ for all t and i. Most research to date assumes that the mortality risk is independent of the financial risk, and insurance products are evaluated accordingly. Such an approach works well if we are interested only in net premiums. If we deal with premiums, the financial market may have an impact on the insurance market, and the independence assumption could be unreasonable. Hence in what follows we introduce the dependency between the index process and the premiums of insurance products and determine the probabilities $e_x^{(j)}(t, i)$ using a copula. The main advantage of using copulas is that they separate a joint distribution function in two parts: the dependence structure and the marginal distribution functions. Copulas provide additional structure to describe the dependence between the index and the insurance products, which is needed to determine the extended martingale measures. Moreover, based on Sklar's Theorem, all multivariate distributions can be expressed using copulas. We also use them because of their mathematical tractability. A comprehensive introduction may be found in Joe (1997) or Nelsen (1999). Frees and Valdez

Figure 2

Probability Tree of Index Process Combined with Term Life Insurance j = 1, Pure Endowment Insurance j = 2, and Endowment Insurance j = 3 Processes between t and t + 1 Given $S(t)$ and $K(x) \geq t$



(1998), Wang (1998), and Venter (2000) have given an overview of copulas and their applications to actuarial science. Cherubini, Luciano, and Vecchiato (2004) presents application of copulas in finance. We remark that the underlying joint distribution is not necessarily inferred by a copula. Using a copula is a convenient way to provide a consistent and tractable pricing framework for equity-linked insurance products. There exists a wide range of copulas that may define a joint cumulative distribution function. They can in general be divided into parametric and nonparametric copulas. The simplest one is the independent copula

$$C^I\ (F_{Y_1}(y_1), F_{Y_2}(y_2)) = F_{Y_1}(y_1)\ F_{Y_2}(y_2), \tag{5.2}$$

where $F_{Y_1}(.)$ and $F_{Y_2}(.)$ are two marginal cumulative distribution functions. Extreme copulas are defined using the upper and lower Frechet bounds, which are given by

$$C^U\ (F_{Y_1}(y_1), F_{Y_2}(y_2)) = \min[F_{Y_1}(y_1), F_{Y_2}(y_2)], \tag{5.3}$$

and

$$C^D\ (F_{Y_1}(y_1), F_{Y_2}(y_2)) = \max[F_{Y_1}(y_1) + F_{Y_2}(y_2) - 1, 0]. \tag{5.4}$$

One of the most important family of parametric copulas are the Archimedean copulas. Among them, the Cook-Johnson (Clayton) copula is widely used in actuarial science because of its desirable properties and simplicity. The Cook-Johnson (Clayton) copula with parameter $\kappa > 0$ is given by

$$C_\kappa^{CJ}\ (F_{Y_1}(y_1), F_{Y_2}(y_2)) = [F_{Y_1}(y_1)^{-\kappa} + F_{Y_2}(y_2)^{-\kappa} - 1]^{-1/\kappa}. \tag{5.5}$$

The Gaussian ($-1 \leq \kappa \leq 1$) copula, which is often used in finance, is defined as

$$C_\kappa^G\ (F_{Y_1}(y_1), F_{Y_2}(y_2)) = \Phi_\kappa(\Phi^{-1}\ (F_{Y_1}(y_1)), \Phi^{-1}(F_{Y_2}(y_2))), \tag{5.6}$$

where Φ_κ is the bivariate standard normal cumulative distribution function with correlation coefficient κ and Φ^{-1} is the inverse of the standard normal cumulative distribution function. Hence, the parameter κ in formulas (5.5) and (5.6) indicates the level of dependence between the index and the policyholder's life.

We now use a copula to introduce a joint cumulative distribution function of the index and the corresponding insurance product. Let $G_x^{(j)}$, $j = 1, 2$, denote this joint conditional cumulative distribution function over time t and $t + 1$:

$$G_x^{(j)}(y_1, y_2; t) = Q_x^{(j)+} \, [S(t+1) \le y_1, W^{(j)}(x, t+1, n) \le y_2 | S(t), K(x) \ge t]. \tag{5.7}$$

As explained above, the marginal cumulative distribution functions of the insurance products and the index are preserved under the extended measures:

$$G_x^{(j)}(\infty, y_2; t) = Q_x^{(j)} \, [W^{(j)}(x, t+1, n) \le y_2 | K(x) \ge t], \tag{5.8}$$

and

$$G_x^{(j)}(y_1, \infty; t) = Q[S(t+1) \le y_1 | S(t)], \tag{5.9}$$

which are determined using (3.4) for $j = 1$, (3.11) for $j = 2$, and (2.7). Let $C_{\kappa(t)}$ be the choice of copula. Then the cumulative distribution function $G_x^{(j)}$ is defined by

$$G_x^{(j)}(y_1, y_2; t) = C_{\kappa(t)} \, (G_x^{(j)}(y_1, \infty; t), G_x^{(j)}(\infty, y_2; t)), \tag{5.10}$$

where $\kappa(t)$ is the free parameter between t and $t + 1$. We remark that $G_x^{(j)}(y_1, \infty; t)$ and hence $G_x^{(j)}(y_1, y_2; t)$ are functions of $S(t)$, but for notational simplicity we suppress $S(t)$.

Using the result presented by Cossette et al. (2002), the martingale probabilities are given by

$$e_x^{(1)}(t, i) = \begin{cases} G_x^{(1)}(S(t)\gamma(t+1, i), V^{(1)}(x, t+1, n); t) \\ -G_x^{(1)}(S(t)\gamma(t+1, i-1), V^{(1)}(x, t+1, n); t), & \text{if } i = 0, \ldots, N, \\ \\ G_x^{(1)}(S(t)\gamma(t+1, i-N-1), 1; t) \\ -G_x^{(1)}(S(t)\gamma(t+1, i-N-2), 1; t) \\ -G_x^{(1)}(S(t)\gamma(t+1, i-N-1), V^{(1)}(x, t+1, n); t) \\ +G_x^{(1)}(S(t)\gamma(t+1, i-N-2), V^{(1)}(x, t+1, n); t), & \text{if } i = N+1, \ldots, 2N+1, \end{cases} \tag{5.11}$$

and

$$e_x^{(2)}(t, i) = \begin{cases} G_x^{(2)}(S(t)\gamma(t+1, i), V^{(2)}(x, t+1, n); t) \\ -G_x^{(2)}(S(t)\gamma(t+1, i-1), V^{(2)}(x, t+1, n); t) \\ -G_x^{(2)}(S(t)\gamma(t+1, i), 0; t) \\ +G_x^{(2)}(S(t)\gamma(t+1, i-1), 0; t), & \text{if } i = 0, \ldots, N, \\ \\ G_x^{(2)}(S(t)\gamma(t+1, i-N-1), 0; t) \\ -G_x^{(2)}(S(t)\gamma(t+1, i-N-2), 0; t), & \text{if } i = N+1, \ldots, 2N+1, \end{cases} \tag{5.12}$$

where $G_x^{(j)}(S(t)\gamma(t+1, -1), .; t) = 0$ and $G_x^{(j)}(., .; t)$ is obtained using (5.10).

For notational convenience, let

$$\mathbf{i}_t = \{i_0, \ldots, i_t\},$$

which represents the index's realization up to time t with

$$S(k) = S(0) \prod_{l=0}^{k} \gamma(l, i_l), \tag{5.13}$$

for $k = 0, \ldots, t$, where $\gamma(0, i_0) = 1$.

Consider now an equity-linked product that pays

$$\begin{cases} D(K(x)+1), & \text{if } K(x) = 0, 1, \ldots, n-1, \\ D(n), & \text{if } K(x) \ge n. \end{cases} \tag{5.14}$$

Note that, in practice, the final payoff $D(.)$ might not be the same function as that of the death benefits, but for simplicity we assume it to be the same. For notational convenience, we sometimes use $D(t, \mathbf{i}_t)$ to specify the index's realization.

Let $P^{(1)}(x, t, n, \mathbf{i}_t)$ denote the premium at time t of the equity-linked contract death benefit given that (x) is still alive and the index process has taken the path $\mathbf{i}_t$. With the martingale structure identified by (5.11), $P^{(1)}(x, t, n, \mathbf{i}_t)$ may be obtained as the expected discounted payoffs

$$P^{(1)}(x, t, n, \mathbf{i}_t) = E_x^{(1)+}\left[\frac{D(K(x) + 1)1_{\{K(x)<n\}}}{B(K(x) + 1)} B(t)|\mathbf{i}_t, K(x) \geq t\right], \tag{5.15}$$

where $E_x^{(1)+}[.]$ represents the expectation with respect to $Q_x^{(1)+}$.

