

A RISK MODEL WITH MULTILAYER DIVIDEND STRATEGY

Hansjörg Albrecher* and Jürgen Hartinger†

ABSTRACT

In recent years various dividend payment strategies for the classical collective risk model have been studied in great detail. In this paper we consider both the dividend payment intensity and the premium intensity to be step functions depending on the current surplus level. Algorithmic schemes for the determination of explicit expressions for the Gerber-Shiu discounted penalty function and the expected discounted dividend payments are derived. This enables the analytical investigation of dividend payment strategies that, in addition to having a sufficiently large expected value of discounted dividend payments, also take the solvency of the portfolio into account. Since the number of layers is arbitrary, it also can be viewed as an approximation to a continuous surplus-dependent dividend payment strategy. A recursive approach with respect to the number of layers is developed that to a certain extent allows one to improve upon computational disadvantages of related calculation techniques that have been proposed for specific cases of this model in the literature. The tractability of the approach is illustrated numerically for a risk model with four layers and an exponential claim size distribution.

1. INTRODUCTION

Consider an insurance portfolio with surplus $U(t)$ at time t and initial surplus $U(0) = u$. Let $N(t)$, the number of claims up to time t , be modeled by a Poisson process with intensity $\lambda > 0$. Further, the claim sizes X_i are assumed to be independent and identically distributed random variables with distribution function P and mean $\mu < \infty$. The premium income and the dividend payout are assumed to depend on the surplus level in the following way: Define k layers $0 = b_0 < b_1 < b_2 < \dots < b_{k-1} < b_k = \infty$. Whenever the surplus $U(t)$ is in layer i , $b_{i-1} \leq U(t) < b_i$, premium is collected with intensity c_i , and dividends are paid out with intensity $0 \leq a_i \leq c_i$, and hence the surplus process $U(t)$ increases with density $(c_i - a_i)$ until a claim causes a jump to a lower layer or the surplus grows to the next higher layer. If we denote the total claim amount up to time t by $S(t) = \sum_{i=1}^{N(t)} X_i$, the dynamics of the surplus process in the i th layer are thus determined by

$$dU(t) = (c_i - a_i) dt - dS(t).$$

Furthermore, the dynamics of the dividend payments at time t is given by

$$dD(t) = a_i dt \quad \text{if } b_{i-1} \leq U(t) < b_i.$$

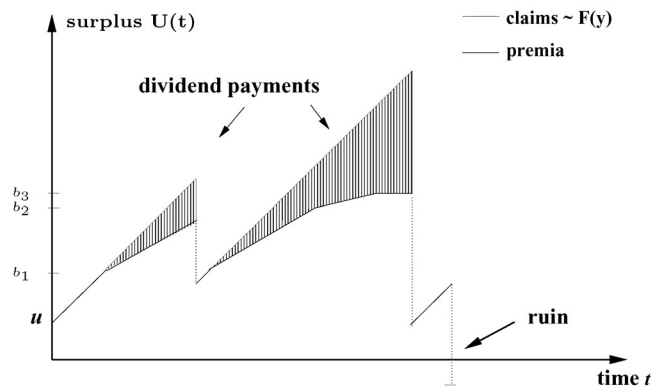
(Figure 1 illustrates a sample path of $U(t)$ for the case $c_1 = c_2 = c_3$ and $0 = a_1 < a_2 < a_3 = c_3$.)

For simplicity, assume that the net profit condition is fulfilled in each layer, i.e.

* Hansjörg Albrecher, PhD, is Associate Professor at the Graz University of Technology, Steyrergasse 30, 8010 Graz, Austria, and Group Leader for Financial Mathematics at the Radon Institute, Austrian Academy of Sciences, Linz, supported by the Austrian Science Fund Project P18392, albrecher@TUGraz.at.

† Jürgen Hartinger, PhD, is a postdoc researcher at the Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, and actuary at the Kärntner Landesversicherung, Domgasse 12, 9020 Klagenfurt, Austria, juergen.hartinger@klv.at.

