Coherent Distortion Risk Measures and Higher-Order Stochastic Dominances

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Abstract

We show that the only coherent distortion risk measure that is consistent with respect to 3-convex order and hence with stochastic dominance of order 3 is the expected value, thus generalizing previous results of Hurlimann and solving a problem posed by Yamai and Yoshiba.

1. Introduction

The problem of the axiomatic foundation of risk measures has received much attention starting with the seminal papers of Artzner et al. (1999) and Delbaen (2000), where the definition of *coherent risk measure* was first provided. A coherent risk measure is a real functional ρ defined on a space of random variables representing possible losses satisfying the following well-known axioms:

1. Monotonicity: if $X \geq 0$, then $\rho(X) \geq 0$
2. Translation invariance: $\rho(X + h) = \rho(X) + h$ for each $h \in R$
3. Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for each $\lambda \geq 0$
4. Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for each X, Y.

The basic contribution of Artzner et al. (1999) and Delbaen (2000) was to characterize coherent risk measures in two ways: either as a minimum amount of capital to be added to the payoff X to achieve an *admissible* position, or as the supremum of the expected loss over a set of so-called *generalized scenarios*. It is interesting to note that the mathematical content of these characterizations was already partly anticipated by Hueber (1981) in the context of robust statistics. Other strictly related previous work in the actuarial setting of the axiomatic foundation of premium principles can be found in Goovaerts, De Vylder, and Haezendonck (1984) and Panjer, Young, and Wang (1997). For a critical discussion of the subadditivity axiom see Dhaene et al. (2004).

Since then, many generalizations and extensions of this notion have been proposed; the most natural is perhaps the notion of *convex risk measure,* in which axioms 3 and 4 are replaced by the weaker axiom of convexity:

$$\rho(\alpha X + (1 - \alpha)Y) \leq \alpha\rho(X) + (1 - \alpha)\rho(Y) \quad \text{for each } X, Y, \text{ for each } \alpha \in (0, 1).$$

Convex risk measures also have a characterization in terms of the supremum of expected losses over a set of generalized scenarios but with an additional penalty term that has been introduced in Föllmer and Schied (2002) and Frittelli and Rosazza (2002). See also Carr, Geman, and Madan (2001) and Deprez and Gerber (1985) for early contributions on convex premium principles.

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Although risk measures are defined on a space of random variables, in many common examples they depend only on the distribution of the random variable X, for example, the expected shortfall or the so-called *distortion risk measures* (see the next section). This has led naturally to the notion of *coherent law-invariant risk measures,* a class of risk measures described in Kusuoka (2001). An analogous result for convex risk measures can be found, for example, in Frittelli and Rosazza (2005).

For this class of risk measures the problem of *consistency* with respect to a given stochastic order arises naturally. Roughly speaking, a risk measure ρ is said to be *consistent* with a given stochastic order $\preceq$ if

$$X \preceq Y \Rightarrow \rho(X) \leq \rho(Y).$$

This means that evaluating risks by means of the risk measure ρ does not contradict evaluating risks by means of the stochastic order $\preceq$.

In Bauerle and Müller (2006) the problem of consistency of a general *convex risk measure* with respect to stochastic dominance of orders 1 and 2 is considered. If the space Ω has atoms, even consistency with respect to these orders is guaranteed only in special cases. However, if Ω is a standard probability space (see Denuit, Lefevre, and Shaked 1998), a convex risk measure is always consistent with stochastic dominance of orders 1 and 2.

In Yamai and Yoshiba (2001), starting from the critical analysis of Value at Risk and Expected Shortfall, the general problem of consistency with respect to stochastic dominance of order n is posed.

In Hürlimann (2004) the problem of consistency of a *coherent distortion risk measure* with respect to the so-called 3-convex order (see Section 3) is considered; the basic result is that the only coherent distortion risk measure consistent with the 3-convex order is associated with the distortion $g(x) = \sqrt{x}$, provided that the set of possible payoffs contains biatomic and Pareto random variables.

We provide an example that shows that if the set of possible payoffs contains more general discrete random variables, then the only coherent distortion risk measure consistent with the 3-convex order is the expected value, thus generalizing Hürlimann's (2004) negative result. Also, since 3-convex order is a sufficient condition for stochastic dominance of order 3, that in turn is a sufficient condition for stochastic dominance of order $n \geq 3$, we show that no nontrivial coherent distortion risk measure is consistent with stochastic dominance of order n for $n \geq 3$, thus answering the question posed in Yamai and Yoshiba (2001).

