Optimal and Simple, Nearly Optimal Rules for Minimizing the Probability of Financial Ruin in Retirement

Kristen S. Moore* and Virginia R. Young†

Abstract

The increasing risk of poverty in retirement has been well documented; it is projected that current and future retirees' living expenses will significantly exceed their savings and income. In this paper, we consider a retiree who does not have sufficient wealth and income to fund her future expenses, and we seek the asset allocation that minimizes the probability of financial ruin during her lifetime. Building on the work of Young (2004) and Milevsky, Moore, and Young (2006), under general mortality assumptions, we derive a variational inequality that governs the ruin probability and optimal asset allocation. We explore the qualitative properties of the ruin probability and optimal strategy, present a numerical method for their estimation, and examine their sensitivity to changes in model parameters for specific examples. We then present an easy-to-implement allocation rule and demonstrate via simulation that it yields nearly optimal ruin probability, even under discrete portfolio rebalancing.

1. Introduction and Motivation

The increasing risk of poverty in retirement has been well documented (Parikh 2003). It is projected that retired Americans' living expenses will exceed their financial resources by $400 billion over the 10-year period 2020–30 (VanDerhei and Copeland 2003). This shortfall is driven by demographic trends, the increased longevity of our aging population, changes in Social Security, inadequate private retirement savings, and the continuing trend toward defined contribution plans such as 401(k)'s, under which the individual, not the employer, assumes all investment and longevity risk. Individuals now have more responsibility for managing their retirement portfolios; in 1998, 62.7% of individuals who participated in a retirement plan had a defined contribution plan as their primary plan, compared to 49.8% in 1993 (Copeland 2002).

In this paper, we seek the investment strategy for a retiree with fixed income and expenses (in real or nominal terms) that minimizes the probability of financial ruin during her lifetime. Beginning with Merton (1969), researchers in economics, finance, and mathematics have studied strategies for maximizing expected utility of consumption or bequest; see Bayraktar and Young (2005b) for additional references on this problem. If one were to maximize expected utility of consumption with a power utility function, then the optimal consumption is a multiple of wealth; therefore, bankruptcy is not possible. Young (2004) and Bayraktar and Young (2005a) studied the connection between this problem and the problem of minimizing the probability of lifetime ruin under consumption that is proportional to wealth (with a positive ruin level). They showed that it is possible to choose the parameters in the two problems so that the consumption and investment strategies are *identical* in the two problems. Therefore, one can reinterpret some results from expected utility maximization and put them in the context of minimizing the probability of ruin.

* Kristen S. Moore is an Assistant Professor in the Department of Mathematics, University of Michigan, Ann Arbor, MI, ksmoore@umich.edu.

† Virginia R. Young is a Professor in the Department of Mathematics, University of Michigan, Ann Arbor, MI, vryoung@umich.edu.

Because a utility function is a subjective measure of an investor's attitude toward risk and wealth, and investors do not know their utility function, utility maximization might be difficult. On the other hand, minimizing the probability of lifetime ruin is an objective and understandable goal. In fact, the Nobel laureate William Sharpe founded a financial services advisory firm that is largely based on using probabilities to provide investment advice.

Several papers in the risk and portfolio management literature revitalized the Roy (1952) Safety-First rule and applied the concept to maximizing the probability of achieving certain investment goals. For example, Browne (1995, 1997, 1999a,b,c) derived the optimal dynamic strategy for a portfolio manager who seeks to maximize the probability of reaching a ''safe'' level before ruin. The probability of lifetime ruin has been studied by Milevsky, Ho, and Robinson (1997), Milevsky and Robinson (2000), Albrecht and Mauer (2002), Orszag (2002), Gerrard, Haberman, and Vigna (2004), Dus, Mitchell, and Maurer (2005), Young (2004), and Milevsky, Moore, and Young (2006).

Young (2004) considered an individual who targets a specific rate of consumption and can invest in and trade dynamically between a risk-free bond and a risky stock or index. Using techniques from stochastic optimal control, Young derived the Hamilton-Jacobi-Bellman (HJB) equation that governs the optimal asset allocation and minimal ruin probability. Under the simplifying assumption of constant force of mortality (or hazard rate function), the HJB equation is a nonlinear ordinary differential equation (ODE), which Young solved in closed form. The result is concise, closed-form expressions for the minimal ruin probability and optimal investment strategy. Indeed, in this work, under the model assumptions, Young prescribed an easy-to-follow investment rule.

The assumption of constant force of mortality is equivalent to assuming that the investor's future lifetime random variable has exponential distribution. Because of the memoryless property of exponential random variables, this assumption is unrealistically simple. Milevsky, Moore, and Young (2006) demonstrated that, for a related problem, the *shape* of the hazard rate function significantly impacts the optimal investment strategy; thus, in this paper, we consider more general mortality assumptions.

Under more general mortality assumptions, the HJB equation that governs the ruin probability and optimal strategy is a nonlinear partial differential equation (PDE) with fixed boundary conditions. We transform this problem to a linear free-boundary problem (FBP), which we solve numerically. In addition, we employ super- and subsolution methods to prove results about the qualitative behavior of smooth solutions to the PDE; these results are useful in the implementation of the numerical method.

By numerically solving the FBP, we are able to approximate the optimal investment rule and ruin probability. However, to implement this method, an investor must know the hazard rate function and rebalance her portfolio continuously. To address this limitation, we prescribe an easy-to-implement investment rule, based on an estimated hazard rate and the results of Young (2004), that yields nearly optimal ruin probability, even under discrete (versus continuous) portfolio rebalancing.

The layout of the paper is as follows. In Section 2, we describe the model and present the HJB equation that governs the ruin probability and optimal strategy. In Section 3, we summarize the results of Young (2004), including the easy-to-follow investment rule. In Section 4, we transform the nonlinear HJB equation to a linear FBP via scaling and duality arguments. In Section 5, we prove the existence of an upper bound on the ruin probability and other qualitative properties of smooth solutions that aid in the implementation of the numerical approximation. In Section 6, we present numerical results for specific examples. In Section 7, we present a modification of the easy-to-follow investment rule of Young (2004). We use simulation to show that this rule yields nearly optimal ruin probability, even under discrete portfolio rebalancing. Our results are consistent with those of Rogers (2001), Rogers and Stapleton (2002), and Browne, Milevsky, and Salisbury (2003). We conclude in Section 8.

2. THE MODEL

In this section, we describe the financial market in which the individual can invest her wealth and formulate the problem of minimizing the probability of lifetime ruin. We consider (x), an individual aged x with future lifetime described by the random variable $T(x)$. Let $\lambda_x(t)$ be the hazard rate function for $T(x)$ (or the *force of mortality* for (x)). Young (2004) considered the special case of constant force of mortality; that is, $T(x)$ has exponential distribution.

In this paper, we assume that the individual consumes wealth at a constant rate of c; this rate might be given in real or nominal units. One can interpret c as the minimum consumption level below which an individual cannot (or will not) reduce her consumption further; therefore, the minimum probability of lifetime ruin that we compute gives the individual a lower bound for her probability of ruin under any consumption function bounded below by c.

Under a fixed hazard rate, Young (2004) and Bayraktar and Young (2005a) considered consumption rates that increase with wealth. We expect that the work in this paper, namely, finding nearly optimal investment strategies under a variable hazard rate, extends to those consumption functions. The reason for this belief is that when Bayraktar and Young (2005a) found the probability of ruin under the consumption rate $c(w) = \bar{c} + \rho(w - d)_+$, in which $\bar{c}$, ρ, and d are positive constants, the optimal investment strategy for wealth below d was qualitatively similar to the strategy when $c(w) \equiv \bar{c}$.

We assume that she invests in a riskless asset whose price at time s, X_s, follows the process $dX_s = rX_s ds$, $X_t = x > 0$, for some fixed force of interest $r \geq 0$. Also, the individual invests in a risky asset whose price at time s, S_s, follows geometric Brownian motion given by

$$\begin{cases} dS_s = \mu S_s ds + \sigma S_s dB_s, \\ S_t = S > 0, \end{cases}$$

in which $\mu > r$, $\sigma > 0$, and B is a standard Brownian motion with respect to a filtration $\{\mathcal{F}_s\}$ of the probability space $(\Omega, \mathcal{F}, P)$. If c is given as a real rate of consumption, then we express r and μ as real rates.