Figure 1
Sample Path of Risk Process $U(t)$



$$(c_i - a_i) > \lambda\mu, \quad i = 1, \dots, k \quad (1.1)$$

The ruin probability for the surplus process $U(t)$ is defined through

$$\psi(u) = \mathbb{P} \left(\inf_{t>0} \{t: U(t) < 0 | U(0) = u\} < \infty \right),$$

and the (more general) discounted penalty function, originally introduced in the seminal paper of Gerber and Shiu (1998), comprises information on the time of ruin and the distribution of both the deficit at ruin and the surplus before ruin.

Variants and special cases of this risk model recently have been studied extensively in the literature. Among these, in the case $k = 2$, $c_1 = c_2$, and $a_1 = 0$, the so-called threshold model is retrieved for $0 < a_2 < c_1$, and the case of a horizontal dividend barrier model is obtained through the further specification $a_2 = c_1$. Apart from its analytical tractability, one particular motivation to study these two models is the fact that for exponential claim sizes, these dividend strategies can be identified as maximizing the expected discounted dividend payout until ruin in the case of bounded and unbounded dividend payment intensity, respectively (see Gerber 1969 and Gerber and Shiu 2006). One should keep in mind, however, that this optimality property does not carry over to arbitrary claim size distributions (see, e.g., Azcue and Muler 2005; Albrecher and Thonhauser 2006; Schmidli 2007). At the same time, the solvency of the insurance portfolio also might be of particular concern, and in such a case it is natural to consider dividend payment policies that depend on the current level of the surplus (larger dividend payments for larger surplus level). The layer model considered in this paper can serve as an example for such dividend payment strategies that still enables an analytical treatment. Analytical formulas are of particular interest for tuning the model toward a target in terms of profitability and safety and for risk management purposes in general.

Looking solely at properties of the surplus process (i.e., $a_i = 0$ for $i = 1, \dots, k$), a surplus-dependent premium rate $c(u)$ already has found considerable attention in the literature (see Schock Petersen (1990) for an early numerical account and Asmussen and Nielsen (1995) for an asymptotic analysis of the resulting risk process based on large deviations). If $c(u)$ is a step function (as in the layer model considered in this paper), an exact expression for the ruin probability was given for two layers (i.e., $k = 2$) in Asmussen (2000). More recently, Lin and Pavlova (2006) derived a system of integro-differential equations for the discounted penalty function for two layers. Based on renewal-theoretic techniques, they showed that below the threshold b_1 , the discounted penalty function is proportional to the discounted penalty function of the classical (one-layer) model plus a product of an exponential and a compound geometric function. Furthermore, they give explicit results for the ruin probability

and the time of ruin when the claims are (mixtures of) exponentials. Lin and Sendova (2006) are able to extend their analysis to a multilayer setting and, for exponential claims, derive explicit formulas for the probability of ruin up to a sequence of constants, which has to be calculated numerically as the solution of a system of linear equations. For the same multilayer model, Zhou (2005) proposes a scheme to compute the ruin probability numerically (in his approach the net profit condition (1.1) is not needed). Furthermore, he uses level crossing techniques to study further ruin-related quantities for $\delta = 0$ and $k = 2$. Laplace transforms of the first passage times across a given level before ruin are investigated in a perturbed risk model in Zhou (2004).

The purpose of this paper is to approach the multilayer model in a generic way, with the aim of unifying and extending some of the results mentioned above. In particular, we intend to contribute to the study of the discounted penalty function and the expected discounted sum of dividend payments up to ruin for this multilayer setting (where the discount rate is denoted by $\delta \geq 0$).

In Section 2 the well-known differential approach will be used to derive a system of integro-differential equations for the discounted penalty function, which will be investigated by means of Laplace transforms. For exponential claim sizes, we give the exact solution for the probability of ruin in this multilayer model (this includes an explicit solution of the linear system of equations for the involved constants in the expression, which had been left to a numerical approach in Lin and Sendova [2006]). Furthermore, for exponential claim sizes the Laplace transform of the ruin time in the multilayer model is rederived in a particularly transparent way. Subsequently, a system of integro-differential equations for the moment-generating function of the discounted dividends is obtained by the infinitesimal approach. The structure of the expected discounted dividends under this multilayer dividend strategy then is given explicitly for exponential claim sizes, whereas the involved constants are expressed as the solution of a system of linear equations. The approach in principle can be generalized to higher moments of dividend payments and Erlang claim sizes along the same lines.