The paper is structured as follows: in Section 2 we review some basic facts and examples about distortion risk measures and their place in the more general context of coherent law-invariant risk measures; in Section 3 we recall the aforementioned results about consistency of coherent distortion risk measures; in Section 4 we prove our main result. Section 5 concludes.

2. DISTORTION RISK MEASURES

Distortion risk measures are a particular class of risk measures that have been extensively studied in actuarial literature in connection with the axiomatic theory of premium calculation; see, for example, Denneberg (1994), Panjer, Young, and Wang (1997), and Wang (2000). We review some basic definitions that can be found, for example, in Denneberg (1994).

A *distortion* is a nondecreasing function g: $[0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$.

If X is a random variable representing a possible financial loss defined on some probability space $(\Omega, \mathcal{F}, P)$, with distribution function $F(x)$ and survival function $S(x) = 1 - F(x)$, the *distortion risk measure* $\rho_g(X)$ associated with the distortion g is given by

$$\rho_g(X) = \int_0^{+\infty} g(S(x))\, dx - \int_{-\infty}^{0} [1 - g(S(x))]\, dx. \tag{2.1}$$

From a mathematical point of view, a distortion risk measure is a Choquet integral with respect to the nonadditive measure $\mu = g \circ P$.

All the standard results about Choquet integrals (see Denneberg 1994) apply to distortion risk measures. In particular:

1. If $X \geq 0$, then $\rho_g(X) \geq 0$
2. $\rho_g(X + a) = \rho_g(X) + a, \forall a \in R$
3. $\rho_g(\lambda X) = \lambda \rho_g(X), \forall \lambda \geq 0$
4. $\rho_g(-X) = -\rho_\gamma(X)$, where $\gamma(x) = 1 - g(1 - x)$ is the *dual distortion* of g
5. ρ_g is *comonotone,* in the sense that if X and Y are comonotone, then $\rho_g(X + Y) = \rho_g(X) + \rho_g(Y)$.

The random losses X and Y are comonotone if there exist two increasing functions h_1 and h_2 and a random loss Z such that $X = h_1(Z)$ and $Y = h_2(Z)$. The notion of comonotonicity is central in risk theory; see, for example, Dhaene et al. (2000, 2004) and references therein.

Many different distortions g have been proposed. Some well-known examples are the following:

EXAMPLE 1

If $g(x) = x^p$ with $p \in (0, 1]$, we have the PH-transform, which has been motivated on an axiomatic basis in Panjer, Young, and Wang (1997).

EXAMPLE 2

If $g_\alpha(x) = \min((x/1 - \alpha), 1)$, we get the usual expected shortfall $ES_\alpha(X)$.

EXAMPLE 3

If $g(x) = N(N^{-1}(x) + a)$, where $N(x)$ is the distribution function of a standard normal and $a \in R$, we get the so-called Wang transform measure, introduced in Wang (2000).

EXAMPLE 4

If $g(x) = x^p(1 - p \ln x)$ $p \in (0, 1]$, we get the so-called lookback distortion measure (*see, e.g., Hürlimann* 1998).

The only property that a distortion risk measure lacks in order to have a *coherent risk measure* in the sense of Artzner et al. (1999) is subadditivity. The subadditivity theorem of Choquet integrals (Denneberg 1994) guarantees that $\rho_g(X + Y) \leq \rho_g(X) + \rho_g(Y)$ if and only if the distortion g is concave. Hence, a distortion risk measure with a concave distortion g is an example of a *coherent distortion risk measure.* Furthermore, by construction a distortion risk measure depends only on the distribution of the random loss X.

It follows that distortion risk measures are a particular case of *law-invariant coherent risk measures,* a general class of coherent risk measures that under some additional hypotheses has been characterized by Kusuoka (2001) in the following theorem:

Theorem 5

Let $(\Omega, \mathcal{F}, P)$ be a standard probability space and consider ρ: $L^\infty(\Omega, \mathcal{F}, P) \rightarrow R$.

The following are equivalent:

1. *ρ is a coherent, law-invariant risk measure, satisfying the Fatou property*
2. *There exists a set $\mathcal{M}_0$ of probability measures on* $[0, 1]$ *such that if $X \in L^\infty$,*

$$\rho(X) = \sup_{m \in \mathcal{M}_0} \left\{ \int_0^1 ES_\alpha(X)\ dm(\alpha) \right\}.$$

REMARK 6

Note that $(\Omega, \mathcal{F}, P)$ is a standard probability space if it is Borel-isomorphic to $([0, 1], B([0, 1]), Leb)$. *In this case it is possible to construct a random variable on Ω with any assigned distribution function F.*

REMARK 7

The Fatou property is a weak continuity property introduced in Delbaen (2000). *The risk measure* ρ *has the Fatou property if* $\rho(X) \leq \lim\inf \rho(X_n)$ *for each sequence* X_n *bounded by* 1 *and converging to* X *in probability.*

The place occupied by coherent distortion risk measures in the general framework of law-invariant risk measures is specified by another result of Kusuoka's (2001).