Finally, we assume that the individual has a preexisting constant rate of income $A < c$, which can represent the income rate from a job, Social Security, or a pension. If the rate of consumption c is real, then we assume that A is also real. Note that the individual's annual shortfall is $c - A$; the present value of this shortfall, in perpetuity, at the risk-free rate r is $(c - A)/r$.

Let W_s be the wealth at time s of the individual, and let π_s be the amount that she invests in the risky asset at time s. It follows that the amount invested in the riskless asset is $W_s - \pi_s$. Thus, wealth follows the process

$$\begin{cases} dW_s = [rW_s + (\mu - r)\pi_s + A - c]ds + \sigma\pi_s dB_s \\ W_t = w \end{cases}. \tag{2.1}$$

Milevsky, Moore, and Young (2006) include the possible purchase of annuities to guarantee part or all of the desired consumption rate c. In this work, however, we restrict our attention to individuals who self-annuitize; that is, we do not include annuities in the set of investments.

By "lifetime ruin," we mean that the individual's wealth hits a specified value $w_l \geq 0$ before she dies. The value w_l could represent assets below which an individual qualifies for social assistance or considers herself impoverished. Denote the minimum probability that the individual outlives her wealth by $\psi(w, t)$, in which the arguments w and t indicate the one conditions the probability on the individual possessing wealth w at time t. This notation is consistent with that used for the ruin of a surplus process of an insurance company (Bowers et al. 1997), namely, $\psi(u)$, the probability of eventual ruin given that the initial surplus is u.

Let τ_l denote the first time that wealth equals w_l, and let τ_d denote the random time of death of (x). Thus, ψ is the minimum probability that $\tau_l < \tau_d$, in which one minimizes with respect to admissible investment strategies $\{\pi_s\}$. A strategy $\{\pi_s\}$ is admissible if it is $\mathcal{F}_s$-progressively measurable and if it satisfies the condition $\int_0^t \pi_s^2 \, ds < \infty$, almost surely, for $t \geq 0$. Thus, one can express ψ as

$$\psi(w, t) = \inf_{\{\pi_s\}} Pr[\tau_l < \tau_d | W_t = w]. \tag{2.2}$$

Since the risk-free asset alone is sufficient to fund the investor's annual shortfall in perpetuity for wealth in excess of $(c - A)/r$, we have that $\psi((c - A)/r, t) = 0$. We define $w_u := (c - A)/r$ and observe that the investor is safe from ruin at this level of wealth. Note also that $\psi(w_l, t) = 1$.

By using the principle of dynamic programming and Itô's Lemma, it follows that for $w \in (w_l, w_u) \times (0, \infty)$, ψ is the unique smooth solution of the following HJB equation:

$$\begin{cases} \lambda_x(t)\psi = \psi_t + (rw + A - c)\psi_w + \min_\pi \left[(\mu - r)\pi\psi_w + \frac{1}{2}\sigma^2\pi^2\psi_{ww} \right], \\ \psi(w_l, t) = 1, \\ \psi(w_u, t) = 0, \\ \lim_{s\to\infty} {}_sp_{x+t} \mathrm{E}[\psi(W_s^*, s)|W_t^* = w] = 0, \end{cases} \tag{2.3}$$

in which W_s^* denotes the optimally controlled wealth at time s. See Young (2004) for a derivation of this equation, and see Bayraktar and Young (2005b) for a related verification theorem.

The optimal investment strategy π^* is given by the first-order condition from the HJB equation (2.3):

$$\pi^*(w, t) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(w, t)}{\psi_{ww}(w, t)}. \tag{2.4}$$

The optimal investment *process* is, therefore, given by

$$\Pi_t^* = \pi^*(W_t^*, t) = -\frac{\mu - r}{\sigma^2} \frac{\psi_w(W_t^*, t)}{\psi_{ww}(W_t^*, t)}. \tag{2.5}$$

3. PRIOR RESULTS FOR A SPECIAL CASE

Young (2004) considers the special case of constant force of mortality, that is, $\lambda_x(t) \equiv \lambda$; this is equivalent to assuming that the random variable $T(x)$ has exponential distribution with parameter λ. Because of the memoryless property of the exponential distribution, this is not a realistic model for human survival. Moreover, Milevsky, Moore, and Young (2006) demonstrated that, for a related problem, the optimal investment strategy was quite sensitive to the shape of the hazard rate function.

Despite these limitations, this special case provides valuable insight into the qualitative properties of the ruin probability and optimal strategy; we discuss this further in Section 5. Moreover, the concise, closed-form optimal investment rule given in Young (2004) is the basis of the easy-to-implement, nearly optimal rule that we describe in Section 7.

Under the assumption of constant force of mortality, ψ is independent of time, and the HJB equation (2.3) is a boundary value problem (BVP) for a nonlinear ODE, which Young solves explicitly. For $w \in (w_l, w_u)$, the ruin probability and optimal strategy are given by

$$\psi(w) = \left(\frac{c - A - rw}{c - A - rw_l} \right)^p, \tag{3.1}$$

$$\pi^*(w) = \frac{\mu - r}{\sigma^2} \frac{c - A - rw}{(p - 1)r}, \tag{3.2}$$

in which

$$p = \frac{1}{2r} \left[(r + \lambda + m) + \sqrt{(r + \lambda + m)^2 - 4r\lambda} \right] > 1, \tag{3.3}$$

and

$$m = \frac{1}{2} \left(\frac{\mu - r}{\sigma} \right)^2.$$

We make the following observations about these results:

- The expression (3.2) above for π^* gives an easy-to-implement investment rule but requires continuous portfolio rebalancing. We call this strategy "continuous p-rebalancing."
- The ruin probability ψ is convex in wealth. Based on this observation, we seek a convex solution to the PDE in the more general case. More precisely, in Section 4, we seek a concave solution to the dual variational inequality. We justify the ansatz of convexity in Section 4.2.
- Note that the ruin probability ψ and the allocation π^* to the risky asset decrease as the wealth w increases. In particular, $\pi^*(w_u) = 0$; thus, once wealth reaches the "safe" level, the individual invests 100% in the risk-free asset.
- ψ and π^* decrease as the hazard rate λ increases.
- $\psi(w_u) = \psi'(w_u) = 0$; the ruin probability "flattens out" at $w = w_u$.

We see in Sections 5 and 6 that several of these properties are preserved under more general mortality.

4. Derivation of the Variational Inequality

In this section, we transform the nonlinear BVP in (2.3) to a linear FBP via the Legendre transform; see Karatzas and Shreve (1998). By exploiting the connection between free-boundary and optimal stopping problems, we recast the FBP as a variational inequality, which we solve numerically in Section 6. We explore qualitative properties of solutions to the HJB equation (and the dual FBP) in Section 5.

4.1 Linearizing the Equation for ψ via Duality

To linearize the HJB equation in (2.3), we first define

$$f(w, t) = {}_tp_x \psi(w, t),$$

in which

$${}_tp_x = \exp\left(-\int_0^t \lambda_x(s)\, ds\right)$$

is the conditional probability that (x) survives until age $x + t$. Then the PDE in (2.3) becomes

$$f_t + (rw + A - c)f_w + \min_{\pi} \left[(\mu - r)\pi f_w + \frac{1}{2}\sigma^2\pi^2 f_{ww} \right] = 0, \tag{4.1}$$

with boundary conditions $f(w_l, t) = {}_tp_x$ and $f(w_u, t) = 0$ and with transversality condition $\lim_{s\to\infty} \mathrm{E}[f(W_s^*, s)|W_t = w] = 0$. This condition can be rewritten as $\lim_{t\to\infty} f(w, t) = 0$ with probability 1 because $0 \le f \le 1$.