One computational disadvantage of the methods available in the literature and the one presented in Section 2 is that for a complete solution (with all involved constants) the obtained recursions have to be evaluated in both directions (i.e., up and down), typically making the approach infeasible for larger values of k or more involved penalty functions and claim distributions. More concretely, the resulting systems of equations for the coefficients in most cases cannot be solved symbolically, so that a representation of the solution in terms of the model parameters is not at hand. In Section 3 we will indicate an alternative approach that improves on this problem to some extent. Rather than going through all the layers recursively at the same time, the explicit solution is calculated successively for an increasing number of layers, always adding one additional layer to the already available solution. In this way a “global” recursion is set up in terms of the number of layers (justifying the name *recursive approach*), and hence the above-mentioned solution of the possibly huge system of linear equations is avoided. The technique in the background is a conditioning on the exit times of the surplus process out of each layer. First, the *time value of upper exit* out of a layer is determined, which is a basic building block for both the discounted penalty and expected dividends. Then recursive formulas are given for these three quantities for general claim sizes. Finally, more explicit results are given for exponential claim sizes.

Throughout the paper the tractability of the approach is illustrated by an example of a risk model with four layers and exponential claim size, with parameters taken from an example of Zhou (2005).

The analysis of this paper in principle can be extended to the case where the claim size distribution also depends on the current layer i (such a model might be appropriate when considering moral hazard behavior of the insurance company; for instance, the determination of claim sizes might be stricter—resulting in modified claim distributions—in the case where the surplus level in the portfolio is low). However, merely for simplification of notation, here we restrict the analysis to the case $P_i(x) \equiv P(x) \forall i = 1, \dots, k$.

The following notation will be used: $\bar{U}(t) := \sup_{0 \leq s \leq t} U(s)$ is the supremum of the surplus up to time t . The subscript k ($k \in \mathbb{N}$) always will refer to the corresponding quantity in a model with k layers (e.g., $U_k(t)$ denotes the surplus at time t in a model with k layers, and $U_{k-1}(t)$ then is the same quantity in

the $(k - 1)$ -layer model resulting from shifting b_{k-1} to infinity). The subscript 1 correspondingly refers to the classical model with premium density c_1 and dividend rate a_1 . Furthermore, let $U_{1,i}(t)$ be the surplus process of the classical (one-layer) model with parameters c_i, a_i (and in general let the subscript $\{\cdot\}_{1,i}$ refer to the corresponding quantity in the classical (one-layer) model with parameters c_i, a_i). Moreover, $\rho_{1,i}$ and $-R_{1,i}$ denote the nonnegative and negative, respectively, solutions of the corresponding Lundberg fundamental equation (in ξ)

$$\lambda + \delta - (c_i - a_i)\xi = \lambda \int_0^\infty e^{-\xi t} dP(t) \quad (1.2)$$

(see Gerber and Shiu 1998). Note that $\rho_{1,i}$ is strictly positive when $\delta > 0$. For $a \leq b$, define the stopping times $\tau^*(u, a, b) = \inf\{t \geq 0 : U(t) \notin [a, b] | U(0) = u\}$,

$$\tau^+(u, a, b) = \begin{cases} \tau^*(u, a, b) & \text{if } U(\tau^*(u, a, b)) = b, \\ \infty & \text{if } U(\tau^*(u, a, b)) < a, \end{cases}$$

and

$$\tau^-(u, a, b) = \begin{cases} \infty & \text{if } U(\tau^*(u, a, b)) = b, \\ \tau^*(u, a, b) & \text{if } U(\tau^*(u, a, b)) < a. \end{cases}$$

Note that the structure of the process implies that upper exits of any interval $[a, b)$ can happen only through (continuous) premium income (hence attaining the value b), whereas lower exits are due to jumps originating from claims. The time of ruin is denoted by $\tau(u) = \tau^-(u, 0, \infty)$. Derivatives at a boundary always are interpreted as the appropriate one-sided derivatives.