Theorem 8

Let $(\Omega, \mathcal{F}, P)$ *be a standard probability space and consider* $\rho\colon L^\infty(\Omega, \mathcal{F}, P) \to R$.

The following are equivalent:

1. ρ *is a coherent, law-invariant risk measure, satisfying the Fatou property and* comonotone
2. *There exists a probability measure* $m \in [0, 1]$ *such that*

$$\rho(X) = \int_0^1 ES_\alpha(X)\, dm(\alpha).$$

But since $ES_\alpha(X)$ is a coherent distortion risk measure corresponding to the distortion $g_\alpha(x)$ of Example 2, we have that

$$\begin{aligned}\rho(X) &= \int_0^1 ES_\alpha(X)\, dm(\alpha) = \int_0^1 [\int_0^{+\infty} g_\alpha(S(t))\, dt - \int_{-\infty}^0 (1 - g_\alpha(S(t)))]\, dm(\alpha) \\ &= \int_0^{+\infty} \int_0^1 g_\alpha(S(t))\, dm(\alpha)\, dt - \int_{-\infty}^0 \int_0^1 (1 - g_\alpha(S(t)))\, dm(\alpha)\, dt.\end{aligned}$$

By posing

$$g(x) = \int_0^1 g_\alpha(x)\, dm(\alpha),$$

we find that $g(x)$ is a distortion and

$$\rho(X) = \int_0^{+\infty} g(S(t))\, dt - \int_{-\infty}^0 [1 - g(S(t))]\, dt = \rho_g(X).$$

Hence the class of coherent distortion risk measures coincides with the class of comonotone law-invariant coherent risk measures.

Furthermore, a general law-invariant coherent risk measure can always be expressed as a supremum of coherent distortion risk measures as in Kusuoka's theorem.

3. STOCHASTIC DOMINANCES AND DISTORTION RISK MEASURES

As noted in the introduction, generally speaking a risk measure is said to be *consistent* with a given stochastic order $\preceq$ if $X \preceq Y \Rightarrow \rho(X) \leq \rho(Y)$. Some care is due to the specification of the order $\preceq$ if the variables X and Y represent losses, since different stochastic orders have different properties under reflection. First note the basic definitions of stochastic dominance:

1. $X \preceq_{st} Y$ if and only if $E[f(X)] \leq E[f(Y)]$ for each nondecreasing f for which both integrals are finite (stochastic dominance of order 1)
2. $X \preceq_{icv} Y$ if and only if $E[f(X)] \leq E[f(Y)]$ for each nondecreasing and concave f for which both integrals are finite (stochastic dominance of order 2).

Since we define risk measures on losses X, we say that a risk measure ρ is consistent with stochastic dominance of order 1 if $-Y \preceq_{st} -X \Rightarrow \rho(X) \leq \rho(Y)$, that is, equivalent to $X \preceq_{st} Y \Rightarrow \rho(X) \leq \rho(Y)$. In the same way ρ is consistent with stochastic dominance of order 2 if $-Y \preceq_{icv} -X \Rightarrow \rho(X) \leq \rho(Y)$, that is, equivalent to $X \preceq_{icx} Y \Rightarrow \rho(X) \leq \rho(Y)$, where the ordering $\preceq_{icx}$ is defined by $X \preceq_{icx} Y$ if and only if $E[f(X)] \leq E[f(Y)]$ for each nondecreasing and convex f for which both integrals are finite (increasing convex order).

Bauerle and Müller (2006) have shown that if the space $(\Omega, \mathcal{F}, P)$ is standard, then a general *convex law-invariant risk measure* with the Fatou property is consistent with respect to stochastic dominance of orders 1 and 2 (Theorems 4.2 and 4.4). Similar results for distortion risk measures were already known and are given in the following theorems.

Theorem 9

Given a nondecreasing distortion g and the associated risk measure ρ_g, then

$$X \preceq_{st} Y \Rightarrow \rho_g(X) \leq \rho_g(Y).$$

The proof is very easy and can be found, for example, in Hardy and Wirch (2003).