On the other hand, let $P^{(2)}(x, t, n, \mathbf{i}_t)$ denote the premium at time t of the equity-linked product accumulation benefit given that (x) is still alive and the index process has taken the path $\mathbf{i}_t$. With the martingale structure identified by (5.12), $P^{(2)}(x, t, n, \mathbf{i}_t)$ may be obtained as the expected discounted payoffs

$$P^{(2)}(x, t, n, \mathbf{i}_t) = E_x^{(2)+}\left[\frac{D(n)1_{\{K(x)\geq n\}}}{B(n)} B(t)|\mathbf{i}_t, K(x) \geq t\right]. \tag{5.16}$$

Let $P(x, t, n, \mathbf{i}_t)$ denote the premium at time t of an n-year equity-linked product issue to (x) with its payoff defined by (5.14). In particular, $P(x, 0, n, \mathbf{i}_0)$ is the amount invested by an insured or, from insurers' point of view, the premium paid by the policyholder. We assume that $P(x, t, n, \mathbf{i}_t)$ may be decomposed in two different premiums; $P^{(1)}(x, t, n, \mathbf{i}_t)$ is the premium to cover the death benefit and $P^{(2)}(x, t, n, \mathbf{i}_t)$ is the premium to cover the accumulation benefit: That is, we assume that

$$P(x, t, n, \mathbf{i}_t) = P^{(1)}(x, t, n, \mathbf{i}_t) + P^{(2)}(x, t, n, \mathbf{i}_t), \tag{5.17}$$

where $P^{(1)}(x, t, n, \mathbf{i}_t)$ and $P^{(2)}(x, t, n, \mathbf{i}_t)$ are obtained using (5.15) and (5.16), respectively.

An alternative approach is to evaluate equity-linked products containing death and accumulation benefits in a unified manner, using the pricing information from the endowment insurance products. In this case the martingale measure $Q_x^{(3)+}$ is such that $\{W^{(3)}(x, t, n)/B(t), t = 0, 1, \ldots\}$ and $\{S(k)/B(k), k = 0, \Delta, \ldots\}$ remain martingales. We may use a similar approach to determine the martingale probabilities $e_x^{(3)}$'s, which are defined by (5.1) with $j = 3$. The joint cumulative distribution function of the index and the endowment insurance may be defined by

$$G_x^{(3)}(y_1, y_2; t) = C_{\kappa(t)}(Q[S(t + 1) \leq y_1|S(t)], Q_x^{(3)}[W^{(3)}(x, t + 1, n) \leq y_2|K(x) \geq t]), \tag{5.18}$$

where $Q_x^{(3)}$ and Q may be obtained using (3.17) and (2.7), respectively. Again, the martingale probabilities are obtained from the joint cumulative distribution function using the result from Cossette et al. (2002). We have

$$e_x^{(3)}(t, i) = \begin{cases} G_x^{(3)}(S(t)\gamma(t + 1, i), V^{(3)}(x, t + 1, n); t) \\ -G_x^{(3)}(S(t)\gamma(t + 1, i - 1), V^{(3)}(x, t + 1, n); t), & \text{if } i = 0, \ldots, N, \\ \\ G_x^{(3)}(S(t)\gamma(t + 1, i - N - 1), 1; t) \\ -G_x^{(3)}(S(t)\gamma(t + 1, i - N - 2), 1; t) \\ -G_x^{(3)}(S(t)\gamma(t + 1, i - N - 1), V^{(3)}(x, t + 1, n); t) \\ +G_x^{(3)}(S(t)\gamma(t + 1, i - N - 2), V^{(3)}(x, t + 1, n); t), & \text{if } i = N + 1, \ldots, 2N + 1, \end{cases} \tag{5.19}$$

where $G_x^{(3)}$ is defined by (5.18).

In this case, the n-year equity-linked product premium $P(x, t, n, \mathbf{i}_t)$ may be obtained as expected discounted payoffs

$$\begin{aligned} P(x, t, n, \mathbf{i}_t) &= P^{(3)}(x, t, n, \mathbf{i}_t) \\ &= E_x^{(3)+}\left[\left\{\frac{D(K(x) + 1)1_{\{K(x)<n-1\}}}{B(K(x) + 1)} + \frac{D(n)1_{\{K(x)\geq n-1\}}}{B(n)}\right\} B(t)|\mathbf{i}_t, K(x) \geq t\right]. \end{aligned} \tag{5.20}$$

Figure 3 presents the dynamic of the equity-linked premiums for time period $[t, t + 1]$.

Bear in mind that the first approach presented in this section evaluates equity-linked products by loading the death and survival probabilities separately. The second approach evaluates the equity-linked product using unified loaded probabilities.

6. EQUITY-LINKED MARTINGALE PROBABILITIES: NUMERICAL EXAMPLE

In this section we compare the martingale probabilities obtained using the different copulas presented previously. Again, the values, $V^{(j)}(x, 0, n)$ ($j = 1, 2, 3$ and $n = 1, \ldots, 5$), are determined using the standard deviation premium principle with a loading factor of 5% to the 1980 U.S. CSO table. The premiums of those insurance product are obtained for a male aged 55. We are still assuming that the short-term rate $r(t)$ is constant over time and is equal to 5%. The index is governed by the modified CRR model introduced previously with $S(0) = 1$ and $N = 1$. The index's volatility is assumed to be constant and is 20%. In other words, $u(t) = e^{\sigma} = 1.22$ and $d(t) = u(t)^{-1} = 0.82$. Using (2.5), the index martingale probability of going up is $\pi(t) = 0.574$. All numbers presented in the tables are multiplied by 100.

Table 5 presents the martingale probabilities obtained from (5.11), (5.12), and (5.19). The cumulative distribution functions $G_{55}^{(j)}$, $j = 1, 2, 3$, are determined using the independent copula (5.2), the upper copula (5.3), and the lower copula (5.4).

We next consider the Cook-Johnson copula (5.5) and the Gaussian copula (5.6). The martingale probabilities $e_{55}^{(j)}$, $j = 1, 2, 3$, are given in Table 6 and Table 7, respectively, for the two copulas.

The independent copula may be obtained by letting $\kappa \to 0$ in the Cook-Johnson copula. Similarly, the Frechet upper bound is obtained by letting $\kappa \to \infty$. This explains that the martingale probabilities with $\kappa(t) = 0.5$ are closer to the independent one than the martingale probabilities obtained using $\kappa(t) = 2$, which are closer to the upper copula. Setting $\kappa(t) = 0$ in the Gaussian copula also leads to the independent copula. The martingale probabilities are between the independent copula and the lower copula when $\kappa(t) = -0.5$. On the other hand, when $\kappa(t) = 0.5$, the martingale probabilities are between the independent copula and the upper copula.

7. SURRENDER OPTION

This section develops pricing formulas for equity-linked contracts that include surrender options. A surrender option allows for a policyholder to surrender his or her contract to the insurance company at the cash surrender value. In general, the policyholder is allowed to surrender his or her policy before its maturity with an imposed surrender charge. We assume that the policyholder may surrender at the beginning of each year. A common practice to evaluate a surrender option is to assume a fixed withdrawal rate in the pricing models. However, evaluating withdrawal rates is difficult because of the lack of general studies and empirical data. Alternatively, similar to Bacinello (2003a,b), a surrender option can be evaluated assuming that the policyholders are rational, which means that policyholders will compare the cash surrender value with the value of the equity-linked contract for the remaining years. If the cash surrender value is equal or greater than the value of the equity-linked contract, they will surrender the policy. This is similar to the valuation of American options (Hull 2000). Hence, surrender options can be evaluated using backward recursion.