2. THE DIFFERENTIAL APPROACH

2.1 The Discounted Penalty Function

The discounted penalty function as introduced by Gerber and Shiu (1998) is in the k -layer model given by

$$m_k(u) = \mathbb{E}[e^{-\delta\tau_k(u)} \varpi(U_k(\tau_k(u)-), |U_k(\tau_k(u))|) I_{\{\tau_k(u) < \infty\}} | U(0) = u]. \quad (2.1)$$

Within each layer, for $i = 1, \dots, k - 1$, and $b_{i-1} \leq u < b_i$, conditioning on the occurrence of a jump within an infinitesimal time interval gives

$$m_k(u) = (1 - \lambda dt)e^{-\delta dt} m_k(u + (c_i - a_i)dt) + \lambda dt e^{-\delta dt} \left(\int_0^{u+(c_i-a_i)dt} m_k(u + (c_i - a_i)dt - \varepsilon) dP(\varepsilon) + \int_{u+(c_i-a_i)dt}^\infty \varpi(u + (c_i - a_i)dt, \varepsilon - u + (c_i - a_i)dt) dP(\varepsilon) \right) + o(dt).$$

Taylor expansion and collection of terms of order dt then yields

$$(c_i - a_i) \frac{\partial}{\partial u} m_k(u) - (\lambda + \delta)m_k(u) + \lambda \left(\int_0^u m_k(u - \varepsilon) dP(\varepsilon) + \int_u^\infty \varpi(u, \varepsilon - u) dP(\varepsilon) \right) = 0, \quad (2.2)$$

where again $u \in [b_{i-1}, b_i)$, ($i = 1, \dots, k - 1$). We are looking for a continuous solution across all layers, so that we get the boundary conditions

$$\lim_{u \rightarrow b_i^+} m_k(u) = \lim_{u \rightarrow b_i^-} m_k(u) \quad (2.3)$$

and

$$\lim_{u \rightarrow \infty} m_k(u) = 0. \quad (2.4)$$

The above derivation of (2.2) is a short alternative proof of Theorem 3.1 of Lin and Sendova (2006) (although their proof based on the renewal approach has the advantage to imply directly the continuity of the solution across the layers, whereas above one has to ask for it).

REMARK 2.1

Note that $m_k(u)$ is not differentiable at the boundaries of the layers. This can be seen directly from rewriting the conditions (2.5) in the form

$$(c_{i+1} - a_{i+1}) \frac{\partial^+}{\partial u} m_k(u)|_{u=b_i} = (c_i - a_i) \frac{\partial^-}{\partial u} m_k(u)|_{u=b_i}.$$

For specific cases (in particular for Erlang distributed claim sizes), the system of integro-differential equations (2.2) together with boundary conditions (2.3) and (2.4) may be solved by a direct approach. We illustrate the idea in the following example.

Example 2.1

Consider the ruin probability $\psi_k(u)$ (i.e., $\varpi \equiv 1$ and $\delta = 0$), $c_i = c$ for all layers, and assume exponential claims ($P(u) = 1 - e^{-\beta u}$). Then for $i = 1, \dots, k-1$, equations (2.2) take the form

$$(c - a_i) \frac{\partial}{\partial u} \psi_k(u) - \lambda \psi_k(u) + \lambda \left(\int_0^u \psi_k(u - z) dP(z) + 1 - P(u) \right) = 0. \quad (2.5)$$

Multiplying (2.5) by the operator $(\partial/\partial u + \beta)$ gives

$$\left((c - a_i) \frac{\partial^2}{\partial u^2} + ((c - a_i)\beta - \lambda) \frac{\partial}{\partial u} \right) \psi_k(u) = 0, \quad u \in [b_{i-1}, b_i]. \quad (2.6)$$

Using the notation

$$\psi_k(u) = \sum_{i=1}^k \psi_k^{(i)}(u) I_{\{b_{i-1} \leq u < b_i\}}, \quad (2.7)$$

where $I_{\mathcal{A}}$ denotes the indicator function of set \mathcal{A} and the function $\psi_k^{(i)}(u)$ is the solution of (2.6) for $u \in [b_{i-1}, b_i)$ for each i , we obtain

$$\psi_k^{(i)}(u) = A_k^{(i)} + C_k^{(i)} e^{-R_{1,i} u}$$

Here

$$-R_{1,i} = \frac{\lambda}{c - a_i} - \beta \quad (2.8)$$

refers to the negative solution of the Lundberg equation (1.2) for $\delta = 0$.