Theorem 10

Given a nondecreasing concave distortion g and the associated risk measure ρ_g, then

$$X \preceq_{icx} Y \Rightarrow \rho_g(X) \leq \rho_g(Y).$$

This theorem states that every coherent distortion risk measure is consistent with respect to second-order stochastic dominance; the proof can be found in Young and Wang (1998). A more refined characterization of *strict consistency* is given in the following theorem (see Hardy and Wirch 2003).

Theorem 11

Given a strictly concave *distortion g and the associated risk measure ρ_g, then*

$$X \prec_{icx} Y \Rightarrow \rho_g(X) < \rho_g(Y).$$

The problem of consistency of risk measures with respect to higher-order stochastic dominances (see, e.g., Levy 1998 for the definitions) has been posed in Yamai and Yoshiba (2001).

Remembering that we have defined distortion risk measures $\rho_g(X)$ in terms of a loss X, we say that a distortion risk measure $\rho_g(X)$ is *consistent with stochastic dominance of order n* if

$$-Y \preceq^n -X \Rightarrow \rho(X) \leq \rho(Y).$$

Some partial results about consistency of distortion risk measures with respect to stochastic dominance of order 3 have been obtained by Hürlimann (2004). More precisely, Hürlimann considers consistency with respect to the 3-convex order as defined in Denuit, Lefevre, and Shaked (1998). It is easy to see that this ordering corresponds to stochastic dominance of order 3 if variables with equal means and variances are considered; hence consistency with respect to 3-convex order is a necessary condition for consistency with respect to stochastic dominance of order 3. We note also that in general stochastic dominance of order n is a sufficient condition for stochastic dominance of order $n + 1$, and hence consistency with respect to stochastic dominance of order n is a necessary condition for consistency with respect to stochastic dominance of order $n + 1$.

Hürlimann proved the following results:

Theorem 12

(*Proposition 4.1 in Hürlimann* 2004) *If the set of possible losses contains all biatomic losses, then a distortion risk measure with g continuous, increasing, and differentiable is consistent with 3-convex order if the and only if the following condition holds:* $(x/1 - x)[1 - g(x)] + g(x) - 2xg'(x) \geq 0$, *for each* $x \in (0, 1)$.

Considering the special case of PH-transforms (see Example 1), he proves the following.

Theorem 13

(*Theorem 6.3 in Hürlimann* 2004) *If the set of possible losses contains also all Pareto (with a generic location and scale parameter allowed) variables, then the only distortion risk measure consistent with 3-convex order corresponds to* $g(x) = \sqrt{x}$ *and* $g(x) = x$.

In the next section we will show that actually under much weaker hypotheses the only coherent distortion risk measure consistent with respect to the 3-convex order (and hence with stochastic dominance of order 3 or of order n with $n > 3$) is given by $g(x) = x$, which corresponds to the expected value.

4. DISTORTION RISK MEASURES AND 3-CONVEX ORDER

Consider the following two possible losses:

$$X = \begin{cases} 1 & p_1 \\ 2 & p_2 \\ 3 & p_3 \\ 4 & p_4 \end{cases} \qquad Y = \begin{cases} 1 & p_1 + \varepsilon \\ 2 & p_2 - 3\varepsilon \\ 3 & p_3 + 3\varepsilon \\ 4 & p_4 - \varepsilon \end{cases} \tag{4.1}$$

with $p_i > 0$, $\Sigma_i\, p_i = 1$, and $\varepsilon > 0$ sufficiently small. This example is a generalization of Example 4.8 in Young and Wang (1998).

We have that $E[X] = E[Y]$ and $E[X^2] = E[Y^2]$, and from the *crossing condition* (see, Denuit, Lefevre, and Shaked 1998 or Young and Wang 1998) it is easy to see that $-Y \succeq_{3-cx} -X$ and hence $-Y \succeq^3 -X$. It is also easy to see that $-Y$ and $-X$ are not ordered by stochastic dominance of orders 1 and 2, while clearly they are ordered by stochastic dominance of order n, for each $n \geq 3$.