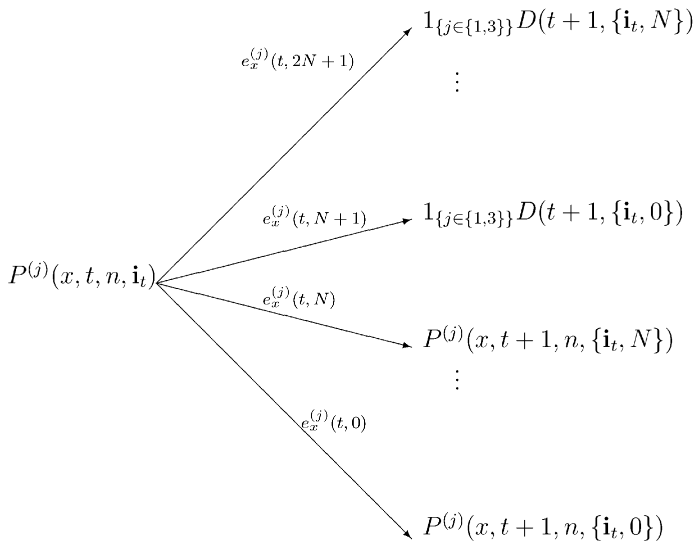
Let $SV(t)$ be the cash surrender value at time t if a policyholder decides to surrender his or her policy at time t. Thus, the cash surrender value for equity-linked contracts may be represented as

$$SV(t) = \delta(t)D(t), \tag{7.1}$$

where $1 - \delta(t)$ is the percentage of surrender charge if the policyholder surrenders at time t, a decreasing function with a value between 0 and 1. For example, for a five-year equity-indexed annuity an insurance company may impose a surrender charge of 1% per year of the remaining years. In this case, $\delta(0) = 95\%$, $\delta(1) = 96\%$, $\delta(2) = 97\%$, $\delta(3) = 98\%$, $\delta(4) = 99\%$, and $\delta(5) = 100\%$. Because of the

Figure 3

Probability Tree of Equity-Linked Products between t and $t + 1$ Given $\mathbf{i}_t$ and $K(x) \geq t$



nonforfeiture laws, the surrender value should be at least as great as the guaranteed minimum account value, which is equal to 3% compounded on 90% of the premium.

A recursive algorithm to evaluate equity-linked contracts without a surrender option is derived based on (5.15) and (5.16):

$$P^{(1)}(x, t, n, \mathbf{i}_t) = \frac{1}{1 + r(t)} \left[\sum_{l=0}^{N} e_x^{(1)}(t, l + N + 1) D(t + 1, \{\mathbf{i}_t, l\}) + \sum_{l=0}^{N} e_x^{(1)}(t, l) P^{(1)}(x, t + 1, n, \{\mathbf{i}_t, l\}) \right], \quad (7.2)$$

Table 5

Martingale Probabilities with Nonparametric Copulas

	$e_{55}^{(1)}(t, i)$				$e_{55}^{(2)}(t, i)$				$e_{55}^{(3)}(t, i)$			
t/i	0	1	2	3	0	1	2	3	0	1	2	3
Independent Copulas												
0	41.90	56.54	0.66	0.89	42.34	57.12	0.23	0.31	41.90	56.54	0.66	0.89
1	41.98	56.65	0.58	0.79	42.18	56.91	0.39	0.53	42.02	56.70	0.54	0.73
2	41.96	56.61	0.61	0.82	42.11	56.82	0.45	0.61	41.97	56.63	0.60	0.81
3	41.92	56.56	0.65	0.87	42.06	56.75	0.51	0.68	41.93	56.57	0.64	0.86
4	41.88	56.50	0.69	0.93	42.01	56.68	0.56	0.76	—	—	—	—
Upper Copulas												
0	42.57	55.88	0.00	1.56	42.03	57.43	0.54	0.00	42.57	55.88	0.00	1.56
1	42.57	56.06	0.00	1.37	41.65	57.43	0.92	0.00	42.57	56.16	0.00	1.28
2	42.57	56.00	0.00	1.43	41.51	57.43	1.06	0.00	42.57	56.03	0.00	1.40
3	42.57	55.92	0.00	1.52	41.38	57.43	1.19	0.00	42.57	55.94	0.00	1.50
4	42.57	55.81	0.00	1.62	41.25	57.43	1.32	0.00	—	—	—	—
Lower Copulas												
0	41.01	57.43	1.56	0.00	42.57	56.90	0.00	0.54	41.01	57.43	1.56	0.00
1	41.20	57.43	1.37	0.00	42.57	56.52	0.00	0.92	41.29	57.43	1.28	0.00
2	41.14	57.43	1.43	0.00	42.57	56.37	0.00	1.06	41.16	57.43	1.40	0.00
3	41.05	57.43	1.52	0.00	42.57	56.24	0.00	1.19	41.07	57.43	1.50	0.00
4	40.94	57.43	1.62	0.00	42.57	56.12	0.00	1.32	—	—	—	—

Table 6

Martingale Probabilities with Cook-Johnson Copula

t/i	$e_{55}^{(1)}(t, i)$ 0	1	2	3	$e_{55}^{(2)}(t, i)$ 0	1	2	3	$e_{55}^{(3)}(t, i)$ 0	1	2	3
						$\kappa(t) = 0.5$						
0	42.13	56.31	0.43	1.12	42.07	57.39	0.50	0.04	42.13	56.31	0.43	1.12
1	42.18	56.45	0.38	0.99	41.74	57.35	0.83	0.09	42.21	56.51	0.36	0.92
2	42.17	56.40	0.40	1.03	41.61	57.33	0.95	0.11	42.18	56.42	0.39	1.01
3	42.14	56.34	0.42	1.09	41.50	57.31	1.06	0.13	42.15	56.35	0.42	1.08
4	42.11	56.26	0.45	1.17	41.40	57.29	1.17	0.15	—	—	—	—
						$\kappa(t) = 2$						
0	42.44	56.00	0.12	1.43	42.03	57.43	0.54	0.00	42.44	56.00	0.12	1.43
1	42.46	56.17	0.11	1.26	41.65	57.43	0.92	0.00	42.47	56.26	0.10	1.18
2	42.45	56.12	0.11	1.32	41.51	57.43	1.06	0.00	42.46	56.14	0.11	1.29
3	42.45	56.04	0.12	1.40	41.38	57.43	1.19	0.00	42.45	56.05	0.12	1.38
4	42.44	55.94	0.13	1.50	41.25	57.43	1.32	0.00	—	—	—	—

where $P^{(1)}(x, n, n, \mathbf{i}_n) = 0$ and

$$P^{(2)}(x, t, n, \mathbf{i}_t) = \frac{1}{1 + r(t)} \sum_{l=0}^{N} e_x^{(2)}(t, l) P^{(2)}(x, t + 1, n, \{\mathbf{i}_t, l\}), \tag{7.3}$$

where $P^{(2)}(x, n, n, \mathbf{i}_n) = D(n, \mathbf{i}_n)$. The time-$t$ value of the contract is given by

$$P(x, t, n, \mathbf{i}_t) = P^{(1)}(x, t, n, \mathbf{i}_t) + P^{(2)}(x, t, n, \mathbf{i}_t), \tag{7.4}$$

and especially $P(x, 0, n, \mathbf{i}_0)$ is the premium of the contract.

We now adjust the previous recursive algorithm to include the surrender option. Let $P_{SV}^{(j)}(x, t, n, \mathbf{i}_t)$, $j = 1, 2$, denote the premium at time t to cover the mortality risk ($j = 1$) or the accumulation benefit ($j = 2$) of an equity-linked contract with a surrender option. Further, let $P_{SV}^{(j)}(x, t+, n, \mathbf{i}_t)$, $j = 1, 2$, denote the time-t value that is equal to $P_{SV}^{(j)}(x, t, n, \mathbf{i}_t)$ given that the policyholder does not surrender at time t. The value of the whole contract $P_{SV}(x, t+, n, \mathbf{i}_t)$ and its premium $P_{SV}(x, t, n, \mathbf{i}_t)$ at time t are defined by adding their corresponding premiums that covers the mortality risk and the accumulation benefits. They are given, respectively, by

Table 7

Martingale Probabilities with Gaussian Copula

t/i	$e_{55}^{(1)}(t, i)$ 0	1	2	3	$e_{55}^{(2)}(t, i)$ 0	1	2	3	$e_{55}^{(3)}(t, i)$ 0	1	2	3
						$\kappa(t) = -0.5$						
0	41.19	57.26	1.38	0.17	42.55	56.91	0.02	0.52	41.19	57.26	1.38	0.17
1	41.34	57.29	1.22	0.15	42.53	56.55	0.04	0.88	41.42	57.30	1.14	0.13
2	41.29	57.28	1.27	0.16	42.52	56.42	0.05	1.02	41.32	57.28	1.25	0.15
3	41.22	57.26	1.35	0.17	42.51	56.30	0.05	1.14	41.23	57.27	1.33	0.17
4	41.13	57.25	1.44	0.19	42.50	56.18	0.06	1.25	—	—	—	—
						$\kappa(t) = 0.5$						
0	42.49	55.96	0.08	1.48	42.07	57.39	0.50	0.04	42.49	55.96	0.08	1.48
1	42.50	56.13	0.07	1.30	41.73	57.35	0.83	0.09	42.51	56.22	0.06	1.22
2	42.50	56.07	0.07	1.36	41.61	57.33	0.96	0.10	42.50	56.10	0.07	1.34
3	42.49	55.99	0.08	1.44	41.50	57.31	1.07	0.12	42.49	56.01	0.07	1.42
4	42.48	55.89	0.08	1.54	41.39	57.29	1.18	0.14	—	—	—	—

$$P_{SV}(x, t+, n, \mathbf{i}_t) = P_{SV}^{(1)}(x, t+, n, \mathbf{i}_t) + P_{SV}^{(2)}(x, t+, n, \mathbf{i}_t), \tag{7.5}$$

and

$$P_{SV}(x, t, n, \mathbf{i}_t) = P_{SV}^{(1)}(x, t, n, \mathbf{i}_t) + P_{SV}^{(2)}(x, t, n, \mathbf{i}_t). \tag{7.6}$$