Next, we assume that ψ, or equivalently f, is convex with respect to w (as is the case when the hazard rate is constant), so we can define the Legendre transform of f as follows (Karatzas and Shreve 1998):

$$\tilde{f}(y, t) = \min_{w > w_1} [f(w, t) + wy]. \tag{4.2}$$

The critical value in the minimization of (4.2) w^* solves the equation $f_w(w, t) + y = 0$; thus, $w^* = I(-y, t)$, in which I is the inverse of f_w with respect to w. It follows that

$$\tilde{f}(y, t) = f[I(-y, t), t] + yI(-y, t). \tag{4.3}$$

Note that

$$\begin{aligned} \tilde{f}_y(y, t) &= -f_w[I(-y, t), t]I_y(-y, t) + I(-y, t) - yI_y(-y, t) \\ &= yI_y(-y, t) + I(-y, t) - yI_y(-y, t) \\ &= I(-y, t). \end{aligned} \tag{4.4}$$

We can retrieve the function f from $\tilde{f}$ by the relationship

$$f(w, t) = \max_{y>0} [\tilde{f}(y, t) - wy]. \tag{4.5}$$

Indeed, the critical value y^* solves the equation $\tilde{f}_y(y, t) - w = 0$; thus, $y^* = -f_w(w, t)$, and

$$\begin{aligned}\tilde{f}(y^*, t) - wy^* &= f[I(-y^*, t), t] + y^*I(-y^*, t) - wy^* \\ &= f[I(f_w(w, t), t), t] - f_w(w, t)I(f_w(w, t), t) + wf_w(w, t) \\ &= f(w, t) - wf_w(w, t) + wf_w(w, t) \\ &= f(w, t),\end{aligned}$$

in which we use equation (4.3) for the first equality. In Section 4.2, we demonstrate that $\tilde{f}$ is concave with respect to y; therefore, our assumption that f is convex is founded.

Next, note that

$$\tilde{f}_{yy}(y, t) = -I_y(-y, t) = -1/f_{ww}[I(-y, t), t], \tag{4.6}$$

and

$$\begin{aligned}\tilde{f}_t(y, t) &= f_w[I(-y, t), t]I_t(-y, t) + f_t[I(-y, t), t] + yI_t(-y, t) \\ &= -yI_t(-y, t) + f_t[I(-y, t), t] + yI_t(-y, t) \\ &= f_t[I(-y, t), t].\end{aligned} \tag{4.7}$$

In the partial differential equation for f, let $w = I(-y, t)$ to obtain

$$f_t[I(-y, t), t] + (rI(-y, t) + A - c)f_w[I(-y, t), t] - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 \frac{(f_w[I(-y, t), t])^2}{f_{ww}[I(-y, t), t]} = 0.$$

Rewrite this equation in terms of $\tilde{f}$ to get

$$\tilde{f}_t(y, t) + (rI(-y, t) + A - c)(-y) - m\frac{(-y)^2}{-1/\tilde{f}_{yy}(y, t)} = 0,$$

or equivalently,

$$\tilde{f}_t(y, t) - ry\tilde{f}_y(y, t) + my^2\tilde{f}_{yy}(y, t) + (c - A)y = 0, \tag{4.8}$$

with boundary conditions given implicitly by $f(w_l, t) = {}_tp_x$ and $f(w_u, t) = 0$. Note that (4.8) is a *linear* partial differential equation. Now, consider the boundary conditions $f(w_l, t) = {}_tp_x$ and $f(w_u, t) = 0$. If $f_w < 0$ is strictly increasing with respect to w, we have $y_l(t) > y_u(t) \geq 0$ for all $t \geq 0$, in which $y_l(t)$ and $y_u(t)$ are defined by

$$y_l(t) = -f_w(w_l, t), \tag{4.9}$$

and

$$y_u(t) = -f_w(w_u, t). \tag{4.10}$$

Thus, the boundary conditions become

$$\tilde{f}(y_l(t), t) = {}_tp_x + w_ly_l(t), \quad \text{for } \tilde{f}_y(y_l, t) = w_l \tag{4.11}$$

and

$$\tilde{f}(y_u(t), t) = w_u y_u(t) \quad \text{for } \tilde{f}_y(y_u(t), t) = w_u. \tag{4.12}$$

The transversality condition $\lim_{t\to\infty} f(w, t) = 0$ with probability 1 becomes $\lim_{t\to\infty} \tilde{f}(y, t) = 0$ with probability 1. Note that the first equations in (4.11) and (4.12) are reminiscent of value-matching conditions, while the second equations are reminiscent of smooth pasting conditions. We exploit this observation in the next section, where we express $\tilde{f}$ as the value function for an optimal stopping problem.

4.2 Optimal Stopping Formulation

Recall that in Section 3, we saw that in the special case of constant hazard rate, the ruin probability ψ is a decreasing, convex function of wealth. Thus, we seek a decreasing, convex solution in the general case. Equivalently, we seek an increasing, concave solution to the dual FBP (4.8), (4.11), (4.12). In this section, we demonstrate that the solution to the dual FBP is indeed concave.

The conditions (4.11) and (4.12) motivate us to define a penalty function u by

$$u(y, t) = \min({}_tp_x + w_l y, w_u y). \tag{4.13}$$

Since u is maximal among those functions that are concave in y and satisfy the boundary conditions in (4.11) and (4.12), it follows that $\tilde{f}(y, t) \le u(y, t)$ for all (y, t) such that $y_u(t) \le y \le y_l(t)$.

Define a stochastic process Y_s by

$$\begin{cases} dY_s = -rY_s\, ds + \dfrac{\mu - r}{\sigma} Y_s\, d\tilde{B}_s \\ Y_t = y > 0, \end{cases} \tag{4.14}$$

in which $d\tilde{B}_s$ is a standard Brownian motion, and consider the optimal stopping problem given by

$$\hat{f}(y, t) = \inf_\tau \mathrm{E}\left[\int_t^\tau (c - A)Y_s\, ds + u(Y_\tau, \tau) \middle| Y_t = y\right]. \tag{4.15}$$

Note that $\hat{f}$ is concave with respect to y. Indeed, because Y_s in (4.14) is given by $Y_s = yh(s)$ with $h(s) = \exp(-(r + m)(s - t) + (\mu - r)/\sigma\, (\tilde{B}_s - \tilde{B}_t))$, the integral in (4.15) can be written as $y \int_t^\tau (c - A)\, h(s)\, ds$, that is, the expression in the expectation is concave with respect to y. Therefore, the infimum over stopping times τ is concave with respect to y.

A candidate solution for $\hat{f}$ is the value function $\tilde{f}$ from Section 4.1. Indeed, Øksendal (2000, Section 10.4) studies such optimal stopping problems and proves a verification theorem that we apply as follows: If we can show that

$$u_t(y, t) - ryu_y(y, t) + my^2u_{yy}(y, t) + (c - A)y \ge 0, \tag{4.16}$$

for $y > y_l(t)$ and for $y < y_u(t)$, and if $\tilde{f}$ is sufficiently regular, then $\hat{f} = \tilde{f}$. Thus, to approximate $\tilde{f}$ numerically, we can use algorithms developed for optimal stopping problems.

It remains for us to verify that inequality (4.16) holds. Indeed, for $y < y_u(t)$, we have that $u(y, t) = (c - A)y/r$, so that

$$u_t(y, t) - ryu_y(y, t) + my^2u_{yy}(y, t) + (c - A)y = -(c - A)y + (c - A)y = 0,$$

so (4.16) holds here. For $y > y_l(t)$, we have that $u(y, t) = {}_tp_x + w_l y$, so that

$$u_t(y, t) - ryu_y(y, t) + my^2u_{yy}(y, t) + (c - A)y = -\lambda_x(t)\, {}_tp_x - ryw_l + (c - A)y,$$

and this expression is nonnegative for all $y > y_l(t)$ if

$$[w_u - w_l]\, ry_l(t) \ge \lambda_x(t)\, {}_tp_x.$$

We check this condition in our numerical examples in Section 6.

Thus, to solve the FBP (4.8), (4.11), (4.12), it is sufficient to solve the variational inequality

$$\max[-\hat{f}_t + ry\hat{f}_y - my^2\hat{f}_{yy} - (c - A)y, \hat{f} - u] = 0. \tag{4.17}$$

Friedman and Shen (2002) prove the existence of a unique, continuous solution to a similar variational inequality. We discuss the numerical solution of (4.17) in Section 6. But first, we examine the qualitative properties of solutions to the HJB equation (2.3) and the dual FBP (4.8), (4.11), (4.12).