It remains to establish the constants in the above representation: from $\psi_k(\infty) = 0$, we have $A_k^{(k)} = 0$. From the continuity conditions (2.3) we immediately have

$$A_k^{(i+1)} + C_k^{(i+1)} e^{-R_{1,i+1} b_i} - A_k^{(i)} - C_k^{(i)} e^{-R_{1,i} b_i} = 0, \quad i = 1, 2, \dots, k-1. \quad (2.9)$$

Using (2.5) and comparing the coefficients of $e^{-\beta u}$, we get after some manipulations

$$-A_k^{(i+1)} - C_k^{(i+1)} \frac{\beta e^{-R_{1,i+1} b_i}}{\beta - R_{1,i+1}} + A_k^{(i)} + C_k^{(i)} \frac{\beta e^{-R_{1,i} b_i}}{\beta - R_{1,i}} = 0, \quad i = 1, 2, \dots, k-1, \quad (2.10)$$

together with $A_k^{(1)} + C_k^{(1)} \beta / (\beta - R_{1,1}) = 1$.

Adding (2.9) and (2.10) we get

$$C_k^{(i+1)} = \frac{(\beta - R_{1,i+1})R_{1,i}}{(\beta - R_{1,i})R_{1,i+1}} e^{(R_{1,i+1}-R_{1,i})b_i} C_k^{(i)} = \frac{\lambda - \beta(c - a_i)}{\lambda - \beta(c - a_{i+1})} e^{\lambda(1/(c-a_i)-1/(c-a_{i+1}))b_i} C_k^{(i)},$$

leading to

$$C_k^{(i)} = \frac{(\beta - R_{1,i})R_{1,1}}{(\beta - R_{1,1})R_{1,i}} e^{\sum_{j=1}^{i-1}(R_{1,j+1}-R_{1,j})b_j} C_k^{(1)} = \frac{\lambda - \beta(c - a_1)}{\lambda - \beta(c - a_i)} e^{\lambda \sum_{j=1}^{i-1}(1/(c-a_j)-1/(c-a_{j+1}))b_j} C_k^{(1)}, \quad i \geq 1.$$

For $i \in \mathbb{N}$, define

$$L_i := R_{1,1} e^{-R_{1,1}b_1} \sum_{j=1}^{i-1} \left(\frac{1}{R_{1,j}} - \frac{1}{R_{1,j+1}} \right) e^{\sum_{l=2}^j (b_{l-1}-b_l)R_{1,l}}. \quad (2.11)$$

Then, again from (2.9), we have

$$A_k^{(i)} = A_k^{(1)} + C_k^{(1)} \frac{\beta}{\beta - R_{1,1}} L_i = A_k^{(1)} + (1 - A_k^{(1)})L_i.$$

Hence,

$$A_k^{(1)} = -\frac{L_k}{1 - L_k} \quad \text{and} \quad C_k^{(1)} = \frac{\beta - R_{1,1}}{\beta} \frac{1}{1 - L_k}.$$

We thus arrive at (2.7) with

$$\begin{aligned} \psi_k^{(i)}(u) &= \frac{1}{1 - L_k} \left(L_i - L_k + \frac{\beta - R_{1,i}}{\beta} \frac{R_{1,1}}{R_{1,i}} e^{\sum_{j=1}^{i-1}(R_{1,j+1}-R_{1,j})b_j} e^{-R_{1,i}u} \right) \\ &= \frac{1}{1 - L_k} \left(L_i - L_k + \frac{R_{1,1}}{R_{1,i}} e^{\sum_{j=1}^{i-1}(R_{1,j+1}-R_{1,j})b_j} \psi_{1,i}(u) \right), \quad i = 1, \dots, k, \end{aligned} \quad (2.12)$$

where according to the notation introduced in Section 1 $\psi_{1,i}(u)$ denotes the ruin probability in the classical (one-layer) model with parameters c_i and a_i . In view of (2.8) and (2.11), equation (2.12) is an explicit formula in terms of the model parameters and hence simplifies the algorithm for the determination of $\psi_k(u)$ in Lin and Sendova (2006, sec. 5). For $k = 2$, formula (2.12) matches the corresponding expression in Lin and Pavlova (2006). Also, Zhou (2005; Example 1 in Section 3 of his paper) gives, in the exponential case, numerical results for $\psi_4(u)$ for some values of u for the parameter set

$$\begin{aligned} \lambda &= 1, \beta = 1, b_1 = 5, b_2 = 10, b_3 = 15, \\ \alpha_1 &= 0, \alpha_2 = 0.1, \alpha_3 = 0.2, \alpha_4 = 0.3, c_i = 1.4 \quad i = 1, \dots, 4 \end{aligned} \quad (2.13)$$