Since the surrender option is an accumulation benefit, it influences only $P_{SV}^{(2)}(x, t+, n, \mathbf{i}_t)$ and $P_{SV}^{(2)}(x, t, n, \mathbf{i}_t)$. In other words, we have

$$\begin{aligned} P_{SV}^{(1)}(x, t+, n, \mathbf{i}_t) &= P_{SV}^{(1)}(x, t, n, \mathbf{i}_t) \\ &= P^{(1)}(x, t, n, \mathbf{i}_t), \end{aligned} \tag{7.7}$$

for all t.

The value $P_{SV}^{(2)}(x, (n-1)+, n, \mathbf{i}_{n-1})$ is the one-year expected discounted payoffs under the corresponding martingale measure, which is given by

$$P_{SV}^{(2)}(x, (n-1)+, n, \mathbf{i}_{n-1}) = \frac{1}{1+r(n-1)} \sum_{l=0}^{N} e_x^{(2)}(n-1, l) D(n, \{\mathbf{i}_{n-1}, l\}). \tag{7.8}$$

The equity-linked contract premium at time $n - 1$, given that (x) is still alive, is the maximum between the contract value and the cash surrender value $SV(n-1)$. Hence, we have

$$P_{SV}(x, n-1, n, \mathbf{i}_{n-1}) = \max[SV(n-1), P_{SV}(x, (n-1)+, n, \mathbf{i}_{n-1})]. \tag{7.9}$$

It follows from (7.5), (7.6), (7.7), and (7.9) that

$$P_{SV}^{(2)}(x, n-1, n, \mathbf{i}_{n-1}) = \max[SV(n-1) - P^{(1)}(x, n-1, n, \mathbf{i}_{n-1}), P_{SV}^{(2)}(x, (n-1)+, n, \mathbf{i}_{n-1})]. \tag{7.10}$$

To complete the recursive algorithm, suppose that the insured is still alive at $t < n - 1$, and the index realization up to time t is given by $\mathbf{i}_t$. The time-t contract premium $P^{(1)}(x, t, n, \mathbf{i}_t)$ covering the mortality risk is given by equation (7.2) since (7.7) holds for all t. The contract value covering the accumulation benefits, which includes the surrender option in the future, is given by

$$P_{SV}^{(2)}(x, t+, n, \mathbf{i}_t) = \frac{1}{1+r(t)} \sum_{l=0}^{N} e_x^{(2)}(t, l) P_{SV}^{(2)}(x, t+1, n, \{\mathbf{i}_t, l\}), \tag{7.11}$$

since the policyholders will be entitled to surrender if they survive to time $t + 1$. Following the same maximization arguments that lead to (7.9) and along with (7.7), we have

$$P_{SV}^{(2)}(x, t, n, \mathbf{i}_t) = \max[SV(t) - P^{(1)}(x, t, n, \mathbf{i}_t), P_{SV}^{(2)}(x, t+, n, \mathbf{i}_t)]. \tag{7.12}$$

Thus, the time-t premium is given by

$$P_{SV}(x, t, n, \mathbf{i}_t) = P^{(1)}(x, t, n, \mathbf{i}_t) + P_{SV}^{(2)}(x, t, n, \mathbf{i}_t), \tag{7.13}$$

and particularly $P_{SV}(x, 0, n, \mathbf{i}_0)$ is the premium of the contract.

A recursive formula to evaluate $P^{(3)}(x, t, n, \mathbf{i}_t)$ without the surrender option is determined using (5.20):

$$\begin{aligned} &P^{(3)}(x, t, n, \mathbf{i}_t) \\ &= \frac{1}{1+r(t)} \left\{ \sum_{l=0}^{N} e_x^{(3)}(t, l+N+1) D(t+1, \{\mathbf{i}_t, l\}) + \sum_{l=0}^{N} e^{(3)}(t, l) P^{(3)}(x, t+1, n, \{\mathbf{i}_t, l\}) \right\}, \end{aligned} \tag{7.14}$$

for $t = 0, \ldots, n-2$, where

$$P^{(3)}(x, n-1, n, \mathbf{i}_{n-1}) = \frac{1}{1+r(n-1)} \sum_{l=0}^{N} \binom{N}{l} \pi(n)^l [1-\pi(n)]^{N-l} D(n, \{\mathbf{i}_{n-1}, l\}). \tag{7.15}$$

The recursive formula needs to be adjusted to consider the case where the policyholder exercises the surrender option. Let $P_{SV}^{(3)}(x, t, n, \mathbf{i}_t)$ denote the time-t premium of an equity-linked contract that includes the surrender option. Further, let $P_{SV}^{(3)}(x, t+, n, \mathbf{i}_t)$ denote the contract value at time t given that the policyholder does not surrender at time t, and, particularly, at time $n-1$ we have

$$P_{SV}^{(3)}(x, (n-1)+, n, \mathbf{i}_{n-1}) = P^{(3)}(x, n-1, n, \mathbf{i}_{n-1}), \tag{7.16}$$

which is defined explicitly in (7.15). This equality is justified by the fact that the payoff $D(n)$ is due with certainty at time n, and there is no surrender option remaining if the policyholder decides to continue the contact at time $n-1$.

The equity-linked product premium at time $n-1$ is the maximum between the contract value and the cash surrender value $SV(n-1)$, which is given by

$$P_{SV}^{(3)}(x, n-1, n, \mathbf{i}_{n-1}) = \max[SV(n-1), P_{SV}^{(3)}(x, (n-1)+, n, \mathbf{i}_{n-1})]. \tag{7.17}$$

Now suppose that the policyholder is still alive at time $t < n-1$ and the index process followed the path $\mathbf{i}_t$. To continue the contract at time t means that if the policyholder dies within one year, the benefit $D(t+1)$ is paid, in the case he or she survives, he or she is entitled to surrender or continue the contract at time $t+1$. Hence, the recursive algorithm is given by

$$P_{SV}^{(3)}(x, t+, n, \mathbf{i}_t) = \sum_{l=0}^{N} \frac{e_x^{(3)}(t, l+N+1)}{1+r(t)} D(t+1, \{\mathbf{i}_t, l\}) + \sum_{l=0}^{N} \frac{e_x^{(3)}(t, l)}{1+r(t)} P_{SV}^{(3)}(x, t+1, n, \{\mathbf{i}_t, l\}). \tag{7.18}$$

The maximization argument that leads to (7.17) may be extended to an intermediate time t to obtain

$$P_{SV}^{(3)}(x, t, n, \mathbf{i}_t) = \max[SV(t), P_{SV}^{(3)}(x, t+, n, \mathbf{i}_t)]. \tag{7.19}$$

Thus, $P_{SV}(x, 0, n, i_0)$ can be derived recursively using (7.18) and (7.19) with starting value given by (7.16).

Note that setting $\delta = 0\%$ implies that premium with surrender option is the same as the premium without the surrender option.

8. VALUATION OF EQUITY-INDEXED ANNUITIES: NUMERICAL EXAMPLES

This section implements numerically the methods we developed previously by considering several types of equity-indexed annuities (EIAs). EIAs appeal to investors because they offer the same protection as conventional annuities by limiting the financial risk, but also provide participation in the equity market. From Lin and Tan (2004) and Tiong (2002), EIA designs may be generally grouped in two broad classes: Annual Reset and Point-to-Point. The index growth on an EIA with the former is measured and locked in each year. Particularly, the index growth with a term-end point design is calculated using the index value at the beginning and the end of each year. The annual yield spread and term yield spread designs reset the growth annually, but a yield spread is deducted from the equity index every year and at the contract's maturity, respectively. The index growth with point-to-point indexing is based on the growth between two time points over the entire term of the annuity. Particularly, the index growth with a high-water-mark feature is calculated to the highest index anniversary value. The cost of the EIA contract is reflected through the participation rate. Hence, the participation rate is expected to be lower for expensive designs.