We end this section by summarizing our results to this point. If we find a solution $\hat{f}$ of (4.17), then by a verification theorem in Øksendal (2000, Section 10.4), $\tilde{f} = \hat{f}$, in which $\tilde{f}$ is the Legendre transform of f; see (4.2). Note that $\tilde{f}$ is concave with respect to y because $\hat{f}$ is concave, so that retrieving f via the inverse Legendre transform makes sense. The concavity of $\tilde{f}$ with respect to y implies that f is convex with respect to w; thus, the solution ψ of (2.3) is convex with respect to w. By a standard verification theorem (Bayraktar and Young 2005b), the minimum probability of ruin is the unique smooth solution of (2.3). Therefore, through this chain of reasoning and corresponding transformations, if we solve (4.17), then we have effectively obtained the minimum probability of ruin ψ.

5. Qualitative Properties of the Ruin Probability and Optimal Strategy

In this section, we explore the qualitative properties of the ruin probability ψ; in particular, we prove an intuitive monotonicity result and upper bound. From this upper bound, we see that the qualitative properties of the solution to the special case ODE of Section 3 are preserved in the general case. Moreover, we prove that the left-hand free boundary $y_u(t)$ is zero, which is useful in the numerical solution of the variational inequality (4.17).

The monotonicity result is intuitive; we prove that an individual with higher hazard rate has lower ruin probability, as she is less likely to outlive her wealth. This result generalizes the monotonicity observed in the fourth bullet point of Section 3 to the time-dependent case.

Theorem 5.1

Let $\lambda_x^1(t) \leq \lambda_x^2(t)$ for all t and let $\psi^i(w, t)$ be the corresponding ruin probabilities, $i = 1, 2$. Then $\psi^1(w, t) \geq \psi^2(w, t)$.

Proof

Let τ_d^i be the future lifetime random variables for (x) under the hazard rates $\lambda_x^i(t)$, $i = 1, 2$. For an admissible investment strategy π, let τ_l^π denote the first time at which wealth hits w_l under the strategy π. Note that if $\tau_l^\pi = \infty$, we have that

$$Pr[\tau_d^i > \tau_l^\pi | W_t = w] = 0;$$

therefore,

$$Pr[\tau_d^i > \tau_l^\pi | W_t = w] = E^{w,t}[e^{-\int_0^{\tau_l^\pi} \lambda_x^i(t)\,dt} \mathbf{1}_{\{\tau_l < \infty\}}].$$

It follows then that

$$\begin{aligned} Pr[\tau_d^1 > \tau_l^\pi | W_t = w] &= E^{w,t}[e^{-\int_0^{\tau_l^\pi} \lambda_x^1(t)\,dt} \mathbf{1}_{\{\tau_l < \infty\}}] \geq E^{w,t}[e^{-\int_0^{\tau_l^\pi} \lambda_x^2(t)\,dt} \mathbf{1}_{\{\tau_l < \infty\}}] \\ &= Pr[\tau_d^2 > \tau_l^\pi | W_t = w] \geq \inf_{\{\pi_s\}} Pr[\tau_d^2 > \tau_l^\pi | W_t = w] \\ &= \psi^2(w, t). \end{aligned}$$

Since the inequality above holds for arbitrary admissible π, taking the infimum over all π yields

$$\psi^1(w, t) \geq \psi^2(w, t).$$

Corollary 5.2

Let ψ and $\bar{\psi}$ be the ruin probabilities associated with hazard rates $\lambda_x(t)$ and λ, respectively, where the constant λ satisfies $\lambda \leq \lambda_x(t)$. Then $\psi(w, t) \leq \bar{\psi}(w)$.

PROOF

This is a straightforward application of Theorem 5.1.

Corollary 5.2 allows us to glean useful qualitative information about the ruin probability ψ in the time-dependent case. Recall that $\bar{\psi}$ is given by (3.1). Since ψ is bounded above by $\bar{\psi}$ and $\bar{\psi}$ ''flattens out'' at $w = w_u$, ψ must also ''flatten out'' at $w = w_u$. More precisely, since $\bar{\psi}(w_u) = \bar{\psi}'(w_u) = \psi(w_u, t) = 0$, it follows that $\psi_w(w_u, t) = 0$. It follows immediately from (2.4) that $\pi^*(w_u, t) = 0$. Thus, at the safe level of wealth, the optimal allocation to the risky asset is zero. Moreover, it follows from (4.10) that $y_u(t) = 0$; that is, that the unknown left-hand free boundary is zero. These results are useful in implementing the numerical method of Section 6.

6. NUMERICAL EXAMPLES

In this section, we consider a numerical example in which we approximate the ruin probability and optimal investment strategy and examine their sensitivity to some of the model parameters. Many of the results are consistent with financial intuition; however, we find that the *shape* of the hazard rate function has a dramatic impact on the optimal investment strategy. This result highlights the importance of using realistic mortality assumptions.

Because (4.17) is similar to the variational inequality associated with pricing an American option, we employ the projected SOR method, as described in Wilmott, Dewynne, and Howison (2000), to find the solution $\hat{f}$ of (4.17) and to recover the free boundary. The projected SOR method is an iterative method for solving the PDE from (4.17), namely,

$$-\hat{f}_t + ry\hat{f}_y - my^2\hat{f}_{yy} - (c - A)y = 0, y \in (y_u(t), y_l(t)), \tag{6.1}$$

subject to the inequality constraint from (4.17), namely,

$$\hat{f} \leq u, y \in (y_u(t), y_l(t)). \tag{6.2}$$

We solve the constrained PDE on a domain that contains the free boundary and then recover the location of the free boundary after computing the solution. The boundary points $y_u(t)$ and $y_l(t)$ are the points at which the inequality in (6.2) changes to equality. Note that in Section 5 we proved that $y_u(t) = 0$; thus we need only recover the location of the right-hand free boundary. The verification theorem of Øksendal (2000, Section 10.4) ensures that $\hat{f}$ also solves the free-boundary problem (4.8), (4.11), and (4.12).

To employ the projected SOR method, we

1. Transform the degenerate problem in (6.1) on $(0, \infty)$ to a nondegenerate problem on $\mathcal{R}$ via the standard transformation $\xi = \ln y$.
2. Solve the transformed variational inequality via the projected SOR method and recover the location of the free boundary.
3. Invert the transforms of Section 4.1 to recover $\psi(w, t)$ from $\hat{f}(y, t)$.
4. Numerically approximate the optimal investment in the risky asset $\pi^*(w, t)$.

We tested this numerical scheme by using it to calculate the probability of lifetime ruin and the corresponding optimal investment strategy for the special case of constant hazard rate, for which a solution is given in closed form by (3.1) and(3.2). Our computed solution matched the closed-form solution, so we are confident in the validity of our numerical scheme.

We begin with a base scenario and then examine the effect on the ruin probability and optimal investment strategy of changing the mortality assumptions and the parameters of the financial model. We take the following as our base scenario:

Base Scenario:

- Consistent with the mortality assumptions in Milevsky and Young (2005) and Huang, Milevsky, and Wang (2004), we use the Gompertz hazard rate $\lambda_x(t) = \exp((x + t - \bar{m})/b)/b$. We choose $\bar{m} = 90$ and $b = 9$; these values approximate the Individual Annuity Mortality 2000 (basic) Table with projection scale G. Note that the hazard rate increases exponentially with age.
- $x = 50$; the investor is 50 years old. Under the mortality assumption described above, her expected future lifetime is 35.32 years.
- $r = 0.02$; the riskless rate of return is 2% over inflation.
- $\mu = 0.06$; the drift on the risky asset is 6% over inflation.
- $\sigma = 0.20$; the volatility of the risky asset is 20%.
- $c = 1$; the individual consumes one unit of real wealth per year.
- $w_l = 0$; the individual considers herself ruined when her wealth reaches 0.
- $A = 0$; without loss of generality, we assume that annuity income is zero.
- It follows that the annual shortfall is $c - A = 1$. The individual is safe from ruin when wealth reaches $w_u = (c - A)/r = 50$.