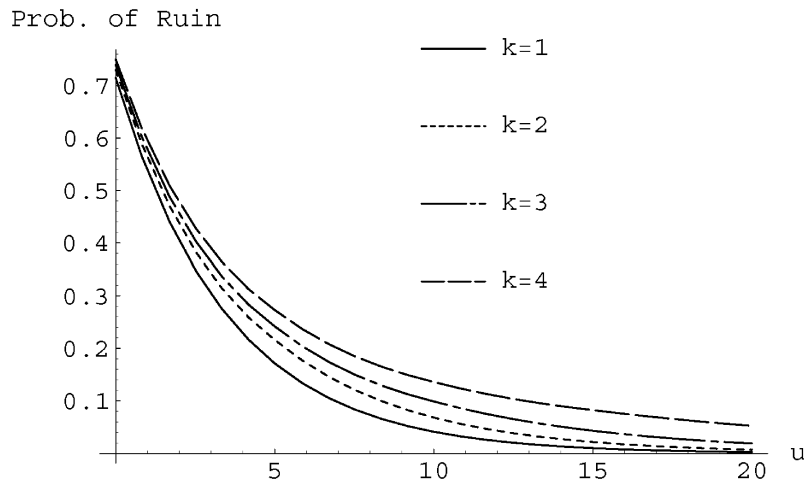
(note that here we interpret the decrease of the premium income of Zhou (2005) as the payout of dividends for constant premiums, which for the ruin process considered by Zhou is equivalent).

From (2.12) one can read off the explicit formula for $\psi_4(u)$ for this case, which (at the corresponding values of u) is in agreement with the numerical values from Zhou (2005) (all values are rounded to their last digit):

$$\psi_4(u) = \begin{cases} 0.123 + 0.627e^{-0.286u} & 0 \leq u \leq 5 \\ 0.0727 + 0.635e^{-0.231u} & 5 \leq u \leq 10 \\ 0.041 + 0.502e^{-0.167u} & 10 \leq u \leq 15 \\ 0.322e^{-0.091u} & 15 \leq u \leq \infty. \end{cases}$$

More generally, Figure 2 depicts the ruin probability as a function of u for $k = 1, \dots, 4$ (where, e.g., $k = 2$ means that the third and fourth layer are shifted to infinity). Hence, the figure illustrates the effect of adding additional (upper) layers in the risk model.

Figure 2
Ruin Probabilities $\psi_k(u)$ for Parameter Set (2.13) and $k = 1, \dots, 4$



REMARK 2.2

The main tool in Example 2.1 was the fact that the reformulation of the integro-differential equations as ordinary differential equations allowed one to find the solution for each layer separately (i.e., instead of using information of the solution from outside the layer (in the integral term), only “local” properties are needed). The remaining task was then to determine the involved coefficients through the continuity assumptions between the solutions in different layers. The latter was achieved through a solution of a system of linear equations (which had a particularly simple structure in the above example). This program still can be carried through when the claim sizes follow an Erlang(n)–distribution, in which case the ordinary differential equations are of order $n + 1$ (but still constant coefficients). Also, more general penalty functions are possible. However, in particular for $\delta > 0$, the solution of the resulting linear system of equations gets highly involved and then usually can be evaluated only numerically.

Another, more general way to approach the system of integro-differential equations (2.2) with boundary conditions (2.3) and (2.4) is to solve it by using Laplace transforms. This can be done in several ways. For the special case of the ruin probability, Kereksha (2004) looked at (2.2) for arbitrary $u > 0$ by introducing, for each i , a correction term that is zero inside the interval $u \in [b_{i-1}, b_i)$ and corrects for the error outside this interval. Formulating the result in terms of Fourier transforms and exploiting analytical projection properties of the latter, he managed to relate these correction terms to each other and express them through truncated Fourier transforms of known quantities (whether these can then be handled depends on the complexity of the considered example). In the following we provide an alternative approach based on the definition of several penalty functions, each “starting” at the height of a layer. This represents an adaptation of a technique used in Albrecher, Claramunt, and Marmól (2005).