The survival functions of X and Y are given by

$$S_X(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ p_2 + p_3 + p_4 & \text{if } 1 < t \leq 2 \\ p_3 + p_4 & \text{if } 2 < t \leq 3 \\ p_4 & \text{if } 3 < t \leq 4 \\ 0 & \text{if } t > 4 \end{cases}$$

and

$$S_Y(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ p_2 + p_3 + p_4 - \varepsilon & \text{if } 1 < t \leq 2 \\ p_3 + p_4 + 2\varepsilon & \text{if } 2 < t \leq 3. \\ p_4 - \varepsilon & \text{if } 3 < t \leq 4 \\ 0 & \text{if } t > 4 \end{cases}$$

According to definition 1, the distorted risk measures $\rho_g(X)$ and $\rho_g(Y)$ are given by

$$\begin{aligned} \rho_g(X) &= \int_0^{+\infty} g(S_X(x))\, dx = 1 + g(p_2 + p_3 + p_4) + g(p_3 + p_4) + g(p_4) \\ &= 1 + g(z) + g(y) + g(x), \\ \rho_g(Y) &= \int_0^{+\infty} g(S_Y(x))\, dx \\ &= 1 + g(p_2 + p_3 + p_4 - \varepsilon) + g(p_3 + p_4 + 2\varepsilon) + g(p_4 - \varepsilon). \end{aligned}$$

By posing $x = p_4$, $y = p_3 + p_4$, $z = p_2 + p_3 + p_4$, we have that

$$\rho_g(X) = 1 + g(z) + g(y) + g(x),$$

$$\rho_g(Y) = 1 + g(z - \varepsilon) + g(y + 2\varepsilon) + g(x - \varepsilon).$$

By construction it is easy to see that $0 < x < y < z < 1$; and each triple $x < y < z$ can be achieved with a proper choice of the probabilities p_i.

Since $-Y \succeq_{3-cx} -X$, to have consistency of the distortion risk measure ρ_g with respect to 3-convex order, we need that $\rho_g(Y) \le \rho_g(X)$, that is,

$$g(z - \varepsilon) + g(y + 2\varepsilon) + g(x - \varepsilon) \le g(z) + g(y) + g(x) \tag{4.2}$$

for each x, y, z with $0 < x < y < z < 1$ and for $\varepsilon > 0$ sufficiently small. From the concavity of g we have that the right and left derivatives g'_+ and g'_- exist everywhere and are both nonincreasing. Furthermore, g'_+ is continuous from the right, g'_- is continuous from the left, and with the exception of an at most numerable set N,

$$g'_+(x) = g'_-(x) = g'(x)$$

(see, e.g., Hörmander 1994, Theorem 1.1.7). Dividing both sides of (4.2) by ε and passing to the limit for $\varepsilon \to 0^+$, we get

$$-g'_-(x) + 2g'_+(y) - g'_-(z) \le 0 \tag{4.3}$$

for each x, y, z such that $0 < x < y < z < 1$.

On the set $D = [0, 1] - N$ we have $g'_+(x) = g'_-(x) = g'(x)$ and hence $g'(y) - g'(x) \le g'(z) - g'(y)$, for each $x, y, z \in D$ with $x < y < z$. By letting $x \to y_-$, since g' is continuous in y, we get $g'(y) \le g'(z)$ for each $y, z \in D$, $y < z$. But from the aforementioned theorem, g' is nonincreasing on D, and hence it follows that g' is constant on D.

Since g'_+ and g'_- coincide with g' on the dense set D and are respectively right continuous and left continuous, it follows that g'_+ and g'_- coincide *everywhere;* hence under the inequality (4.3) g is differentiable everywhere, and its derivative is a constant. From the condition $g(0) = 0$ and $g(1) = 1$ it follows that $g(x) = x$.

Hence we have proved the following theorem.

Theorem 14

If the set of possible losses contains all discrete variables as in (4.1), *then the only coherent distortion risk measure that is consistent with respect to the 3-convex order is the expected value.*

We note that no assumption on the differentiability of the distortion g is required, only the concavity that follows from its coherence. We also note that the problem of consistency is still open if only *continuous* losses are allowed, thus ruling out discrete variables as in (4.1).

5. CONCLUSIONS AND DIRECTIONS FOR FURTHER RESEARCH

The preceding result shows that when we move from consistency with respect to stochastic dominance of order 2 to consistency with respect to higher-order stochastic dominances, the requirement of consistency becomes very strong and makes the risk measure ‘‘collapse’’ on the expected value. The role played by distortion risk measures in the general setting of coherent law-invariant risk measures (see Section 1) enlightened by Kusuoka’s (2001) results strongly suggests that an analogous result should hold for general coherent law-invariant risk measures, but we don’t have a proof at the moment.

It is not even clear at the moment if there are nontrivial convex risk measures that are consistent with higher-order stochastic dominances.

Our results also seem interesting in connection with the relationships between the usual stochastic dominance and the so-called dual stochastic dominances coming from Yaari’s dual theory of choice under uncertainty considered in Young and Wang (1998).

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