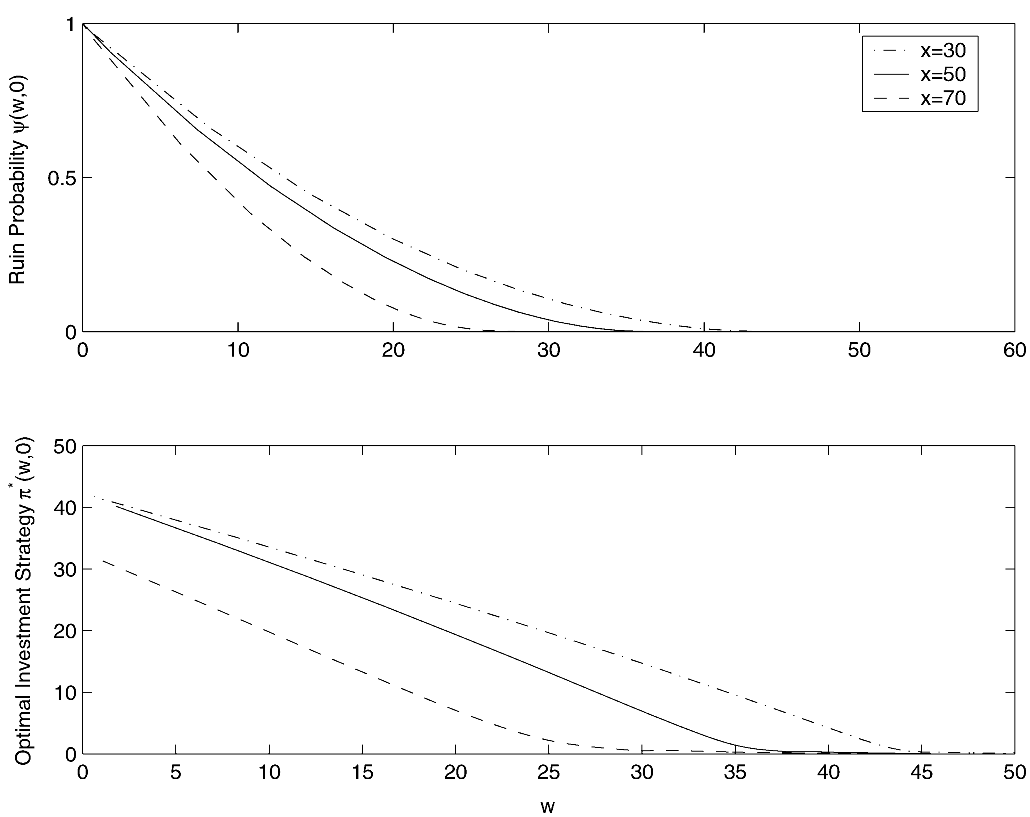
In the experiments that follow, we examine the impact on the ruin probability and optimal investment strategy of varying individual parameters from the values given above.

EXAMPLE 6.1, IMPACT OF ATTAINED AGE

Figure 1 shows the ruin probability $\psi(w, 0)$ and optimal investment in the risky asset $\pi^*(w, 0)$ for the base scenario described above (that is, for $x = 50$) as well as for ages $x = 30$ and 70. Observe that $\psi(w_l, 0) = 1$ and $\psi(w_u, 0) = \psi_w(w_u, 0) = 0$. We note that the ruin probability and optimal investment

Figure 1

Ruin Probabilities and Optimal Investment Strategies as Attained Age *x* Is Varied



in the risky asset decrease as age increases. Thus, a younger investor with wealth $w \in (w_l, w_u)$ is more likely to ruin than an older investor with the same wealth because the younger investor has a greater probability of survival for each future year; that is, the older investor is more likely to die and thereby not ruin. In addition, the younger individual will invest more in the risky asset than an older individual with the same wealth. This occurs because the investor with the longer expected horizon needs the (potentially) higher return of the risky asset because she needs her wealth to last longer. On the other hand, the investor with the shorter expected horizon can invest more in the risk-free asset because the lower return is more likely to be sufficient to fund her lifetime consumption. This investment pattern is consistent with the conventional wisdom that investors with longer horizons can be more aggressive in assuming risk (see Siegel 1994, for example). We found similar results under power utility in Moore and Young (2006). Samuelson (1963, 1989a,b) and Bodie (1995) challenge this view.

In general, if an individual faces a greater risk of ruin, then she will invest more in the risky asset. We see this phenomenon demonstrated in that the younger individual invests more in the risky asset. We also see it demonstrated in that the amount invested in the risky asset increases as wealth decreases (and ruin becomes more likely). Note that at low values of wealth, the optimal strategy is a heavily leveraged position in the risky asset; the investor employs a desperate strategy to avoid ruin. Young and Bayraktar (2005b) study this phenomenon by restricting borrowing and considering alternative objective functions.

EXAMPLE 6.2, IMPACT OF STOCK VOLATILITY

We next examine the impact of changing the volatility σ of the stock return. Specifically, we consider $\sigma = 0.1$, 0.2, and 0.3 (see Fig. 2). We observe that for fixed wealth, the ruin probability increases and

Figure 2

Ruin Probabilities and Optimal Investment Strategies as Volatility σ of the Risky Asset Is Varied

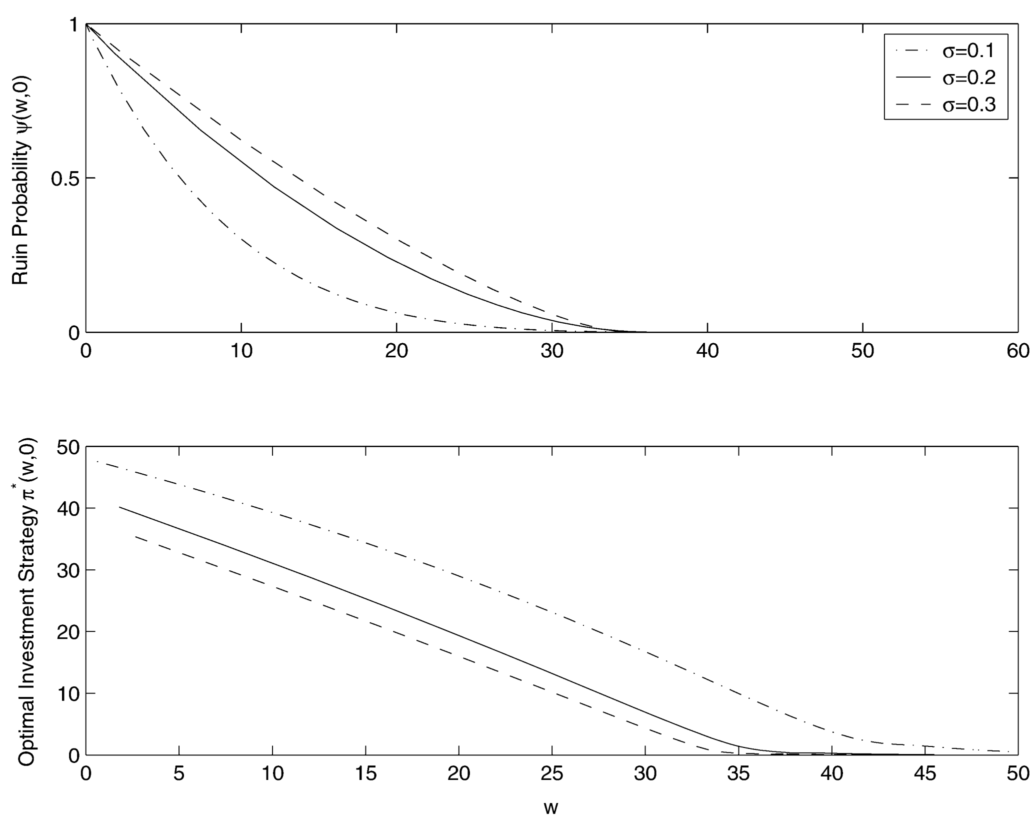
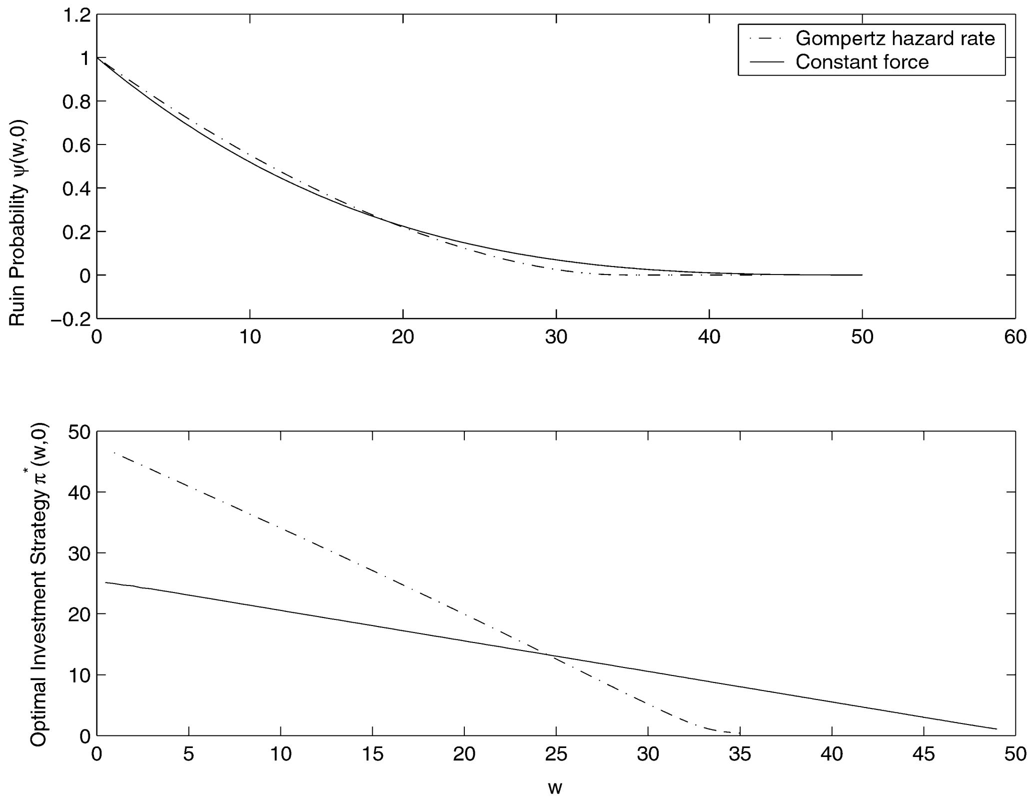