Our examples involve five-year EIAs issued to a male aged 55 with minimum interest rate guarantee of either 3% on 100% of the premium or 3% on 90% of the premium. For illustration purposes, we assume that the insurance product values, $V^{(j)}(x, 0, n)$, $j = 1, 2, 3$ and $n = 1, 2, 3, 4, 5$, are determined using the standard deviation premium principle with a loading factor of 5.00% to the 1980 US CSO table. We also assume, unless specified, that the short-term rate $r(t)$ is still constant over time and is equal to 5%. The index is governed by the modified CRR model introduced previously with $S(0) = 1$ and where the number of trading dates N, unless specified, is 4. The index's volatility is assumed to be constant and is either 20% and 30%. In other words, $u(t) = e^{\sigma/\sqrt{N}}$ ($\sigma = 0.2, 0.3$) and $d(t) = u(t)^{-1}$.

Using (2.5) with $N = 4$, the index martingale probability of going up is either $\pi(t) = 0.536$ if $\sigma = 0.2$ or $\pi(t) = 0.503$ if $\sigma = 0.3$. We also assume that surrender value $SV(t)$ is zero, meaning that there no surrender. The analysis is performed using the point-to-point EIA class with term-end point and high-water-mark designs and also the term-end point design from the annual reset class.

8.1 Point-to-Point

We first consider one of the simplest classes of EIAs, known as the point-to-point. Their payoffs in year t can be represented by

$$D(t) = \max[\min[1 + \alpha R(t), (1 + \zeta)^t], \beta(1 + g)^t], \tag{8.1}$$

where α represents the participation rate and the ''gain'' $R(t)$ needs to be defined depending on the design. It also provides a protection against the loss from a down market $\beta(1 + g)^t$. The cap rate $(1 + \zeta)^t$ reduces the cost of such contract since it imposes an upper bound on the maximum return.

As explained in Lin and Tan (2004), an EIA is evaluated through its participation rate α. Without loss of generality we suppose that the initial value of EIA contracts is one monetary unit. The present value of the EIA is a function of the participation rate through the payoff function $D(n)$, $n = 1, 2, \ldots$. We then solve for α, the critical participation rate, such that $P(x, 0, n, \mathbf{i}_0) = 1$, where $P(x, 0, n, \mathbf{i}_0)$ is obtained using (5.17) for the first approach or using (5.20) for the second approach by holding all other parameter values constant.

8.1.1 Term-End Point

In practice, various designs for $R(t)$ have been proposed. The term-end point design is the simplest crediting method. It measures the index growth from the start to the end of a term. The index on the day the contract is issued is taken as the starting index, and the index on the day the policy matures or the time of death is taken as the ending index. Hence, the ''gain'' provided by the point-to-point EIA with term-end point may expressed as

$$R(t) = \frac{S(t)}{S(0)} - 1. \tag{8.2}$$

The EIA payoff given in (8.1) is defined by

$$D(t, \mathbf{i}_t) = \max\left[\min\left[1 + \alpha\left(\prod_{l=0}^{t} \gamma(l, i_l) - 1\right), (1 + \zeta)^t\right], \beta(1 + g)^t\right]. \tag{8.3}$$

Table 8 gives the critical participation rates based on equations (7.4) for the first approach and from (7.14) for the second approach over various numbers of trading dates ($N = 1, 2, 4, 8$). We still suppose that the short-term rate is equal to 5%, and the index's volatility is set to either 20% or 30%. We present the participation rates of five-year EIA contracts with the term-end design without cap rate ($\zeta = \infty$) for a policyholder aged 55. We consider two types of minimum guarantees: $\beta = 90\%$ and $\beta = 100\%$ and both with $g = 3\%$.

Under the independent copula, the difference between the participation with $N = 1$ and $N = 8$ is less than 2%. However, the difference in the participation levels when $N = 1$ and $N = 8$ can be greater than 6% with the lower copula. For example, we consider a three-year EIA contract with the same set of parameters, more precisely $\sigma = 0.3$ and $\beta = 100\%$; the participation rates, under the independent copula for various N, are 34.45% ($N = 1$), 33.35% ($N = 8$), 33.22% ($N = 25$), and 33.32% ($N = 50$) using the second approach. The participation rate converges and eight trading dates per year provide a good approximation. As expected, the participation rates for $\beta = 100\%$ are lower than the corresponding values with $\beta = 90\%$.

The stochastic ordering pointed out previously with the martingale probabilities is also observed with the term-end point design for the first approach. The width of participation rate bands for the first approach increase with the number of trading dates. Here the participation rate band represents the

Table 8

Point-to-Point with Term-End Design for Various Values of *N*

σ	N	First Approach							Second Approach		
		C^I	C^U	C^D	$C^{CJ}_{0.5}$	C^{CJ}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	1	61.63	58.42	65.50	59.76	58.72	64.83	58.86	69.64	69.56	69.76
	2	61.99	57.16	67.43	59.82	58.39	66.04	58.28	70.26	70.06	70.17
	4	61.93	55.04	69.43	59.63	57.98	66.53	57.64	70.13	69.79	69.82
	8	61.99	53.69	71.55	59.65	57.93	66.88	57.41	70.15	69.75	69.77
30%	1	48.18	45.15	51.71	46.56	45.57	51.16	45.56	54.59	54.52	54.66
	2	49.00	44.29	54.18	47.10	45.79	52.85	45.45	55.67	55.47	55.57
	4	48.91	42.11	56.39	46.91	45.45	53.29	44.85	55.53	55.16	55.25
	8	48.95	40.94	58.43	46.91	45.38	53.59	44.64	55.53	55.10	55.19
3% Minimum Guarantee on 100% Premium											
20%	1	44.54	42.65	46.84	43.40	42.81	46.44	42.91	54.51	54.64	54.35
	2	45.25	42.18	48.56	43.91	43.01	47.78	42.88	54.23	54.19	53.97
	4	44.20	40.16	48.28	43.00	42.13	46.71	41.76	53.36	53.14	52.91
	8	44.42	39.08	50.17	43.14	42.17	47.31	41.63	53.51	53.19	53.05
30%	1	32.86	31.02	35.07	31.84	31.25	34.72	31.26	41.16	41.26	41.07
	2	32.94	30.63	35.36	32.05	31.41	34.77	31.21	39.62	39.56	39.41
	4	32.26	28.34	36.40	31.22	30.45	34.63	30.00	39.79	39.55	39.44
	8	32.46	27.70	37.58	31.48	30.73	34.84	30.18	39.49	39.17	39.12

difference between the participation rates obtained from the lower copula and the upper copula. For example, we consider the set of parameters as the one introduced for the three-year EIA contract; the width of the bands for various N are 1.70% ($N = 1$), 5.87% ($N = 8$), 6.61% ($N = 25$), and 6.69% ($N = 50$) using the first approach. The participation rate converges slowly for the extreme dependence cases compared with the corresponding α under the independent copula. The dependence effects for the second approach are negligible since there is a natural ''hedging'' between the death and accumulation benefits.

As we increase the volatility of the index, the participation rate decreases since a higher volatility leads to more valuable embedded financial options. Generally, the width of the bands is function of the participation level. In other words, a higher participation level leads to a wider band.

In Table 9 we present the impact of the level of interest rates on the critical participation rates. We suppose that the short-term rate is equal to either 5%, 6%, 7%, or 8%, while the index's volatility remains 20% or 30%. The number of trading dates for the index is $N = 4$. Considering the five-year EIA contracts with the term-end design without cap rate, we use two types of minimum guarantees where β is either 90% or 100% with $g = 3\%$.

When increasing the interest rate from 5% to 8%, the participation rates for the case $\beta = 90\%$ increase by around 20%, and they increase by 25% for $\beta = 100\%$. Recall, however, that this assumption affects only the index martingale probabilities. Indeed, the index martingale probability of going up goes from $\pi(t) = 0.536$ to $\pi(t) = 0.572$ for $\sigma = 0.2$ and from $\pi(t) = 0.503$ to $\pi(t) = 0.527$ for $\sigma = 0.3$ when the short-term rates are, respectively, 5% and 8%. The width of the bands also increases as the short-term rates increase since the participation rates increase. Note that the participation rates for the second approach are similar to the ones presented by Lin and Tan (2004) with comparable interest rates.