Figure 3
Shape of Hazard Rate Function, Showing a Dramatic Impact on Optimal Strategy, Even for "Similar" Mortality Assumptions



the optimal investment in the risky asset decreases with σ. This is consistent with our financial intuition.

EXAMPLE 6.3, IMPACT OF THE HAZARD RATE

In this example, we examine the significance of the mortality assumption. We contrast the mortality assumption described in the base scenario above with a constant hazard rate ($\lambda = 0.0283$) chosen to yield the same expected future lifetime. Figure 3 shows that, although the impact on the ruin probability is small, the optimal strategies differ markedly. In effect, it is possible to achieve nearly the same minimal ruin probability, regardless of the shape of the hazard rate, but the strategy required to "steer" the probability to the minimum differs significantly, depending on the hazard rate. At low wealth, the individual with increasing (Gompertz) hazard rate behaves more like a "long horizon" investor, while at high wealth, she behaves like a "short horizon" investor. To understand this result, we remark that ${}_tp_{50}$, the probability of a 50-year-old surviving another t years is higher under the Gompertz hazard rate than under the constant hazard rate for t less than about 42. Thus, at low wealth, the Gompertz individual must invest more in the risky asset. At high wealth, the Gompertz individual can invest more conservatively because the increasing hazard rate will eventually take over to prevent ruin.

7. A SIMPLE, NEARLY OPTIMAL STRATEGY

In Section 6, we showed that under a general hazard rate assumption, one can compute the optimal investment strategy and corresponding ruin probability by numerically solving the dual variational inequality (4.17). However, investors do not know their hazard rate function in parametric form. Thus, in this section, we examine the effectiveness of a modified strategy based on the easy-to-implement continuous p-rebalancing rule given in (3.2) with the hazard rate estimated from readily available mortality data. We also examine the impact of discrete versus continuous portfolio rebalancing. Using simulation, we find that the modified, easy-to-implement rule yields nearly optimal ruin probability, even under discrete rebalancing.

Consider an investor who has access to mortality data such as mortality rates or future life expectancy. Suppose that on her birthday she estimates a constant hazard rate λ_x for the year from the data and then employs discrete or continuous p-rebalancing as in (3.2) assuming that the estimated λ_x would apply for the rest of her life. (We call this ''approximate p-rebalancing.'') Thus, she uses the same value of p for the whole year. On her next birthday, she consults the mortality data again and updates her estimate of λ_x (and hence of p).

In this section, we use simulation to calculate the probability of lifetime ruin for an individual who employs this simple strategy.

7.1 Details of the Simulation

We use the same parameter values as in the base scenario of Section 6. For mortality, we compute life table values based on the Gompertz hazard rate of Section 6 and consider four methods of estimating an annual constant hazard rate λ_x:

1. λ_x is computed each year directly from the parametric form of the hazard rate. More specifically, $\lambda_x \approx \lambda_x(0)$, where $\lambda_x(0)$ is given by the Gompertz hazard rate function.
2. $\lambda_x \approx 1/(\mathrm{E}[T(x)])$, where $\mathrm{E}[T(x)]$ is the future life expectancy of(x).
3. $\lambda_x \approx q_x$, where q_x is the probability that (x) dies in the next year.
4. $\lambda_x \approx -\ln(p_x)$, where $p_x = 1 - q_x$.

See Section 3.6 of Bowers et al. (1997) for explanations of these approximations.

Figures 4 and 5 show the estimated constant annual hazard rate versus the parametric form for $t \in [0, 80]$ for an individual who is aged 50 at time $t = 0$. All of the approximations are good on a portion of the domain, but method 2 above overstates the actual value of the hazard rate for small t. On the other hand, method 3 understates $\lambda_x(t)$ for large t.

We consider a 50-year-old individual with initial wealth w_0. We simulate the future lifetime random variable $T(x)$ to determine the time of death. We assume that the investor employs discrete or continuous p-rebalancing based on the estimated annual hazard rate λ_x and we simulate the suboptimally controlled wealth and check daily whether financial ruin or death occurs first. We estimate the ruin probability ψ^{sim} as the proportion of the population for whom ruin precedes death.

Finally, we measure the effectiveness of this suboptimal strategy by comparing ψ^{sim} with the ruin probability for a 50-year-old investor with the same initial wealth who follows the optimal strategy computed as in Section 6 (by numerically solving the dual variational inequality). For comparison, we also compute the ruin probability for an individual who invests 100% in the risk-free asset.

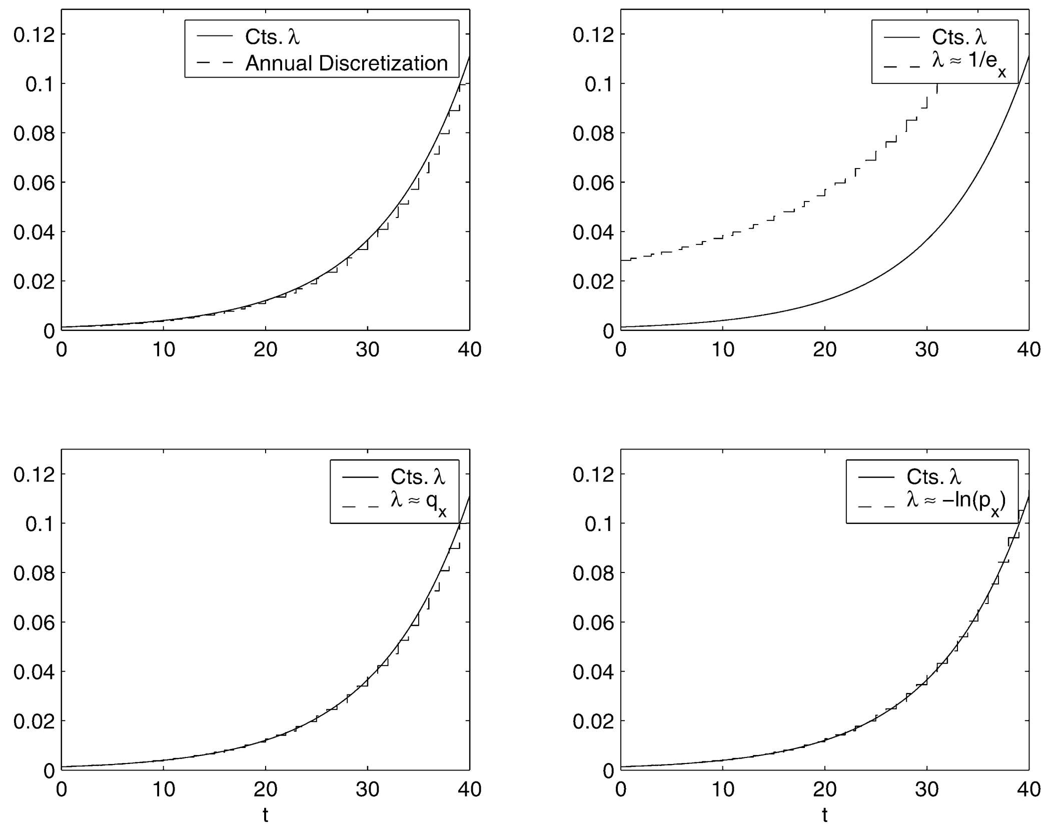
We consider three different levels of initial wealth in order to test the effectiveness of the suboptimal strategy at low, intermediate, and high ruin probabilities. In the simulation results that follow, we used $N = 20{,}000$.