Table 10 considers the effect of different cap rates on the five-year EIA contracts for the same set of parameters. Particularly, the short-term rate is equal to 5%, and the index's volatility is set to either 20% or 30%. We present the participation rates based on equations (7.4) for the first approach and from (7.14) for the second approach, where the cap rate ζ is either 12%, 15%, 20%, or ∞. We still consider two types of minimum guarantees: $\beta = 90\%$ and $\beta = 100\%$ and both with $g = 3\%$.

As expected, imposing a ceiling on the equity return that can be credited increases the participation rates. Furthermore, the magnitude of the increments is more significant in a high-volatility market.

Table 9

Point-to-Point with Term-End Design for Various Interest Rate Levels

		First Approach							Second Approach		
σ	$r(t)$	C^I	C^U	C^D	$C^{CJ}_{0.5}$	C^{CJ}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	5%	61.93	55.04	69.43	59.63	57.98	66.53	57.64	70.13	69.79	69.82
	6%	69.62	62.31	77.55	67.09	65.29	74.53	65.02	77.17	76.94	76.72
	7%	75.57	68.06	83.71	72.89	70.98	80.64	70.82	82.57	82.42	82.00
	8%	80.20	72.63	88.38	77.41	75.43	85.32	75.38	86.71	86.63	86.06
30%	5%	48.91	42.11	56.39	46.91	45.45	53.29	44.85	55.53	55.16	55.25
	6%	56.59	49.04	64.86	54.29	52.64	61.48	52.04	62.84	62.55	62.44
	7%	63.00	54.92	71.82	60.48	58.67	68.27	58.09	68.93	68.71	68.42
	8%	68.37	59.93	77.56	65.66	63.73	73.92	63.19	74.01	73.85	73.41
3% Minimum Guarantee on 100% Premium											
20%	5%	44.20	40.16	48.28	43.00	42.13	46.71	41.76	53.36	53.14	52.91
	6%	56.28	51.06	61.65	54.60	53.40	59.62	53.06	64.59	64.49	63.96
	7%	65.52	59.58	71.69	63.50	62.07	69.40	61.80	73.04	73.03	72.32
	8%	72.57	66.26	78.68	70.36	68.78	76.45	68.59	79.09	79.14	78.25
30%	5%	32.26	28.34	36.40	31.22	30.45	34.63	30.00	39.79	39.55	39.44
	6%	43.37	38.00	48.69	41.85	40.75	46.59	40.21	49.87	49.72	49.43
	7%	51.81	46.04	57.86	50.11	48.89	55.41	48.37	57.74	57.65	57.17
	8%	58.71	52.27	65.48	56.74	55.34	62.79	54.82	64.35	64.32	63.67

This is because the effect of the volatility diminishes as the cap rate decreases, and hence the behavior of the EIA payoff is similar for different ranges of volatilities. This is particularly observable when $\beta = 90\%$.

Table 11 considers the effect on the critical participation rates as we vary the surrender values. For these examples, we suppose that the surrender value defined by (7.1) is obtained using either $\delta_1(t) = 0$, $\delta_2(t) = 1 - 1\%(5 - t)$ (1% per year), $\delta_3(t) = 1 - 0.5\%(5 - t)$ (0.5% per year), or $\delta_4 = 100\%$ for $t = 0, \ldots, 5$. The EIA contract premiums, without cap rate, are obtained using (7.13) for the first approach and (7.19) for the second approach. The set of parameters is the same as the previous example except we set $\zeta = \infty$.

Table 10

Point-to-Point with Term-End Design for Various Cap Rates

		First Approach							Second Approach		
σ	ζ	C^I	C^U	C^D	$C^{CJ}_{0.5}$	C^{CJ}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	12%	72.18	62.45	83.07	68.34	65.75	78.98	65.89	87.15	88.67	86.15
	15%	65.46	57.92	74.23	62.57	60.58	70.92	60.50	76.31	77.46	75.66
	20%	62.51	55.83	70.39	60.17	58.52	67.32	58.30	71.52	72.22	71.07
	∞	61.93	55.04	69.43	59.63	57.98	66.53	57.64	70.13	69.79	69.82
30%	12%	73.97	60.54	88.74	68.90	65.52	83.15	65.43	94.43	100.78	91.78
	15%	59.20	50.39	69.04	56.08	53.93	65.20	53.65	71.25	73.95	69.82
	20%	52.21	44.86	60.54	49.78	48.07	57.21	47.66	60.60	61.78	60.08
	∞	48.91	42.11	56.39	46.91	45.45	53.29	44.85	55.53	55.16	55.25
3% Minimum Guarantee on 100% Premium											
20%	12%	46.38	42.28	51.26	44.83	43.87	49.41	43.69	58.90	59.95	58.11
	15%	44.81	40.87	49.14	43.50	42.63	47.48	42.36	55.14	55.81	54.54
	20%	44.27	40.21	48.35	43.07	42.19	46.78	41.83	53.67	53.86	53.16
	∞	44.20	40.16	48.28	43.00	42.13	46.71	41.76	53.36	53.14	52.91
30%	12%	37.96	32.59	44.01	36.18	34.94	41.58	34.60	50.73	51.88	50.02
	15%	34.59	30.29	39.29	33.29	32.38	37.36	32.02	44.82	46.04	44.03
	20%	32.87	28.82	37.18	31.74	30.92	35.37	30.50	41.21	41.75	40.76
	∞	32.26	28.34	36.40	31.22	30.45	34.63	30.00	39.79	39.55	39.44

Table 11
Point-to-Point with Term-End Design for Various Surrender Values

σ	*SV*	First Approach							Second Approach		
		C^I	C^U	C^D	$C^{CI}_{0.5}$	C^{CI}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	δ_1	61.93	55.04	69.43	59.63	57.98	66.53	57.64	70.13	69.79	69.82
	δ_2	61.53	54.71	67.65	59.27	57.64	65.69	57.28	69.44	69.05	69.25
	δ_3	59.30	54.10	63.86	57.88	56.77	62.37	56.28	66.94	66.60	66.71
	δ_4	52.39	48.98	55.46	51.50	50.76	54.46	50.29	60.25	60.14	60.14
30%	δ_1	48.91	42.11	56.39	46.91	45.45	53.29	44.85	55.53	55.16	55.25
	δ_2	48.28	41.72	53.25	46.46	45.02	51.39	44.42	54.48	54.04	54.26
	δ_3	45.97	41.03	49.36	44.79	43.84	48.21	43.24	51.79	51.50	51.60
	δ_4	40.50	37.35	43.24	39.80	39.19	42.23	38.70	46.34	46.24	46.24
3% Minimum Guarantee on 100% Premium											
20%	δ_1	44.20	40.16	48.28	43.00	42.13	46.71	41.76	53.36	53.14	52.91
	δ_2	42.00	38.76	44.68	41.19	40.54	43.78	40.12	50.23	49.89	49.96
	δ_3	36.16	34.12	37.83	35.75	35.38	37.23	35.01	44.11	43.96	44.01
	δ_4	20.25	19.48	20.71	20.16	20.04	20.55	19.87	29.86	29.86	29.86
30%	δ_1	32.26	28.34	36.40	31.22	30.45	34.63	30.00	39.79	39.55	39.44
	δ_2	30.11	27.02	32.30	29.46	28.91	31.49	28.46	36.48	36.19	36.27
	δ_3	25.29	23.39	26.75	24.95	24.64	26.18	24.29	31.29	31.16	31.21
	δ_4	13.81	13.15	14.21	13.74	13.65	14.05	13.51	20.88	20.88	20.88

The participation rates decrease by at least a half to cover the cost related to a contract without surrender charge δ_4 when $\beta = 100\%$. In this case the width of the band is approximately 1%. However, the surrender values for $\beta = 90\%$ have less impact on the participation rates since the insurer is less exposed to the financial guarantees. The surrender value that decreases by 1% per year reduces the participation rates by approximately 2% regardless of the scenario.

8.1.2 High-Water Mark

A more exotic structure that has appeared in EIAs is the high-water mark design. The interest earned at the end of term is based on the growth rate of the highest index value reached during the life of the policy over the index value at the beginning of the contract. The sampling frequency of the index is usually daily, monthly, or yearly. Those numerical examples focus on the yearly sampling since we need additional structures to evaluate other frequencies. The reader should refer to Gaillardetz (2006) for the daily and monthly sampling frequencies. The return may be expressed by

$$R(t) = \max_{k \in \{0,1,\ldots,t\}} \frac{S(k)}{S(0)} - 1. \tag{8.4}$$

The EIA payoff given in (8.1) is defined by

$$D(t, \mathbf{i}_t) = \max\left[\min\left[1 + \alpha\left(\max_{k \in \{0,1,\ldots,t\}} \prod_{l=0}^{k} \gamma(l, i_l) - 1\right), (1 + \zeta)^t\right], \beta(1 + g)^t\right]. \tag{8.5}$$

In Tables 12 and 13 we give the critical participation rates for various cap rates and surrender values, respectively, based on five-year EIA contracts with high-water mark design. Table 12 assumes EIA contracts without surrender value where the cap rate is either 12%, 15%, 20%, or ∞, while Table 13 assumes that the surrender value is either δ_1, δ_2, δ_3, or δ_4 with $\zeta = \infty$. In these numerical illustrations, we consider the same set of parameters as in the previous section ($r(t) = 5\%$, $\sigma = 20\%, 30\%$). Again, we present two types of minimum guarantees where β is either 90% or 100% with $g = 3\%$. The participation rates are obtained using (7.13) for the first approach and (7.19) for the second approach. We may use δ_1 for Table 12 or use the recursive formulas given by (7.4) and (7.14) for the first and the second approaches, respectively.

Table 12
Point-to-Point with High-Water Mark Design for Various Cap Rates

σ	ζ	First Approach					Second Approach				
		C^I	C^U	C^D	$C^G_{-0.5}$	$C^G_{0.5}$	C^I	C^U	C^D	$C^G_{-0.5}$	$C^G_{0.5}$
3% Minimum Guarantee on 90% Premium											
20%	12%	54.08	50.25	57.61	56.39	51.67	62.52	64.71	60.90	61.52	63.74
	15%	51.83	48.16	55.22	54.02	49.57	59.01	60.60	57.69	58.21	59.88
	20%	50.91	47.12	54.33	53.10	48.61	57.63	58.69	56.45	56.93	58.25
	∞	50.76	46.89	54.17	52.94	48.44	57.29	57.90	56.18	56.65	57.72
30%	12%	43.58	39.98	47.05	45.70	41.42	51.26	53.91	49.84	50.33	52.37
	15%	39.85	36.64	43.04	41.75	37.95	46.41	48.55	45.01	45.50	47.48
	20%	38.02	34.78	41.19	39.87	36.16	43.31	44.91	42.29	42.62	44.12
	∞	37.24	33.85	40.47	39.11	35.33	42.12	42.92	41.27	41.57	42.61
3% Minimum Guarantee on 100% Premium											
20%	12%	40.78	38.31	43.15	42.27	39.28	49.80	51.10	48.81	49.19	50.48
	15%	39.98	37.41	42.37	41.47	38.45	47.91	48.88	47.08	47.40	48.44
	20%	39.73	37.04	42.15	41.24	38.14	47.17	47.77	46.45	46.74	47.54
	∞	39.70	37.02	42.12	41.21	38.13	47.09	47.49	46.39	46.67	47.38
30%	12%	29.65	27.43	31.95	30.98	28.36	38.26	40.14	37.25	37.59	39.14
	15%	28.25	26.07	30.65	29.57	27.02	35.23	36.53	34.51	34.75	35.84
	20%	27.63	25.30	29.91	28.91	26.36	33.69	34.65	33.08	33.28	34.18
	∞	27.45	25.11	29.73	28.73	26.16	33.13	33.70	32.59	32.77	33.48

The high-water mark design is more expensive than the term-end point design. The participation rates observed for the high-water mark design are consistently lower than the respective rates in the term-end point design. Under the second approach, the participation rates appear in reverse order, comparatively to those obtained from the first approach. This is because the death benefits are relatively more valuable than the accumulation benefits. There is no difference between the high-water mark design and the term-end point design for $\beta = 100\%$ when considering the EIA contract without surrender charge δ_4.

Table 13
Point-to-Point with High-Water Mark Design for Various Surrender Values

σ	SV	First Approach					Second Approach				
		C^I	C^U	C^D	$C^G_{-0.5}$	$C^G_{0.5}$	C^I	C^U	C^D	$C^G_{-0.5}$	$C^G_{0.5}$
3% Minimum Guarantee on 90% Premium											
20%	δ_1	50.76	46.89	54.17	52.94	48.44	57.29	57.90	56.18	56.65	57.72
	δ_2	43.87	41.03	46.42	45.55	42.11	49.53	49.92	49.12	49.23	49.78
	δ_3	40.76	38.19	43.10	42.30	39.18	46.38	46.70	46.13	46.19	46.57
	δ_4	36.45	34.29	38.30	37.70	35.14	42.99	43.27	42.89	42.90	43.14
30%	δ_1	37.24	33.85	40.47	39.11	35.33	42.12	42.92	41.27	41.57	42.61
	δ_2	30.96	28.53	33.24	32.33	29.56	34.99	35.52	34.65	34.73	35.29
	δ_3	28.33	26.16	30.39	29.56	27.09	32.32	32.78	32.11	32.15	32.57
	δ_4	24.73	22.90	26.37	25.73	23.70	29.35	29.77	29.23	29.24	29.55
3% Minimum Guarantee on 100% Premium											
20%	δ_1	39.70	37.02	42.12	41.21	38.13	47.09	47.49	46.39	46.67	47.38
	δ_2	36.40	34.22	38.40	37.66	35.11	42.59	42.91	42.28	42.37	42.80
	δ_3	32.95	30.92	34.69	34.06	31.76	39.11	39.32	38.96	39.00	39.23
	δ_4	20.25	19.48	20.70	20.55	19.87	29.86	29.86	29.86	29.86	29.86
30%	δ_1	27.45	25.11	29.73	28.73	26.16	33.13	33.70	32.59	32.77	33.48
	δ_2	24.68	22.82	26.49	25.71	23.66	29.14	29.56	28.89	28.94	29.38
	δ_3	21.83	20.23	23.33	22.69	20.96	26.22	26.54	26.08	26.10	26.39
	δ_4	13.81	13.15	14.19	14.04	13.51	20.62	20.71	20.60	20.61	20.66

8.2 Annual Reset

We now consider the most popular class of EIAs, known as the annual reset. They appeal to investors because they offer similar features as the point-to-point class; however, the interest credited to a annual reset EIA contract cannot be lost. This ‘‘lock-in’’ feature protects the investor against a poor performance of the index over a particular year. The payoff of this type of EIA contracts is defined by

$$D(t) = \max\left[\prod_{l=1}^{t} \max[\min[1 + \alpha R(l) - \nu, (1 + \zeta)], 1], \beta(1 + g)^t\right], \tag{8.6}$$

where $R(l)$ represents the realized ‘‘gain’’ in year l, which varies from product to product.

The cases where ν is set to 0 are known as annual reset EIAs, and the cases where $\nu > 0$ are known as annual yield spreads. Furthermore, the participation levels in those cases $\nu > 0$ are typically 100%. As mentioned previously, in the case of annual reset, we fix $\nu = 0$ and determine the critical participation rate α while fixing g, β, and ζ. In the traditional yield spread ν needs to be determined such that the cost of EIA embedded options are covered by the initial premium while fixing $\alpha = 100\%$, g, β, and ζ.

8.2.1 Term-End Point

In the case of annual reset EIA with term-end point, the index return is calculated each year by comparing the indices at the beginning and ending policy anniversaries. Hence, the participation rate may be expressed as

$$R(t) = \frac{S(t)}{S(t - 1)} - 1. \tag{8.7}$$

In this case, the EIA payoff given in (8.6) is defined by

$$D(t, \mathbf{i}_t) = \max\left[\prod_{l=1}^{t} \max[\min[1 + \alpha(\gamma(l, i_l) - 1) - \nu, (1 + \zeta)], 1], \beta(1 + g)^t\right]. \tag{8.8}$$