7.2 The Effectiveness of the Simple, Nearly Optimal Strategy

Table 1 shows the results of our tests. We include the standard error $\sigma_{\psi^{sim}} = \sqrt{\psi(1 - \psi)/N}$ of our point estimator ψ^{sim}. We summarize the results below.

1. The ruin probability does increase under approximate p-rebalancing, but the effect is not large. For example, if an individual has ruin probability of 79% under the optimal strategy, employing approximate p-rebalancing increases the ruin probability to 80–82%, depending on the method of estimating λ_x and the frequency of rebalancing. If the optimal ruin probability is 1%, the suboptimal ruin probability increases to 3–8%.

Figure 4
Actual and Approximate Hazard Rates for $x = 50$ and $t \in [0, 40]$



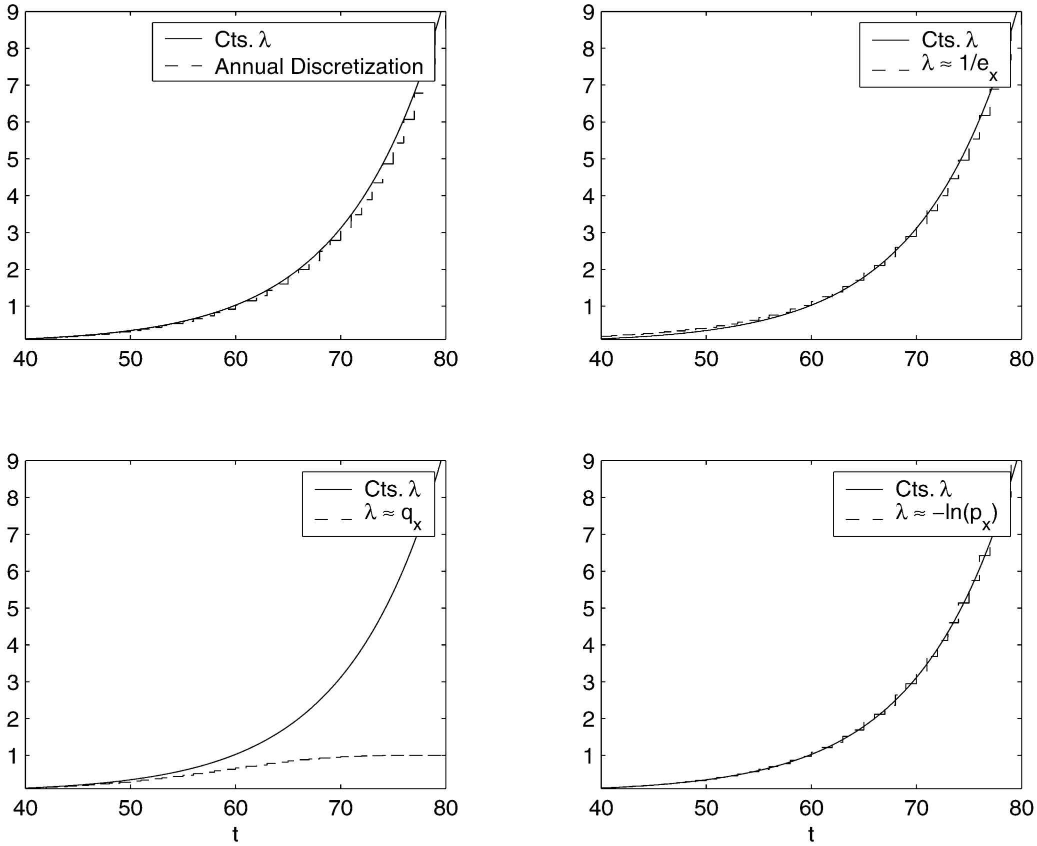
2. The effect of discrete versus continuous rebalancing is small compared to the effect of parameter estimation. This is consistent with the results of Rogers (2001), Rogers and Stapleton (2002), and Browne, Milevsky, and Salisbury (2003).
3. At low ruin probability, the approximation $\lambda_x = 1/(E[T(x)])$ yields the best results; at the high ruin probability, it yields the worst.
4. At 1% ruin probability, investing 100% in the risk-free asset is a better strategy than approximate p-rebalancing using a ''bad'' estimate of λ_x.

8. CONCLUSION AND FUTURE RESEARCH

Because of the increased longevity of our aging population and the continuing trend toward defined contribution retirement plans, under which the individual investor assumes all investment and longevity risk, a growing number of people are faced with the risk of outliving their retirement savings. In this paper, we presented a method for computing the asset allocation that minimizes the probability of lifetime ruin. This investigation is timely, and our choice of ruin probability as the optimization criterion is appropriate in light of this increased longevity risk. Moreover, ruin probability minimization is arguably a less subjective and more understandable goal than utility maximization. Young (2004) and Bayraktar and Young (2005a) studied the connections between the two optimization criteria.

We found that the ruin probability and optimal strategies respond in an intuitive and predictable way to changes in the model parameters. In particular, we found that investors with a longer horizon should invest more in the risky asset in order to minimize their ruin probability. In addition, we found that the shape of the hazard rate function has a significant qualitative impact on the optimal strategy.

Figure 5
Actual and Approximate Hazard Rates for x = 50 and $t \in [40, 80]$



While it is possible to achieve nearly the same ruin probability regardless of the shape of the hazard rate function, the investment strategy required to ‘‘steer’’ the probability to the minimum differs significantly. Therefore, it is important to consider the shape of the hazard rate function in determining one’s investment strategy to minimize the probability of lifetime ruin.

On the other hand, the method of solving a variational inequality that allows for a general hazard rate function, as in Section 6, is not available to most investors. To provide individual investors with investment advice that they can implement more easily, we examined the effectiveness of a modified strategy based on the continuous p-rebalancing rule given in (3.2) with the hazard rate estimated from readily available mortality data. We examined the impact of discrete versus continuous portfolio rebalancing. By using simulation, we found that the modified, easy-to-implement rule yielded nearly optimal ruin probability, even under discrete rebalancing. Therefore, one can modify the simple investment rule obtained by Young(2004) under constant force of mortality to apply in a more general setting to give nearly optimal results.

Because of the the long investment horizon in this problem, the assumptions of constant risk-free rate and volatility and deterministic mortality are not realistic. We propose to address these limitations in future work.

9. ACKNOWLEDGMENTS

We thank Masahiko Egami and Erhan Bayraktar for their help in this work. We also thank two anonymous referees for their useful comments.

Table 1

Scenario 1: High Ruin Probability (79%)

Initial Wealth	4.3787
Optimal Ruin Probability	79.04%
Ruin Probability 100% Riskless	99.22%
Standard Error	0.29%

	Frequency of Rebalancing			
Approximation Method	**Continuous**	**Weekly**	**Monthly**	**Annual**
$\lambda_{x(0)}$	80.25%	80.45%	80.11%	81.41%
$\lambda \approx 1/e_x$	81.56	81.62	81.93	82.15
$\lambda \approx q_x$	79.80	80.34	80.06	80.80
$\lambda \approx -\ln(p_x)$	79.75	80.65	80.47	81.38

Scenario 2: Intermediate Ruin Probability (40%)

Initial Wealth	14.1223
Optimal Ruin Probability	40.03%
Ruin Probability 100% Riskless	93.94%
Standard Error	0.35%

	Frequency of Rebalancing			
Approximation Method	**Continuous**	**Weekly**	**Monthly**	**Annual**
$\lambda_{x(0)}$	45.62%	46.22%	46.00%	48.35%
$\lambda \approx 1/e_x$	45.25	44.86	45.26	45.97
$\lambda \approx q_x$	45.33	45.66	45.54	47.17
$\lambda \approx -\ln(p_x)$	45.60	44.68	45.46	47.77

Scenario 3: Low Ruin Probability (1%)

Initial Wealth	31.6596
Optimal Ruin Probability	1.02%
Ruin Probability 100% Riskless	4.62%
Standard Error	0.07%

	Frequency of Rebalancing			
Approximation Method	**Continuous**	**Weekly**	**Monthly**	**Annual**
$\lambda_{x(0)}$	7.74%	7.69%	7.54%	8.47%
$\lambda \approx 1/e_x$	3.05	3.09	3.57	3.88
$\lambda \approx q_x$	7.50	7.75	8.04	8.83
$\lambda \approx -\ln(p_x)$	7.54	7.71	7.60	8.49

REFERENCES

ALBRECHT, PETER, AND RAIMOND MAURER. 2002. Self-annuitization, Consumption Shortfall in Retirement and Asset Allocation: The Annuity Benchmark. *Journal of Pension Economics and Finance* 1: 269–88.

BAYRAKTAR, ERHAN, AND VIRGINIA R. YOUNG. 2005a. Correspondence between Lifetime Minimum Wealth and Utility of Consumption. Working paper, Department of Mathematics, University of Michigan.

———. 2005b. Minimizing the Probability of Lifetime Ruin under Borrowing Constraints. Working paper, Department of Mathematics, University of Michigan.

BODIE, ZVI. 1995. On the Risks of Stocks in the Long Run. *Financial Analysts Journal* 51(3): 18–22.

BOWERS, NEWTON L., HANS U. GERBER, JAMES C. HICKMAN, DONALD A. JONES, AND CECIL J. NESBITT. 1997. *Actuarial Mathematics* 2nd ed. Schaumburg, IL: Society of Actuaries.

BROWNE, SID. 1995. Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research* 20(4): 937–58.

———. 1997. Survival and Growth with a Liability: Optimal Portfolio Strategies in Continuous Time. *Mathematics of Operations Research* 22(2): 468–93.

———. 1999a. Beating a Moving Target: Optimal Portfolio Strategies for Outperforming a Stochastic Benchmark. *Finance and Stochastics* 3: 275–94.

———. 1999b. The Risk and Rewards of Minimizing Shortfall Probability. *Journal of Portfolio Management* 25(4): 76–85.

———. 1999c. Reaching Goals by a Deadline: Digital Options and Continuous-Time Active Portfolio Management. *Advances in Applied Probability* 31: 551–77.

BROWNE, SID, MOSHE A. MILEVSKY, AND THOMAS S. SALISBURY. 2003. Asset Allocation and the Liquidity Premium for Illiquid Annuities. *Journal of Risk and Insurance* 70(3): 509–26.

COPELAND, CRAIG. 2002. An Analysis of the Retirement and Pension Plan Coverage Topical Module of SIPP. Issue Brief of the Employee Benefit Research Institute, www.ebri.org, May.

DUS, IVICA, OLIVIA S. MITCHELL, AND RAIMOND MAURER. 2005. Betting on Death and Capital Markets in Retirement: A Shortfall Risk Analysis of Life Annuities. National Bureau of Economic Research, NBER Working Paper 11271.

FRIEDMAN, AVNER, AND WEIXI SHEN. 2002. A Variational Inequality Approach to Financial Valuation of Retirement Benefits Based on Salary. *Finance and Stochastics* 6(3): 273–302.

GERRARD, RUSSELL, STEVEN HABERMAN, AND ELENA VIGNA. 2004. Optimal Investment Choices Post-retirement in a Defined Contribution Pension Scheme. *Insurance: Mathematics and Economics* 35: 321–42.

HUANG, HUAXIONG, MOSHE A. MILEVSKY, AND JIN WANG. 2004. Ruined Moments in Your Life: How Good Are the Approximations? *Insurance: Mathematics and Economics* 34(3): 421–47.

KARATZAS, IOANNIS, AND STEVEN E. SHREVE. 1998. *Methods of Mathematical Finance.* New York: Springer.

MERTON, ROBERT C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case. *Review of Economics and Statistics* 51: 247–57.

MILEVSKY, MOSHE A., KWOK HO, AND CHRIS ROBINSON. 1997. Asset Allocation via the Conditional First Exit Time or How to Avoid Outliving Your Money. *Review of Quantitative Finance and Accounting* 9(1): 53–70.

MILEVSKY, MOSHE A., KRISTEN S. MOORE, AND VIRGINIA R. YOUNG. 2006. Asset Allocation and Annuity-Purchase Strategies to Minimize the Probability of Financial Ruin. *Mathematical Finance,* to appear.

MILEVSKY, MOSHE A., AND CHRIS ROBINSON. 2000. Self-annuitization and Ruin in Retirement, with Discussion. *North American Actuarial Journal* 4(4): 112–29.

MILEVSKY, MOSHE A., AND VIRGINIA R. YOUNG. 2005. Annuitization and Asset Allocation. Working paper, Schulich School of Business, York University.

MOORE, KRISTEN S., AND VIRGINIA R. YOUNG. 2006. Optimal Insurance in Continuous Time. *Insurance: Mathematics and Economics,* to appear.

ØKSENDAL, BERNT. 2000. *Stochastic Differential Equations: An Introduction with Applications.* 5th ed. Berlin: Springer.

ORSZAG, J. MICHAEL. 2002. Ruin in Retirement: Running Out of Money in Drawdown Programs. Watson Wyatt Worldwide Technical Report RU06.

PARIKH, ALAN N. 2003. The Evolving U.S. Retirement System. *The Actuary,* March: 2–6.

ROGERS, L. CHRIS G. 2001. The Relaxed Investor and Parameter Uncertainty. *Finance and Stochastics* 5: 131–54.

ROGERS, L. CHRIS G., AND EMILY J. STAPLETON. 2002. Utility Maximisation with a Time Lag in Trading. *Applied Optimization* 74: 249–69.

ROY, ANDREW D. 1952. Safety First and the Holding of Assets. *Econometrica* 20: 431–39.

SAMUELSON, PAUL A. 1963. Risk and Uncertainty: The Fallacy of the Law of Large Numbers. *Scientia* 98: 108–13.

———. 1989a. The Judgment of Economic Science on Rational Portfolio Management: Indexing, Timing, and Long-Horizon Effects. *Journal of Portfolio Management* 16: 3–12.

———. 1989b. A Case at Last for Age-Phased Reduction in Equity. *Proceedings of the National Academy of Sciences* 86: 9048–51.

SIEGEL, JEREMY J. 1994. *Stocks for the Long Run.* Burr Ridge, IL: Irwin.

VANDERHEI, JACK, AND CRAIG COPELAND. 2003. Can America Afford Tomorrow's Retirees: Results from the EBRI-ERF Retirement Security Projection Model. Issue Brief of the Employee Benefit Research Institute, www.ebri.org, November.

WILMOTT, PAUL, JEFF DEWYNNE, AND SAM HOWISON. 2000. *Option Pricing: Mathematical Models and Computation.* Oxford: Oxford Financial Press.

YOUNG, VIRGINIA R. 2004. Optimal Investment Strategy to Minimize the Probability of Lifetime Ruin. *North American Actuarial Journal* 8(4): 105–26.

Discussions on this paper can be submitted until April 1, 2007. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.