Table 14

Annual Reset with Term-End Design for Various Cap Rate

σ	ζ	First Approach							Second Approach		
		C^I	C^U	C^D	$C^{CJ}_{0.5}$	C^{CJ}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	18%	46.40	42.70	49.43	45.49	44.68	48.55	44.10	54.33	54.27	54.36
	20%	43.79	40.08	47.13	42.87	42.04	46.05	41.34	51.72	51.66	51.75
	22%	42.57	40.08	44.82	42.01	41.50	44.00	41.07	49.11	49.04	49.14
	∞	42.57	40.08	44.79	42.01	41.50	44.00	41.07	47.39	47.36	47.42
30%	18%	36.54	33.91	38.68	35.89	35.29	38.10	34.87	42.21	42.15	42.24
	20%	35.09	32.14	37.45	34.44	33.82	36.72	33.33	40.76	40.70	40.79
	22%	33.64	30.37	36.23	32.99	32.36	35.34	31.80	39.32	39.25	39.35
	∞	31.07	28.85	33.05	30.66	30.25	32.24	29.85	34.63	34.59	34.65
3% Minimum Guarantee on 100% Premium											
20%	18%	38.59	35.78	41.34	37.90	37.27	40.38	36.64	47.90	47.83	47.88
	20%	37.74	35.78	39.45	37.35	36.98	38.80	36.62	44.90	44.80	44.88
	22%	37.74	35.78	39.45	37.35	36.98	38.80	36.62	43.10	43.03	43.09
	∞	37.74	35.78	39.45	37.35	36.98	38.80	36.62	43.10	43.03	43.09
30%	18%	30.26	27.65	32.30	29.75	29.27	31.60	28.82	37.33	37.28	37.32
	20%	28.49	25.50	30.77	27.98	27.49	29.89	26.96	35.57	35.49	35.57
	22%	26.73	24.95	29.24	26.44	26.15	28.18	25.81	33.80	33.70	33.80
	∞	26.73	24.95	28.28	26.44	26.15	27.60	25.81	30.84	30.76	30.84

In Tables 14 and 15 we give the participation rates for various cap rates and surrender values, respectively, based on the term-end point design (annual reset class). In this numerical illustration, we consider the same set of parameters as in the high-water mark design case. Table 14 assumes EIA contracts without surrender value (δ_1), where the cap rate is either 18%, 20%, 22%, or ∞, while Table 15 assumes that the surrender value is either δ_1, δ_2, δ_3, or δ_4 with $\zeta = \infty$. We use different cap rates since the embedded financial guarantees are too expensive. Indeed, the policyholder would need more compensation than what the cap rates may allow. The participation rates are obtained using (7.13) for the first approach and (7.19) for the second approach.

The annual reset with term-end point design is more expensive than the point-to-point with the term-end point design as well as the high-water mark design. The participation rates observed for the high-water mark design are consistently higher than the respective rates in the previous EIAs. There is no difference between annual reset and the point-to-point design for $\beta = 100\%$ when considering the EIA contract without surrender charge δ_4. The effect of the different surrender values on the participation rates is more sensible than the other designs presented previously. The cost related to δ_2 is zero as well as for δ_3 when $\beta = 90\%$. However, as for the previous designs, the participation rates for $\beta = 100\%$ are almost halved for the full surrender option δ_4.

Table 16 considers an annual yield spread EIA with term-end point design for various cap rates ($\zeta = 18\%, 20\%, 22\%, \infty$) while there is no surrender value. Using the same set of parameters, we find ν such that (7.4) for the first approach and (7.14) for the second approach are equal to 1 by setting $\alpha = 100\%$.

The pattern of the yield spread is not similar to the one presented by the participation rates (Table 14). The participation rates for $\sigma = 20\%$ remain the same while the cap rate goes from 22% to ∞. However, the yield spreads still increase for this particular case.

9. CONCLUSIONS

The purpose of this paper is to present a market consistent valuation technique for equity-linked products that can be implemented in practice. To this end, we introduce age-dependent, mortality risk-adjusted martingale probability measures for each of the term life, pure endowment, and endowment insurances using their corresponding premiums. This is an improvement over the traditional approach,

Table 15

Annual Reset with Term-End Design for Various Surrender Values

		First Approach							Second Approach		
σ	*SV*	C^I	C^U	C^D	$C^{CI}_{0.5}$	C^{CI}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	δ_1	42.57	40.08	44.79	42.01	41.50	44.00	41.07	47.39	47.36	47.42
	δ_2	42.57	40.08	44.79	42.01	41.50	44.00	41.07	47.39	47.36	47.42
	δ_3	42.57	40.08	44.79	42.01	41.50	44.00	41.07	47.39	47.36	47.42
	δ_4	38.47	36.40	39.78	38.10	37.66	39.43	37.27	46.63	46.58	46.66
30%	δ_1	31.07	28.85	33.05	30.66	30.25	32.24	29.85	34.63	34.59	34.65
	δ_2	31.07	28.85	33.05	30.66	30.25	32.24	29.85	34.63	34.59	34.65
	δ_3	31.07	28.85	33.05	30.66	30.25	32.24	29.85	34.63	34.59	34.65
	δ_4	28.30	26.41	29.43	28.02	27.67	29.08	27.30	33.93	33.88	33.96
3% Minimum Guarantee on 100% Premium											
20%	δ_1	37.74	35.78	39.45	37.35	36.98	38.80	36.62	43.10	43.03	43.09
	δ_2	37.60	35.64	39.31	37.21	36.85	38.66	36.49	42.92	42.85	42.92
	δ_3	35.47	33.68	36.87	35.16	34.83	36.36	34.48	42.20	42.07	42.22
	δ_4	20.25	19.45	20.71	20.15	20.02	20.55	19.85	29.86	29.86	29.86
30%	δ_1	26.73	24.95	28.28	26.44	26.15	27.60	25.81	30.84	30.76	30.84
	δ_2	26.64	24.85	28.17	26.35	26.06	27.50	25.72	30.70	30.61	30.71
	δ_3	25.05	23.47	26.15	24.82	24.57	25.69	24.25	29.82	29.72	29.84
	δ_4	13.81	13.10	14.20	13.74	13.64	14.05	13.49	20.88	20.88	20.88

Table 16

Annual Yield Spread with Term-End Design for Various Cap Rates

σ	ζ	First Approach							Second Approach		
		C^I	C^U	C^D	$C^{CJ}_{0.5}$	C^{CJ}_{2}	$C^{G}_{-0.5}$	$C^{G}_{0.5}$	C^I	C^U	C^D
3% Minimum Guarantee on 90% Premium											
20%	18%	11.87	12.69	11.20	12.07	12.25	11.39	12.38	10.11	10.13	10.10
	20%	12.45	13.36	11.71	12.65	12.83	11.94	12.99	10.69	10.70	10.68
	22%	13.02	14.04	12.22	13.23	13.42	12.50	13.60	11.27	11.28	11.26
	∞	16.20	17.37	15.26	16.36	16.54	15.68	16.77	14.84	14.85	14.83
30%	18%	22.20	23.12	21.45	22.43	22.64	21.66	22.79	20.22	20.24	20.21
	20%	22.71	23.74	21.88	22.94	23.15	22.14	23.32	20.72	20.75	20.71
	22%	23.22	24.36	22.31	23.45	23.66	22.62	23.86	21.23	21.25	21.22
	∞	30.70	33.27	29.01	30.93	31.19	29.93	31.70	29.11	29.13	29.10
3% Minimum Guarantee on 100% Premium											
20%	18%	13.60	14.38	12.99	13.75	13.89	13.20	14.03	11.53	11.55	11.54
	20%	14.27	15.44	13.58	14.42	14.60	13.85	14.84	12.20	12.22	12.20
	22%	15.14	16.54	14.18	15.35	15.55	14.50	15.83	12.87	12.89	12.87
	∞	19.68	21.30	18.36	19.85	20.05	19.03	20.43	17.02	17.11	17.01
30%	18%	24.40	25.31	23.69	24.58	24.74	23.93	24.90	21.93	21.94	21.93
	20%	25.02	26.07	24.22	25.20	25.37	24.53	25.56	22.54	22.57	22.54
	22%	25.64	26.82	24.76	25.82	25.99	25.13	26.21	23.16	23.20	23.16
	∞	43.33	49.04	38.14	43.78	44.39	41.27	45.73	34.95	36.01	34.92

under which diversification of mortality risk is assumed. Equity-linked martingale measures are derived using copulas that combined the premium information from the insurance market and the financial market. Although the choice of a copula is somewhat arbitrary, with additional premium information from certain equity-linked products, we would be able to narrow down the choices. We present two different pricing approaches for equity-linked products. The first approach evaluates death benefits and accumulation/survival benefits separately. In the second approach, we evaluate the death benefits and the survival benefits in a unified manner by using the endowment insurance products to define the martingale measure. The early withdrawal risk is also discussed since it is a relevant issue when pricing EIAs. Similar to American options, we presented a recursive algorithm based on maximization arguments using a predetermined cash surrender value. A detailed numerical analysis is then performed for various existing EIAs in the North American market.

Our methodology may be used to evaluate variable annuities (segregated fund contracts in Canada) because of the similarity in payoff structure between EIAs and VAs. Furthermore, our approach may also be used to evaluate universal life insurances, variable universal life insurances, and other equity-linked products.

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