Similarly to (2.7), use the notation

$$m_k(u) = \sum_{i=1}^k m_k^{(i)}(u)I_{\{b_{i-1} \leq u < b_i\}},$$

where, for each i , the function $m_k^{(i)}(u)$ is set to 0 for $u \in [0, b_{i-1})$ and for $u \geq b_{i-1}$ is defined by the equation

$$(c_i - a_i) \frac{\partial}{\partial u} m_k^{(i)}(u) - (\lambda + \delta) m_k^{(i)}(u) + \lambda \left(\int_0^{u-b_{i-1}} m_k^{(i)}(u - z) dP(z) + \int_{u-b_{i-1}}^u m_k(u - z) dP(z) + \int_u^\infty \varpi(u, z - u) dP(z) \right) = 0. \quad (2.14)$$

For $i = 1, \dots, k$, define the Laplace transforms $\tilde{m}_k^{(i)}(s) = \int_0^\infty e^{-su} m_k^{(i)}(u) du$ and $\tilde{p}(s) = \int_0^\infty e^{-su} dP(u)$. Multiplying (2.14) by e^{-su} and integrating from b_{i-1} to ∞ , one then obtains

$$\tilde{m}_k^{(i)}(s) = \frac{m_k(b_{i-1})(c_i - a_i)e^{-sb_{i-1}} - \lambda \left(\int_{b_{i-1}}^\infty e^{-su} \left(\int_{u-b_{i-1}}^u m_k(u - z) + \int_u^\infty \varpi(u, z - u) \right) dP(z) du \right)}{(c_i - a_i)s - (\lambda + \delta) + \lambda \tilde{p}(s)}. \quad (2.15)$$

Clearly, this provides an iterative solution algorithm by determining the Laplace transform of the discounted penalty function in each layer, starting with the lowest level. Whenever one can explicitly invert this Laplace transform in a certain layer, the solution for the next layer can be determined up to constants. In a final step the boundary conditions then can be used to determine the value of these constants. The following example illustrates such a procedure for a two-layer model with exponential claim sizes and penalty function $\varpi \equiv 1$, rederiving a result of Lin and Pavlova (2006) in this alternative way.

Example 2.2

Consider the time value of ruin in the k -layer model $m_k(u)$ (i.e., $\varpi \equiv 1$ and $\delta \geq 0$). For exponentially distributed claim sizes ($P(u) = 1 - e^{-\beta u}$), we have

$$\tilde{m}_k^{(i)}(s) = \frac{m_k(b_{i-1})(c_i - a_i)e^{-sb_{i-1}} - \lambda \left(\int_{b_{i-1}}^\infty e^{-su} \int_{u-b_{i-1}}^u m_k(u - z) \beta e^{-\beta z} dz du + \frac{e^{-(\beta+s)b_{i-1}}}{\beta + s} \right)}{(c_i - a_i)s - (\lambda + \delta) + \lambda \frac{\beta}{\beta + s}}. \quad (2.16)$$

These Laplace transforms and their inverses now may be calculated up to constants successively, for $i = 1, \dots, k$. Finally using continuity, we get a system of linear equations to establish the values at the boundaries b_i , $i = 0, \dots, k - 1$.

Let us have a look at the two-layer model: Equation (2.16) for $i = 1$ yields

$$\tilde{m}_2^{(1)}(s) = \frac{(s + \beta)m_2(0) - \frac{(\beta + \rho_{1,1})(\beta - R_{1,1})}{\beta}}{(s - \rho_{1,1})(s + R_{1,1})}$$

or

$$m_2^{(1)}(u) = \frac{1}{\rho_{1,1} + R_{1,1}} \left((\beta + \rho_{1,1}) \left(m_2(0) - \frac{\beta - R_{1,1}}{\beta} \right) e^{\rho_{1,1}u} - (\beta - R_{1,1}) \left(m_2(0) - \frac{\beta + \rho_{1,1}}{\beta} \right) e^{-R_{1,1}u} \right).$$

Equation (2.15) for $i = 2$ gives

$$\tilde{m}_2^{(2)}(s) = e^{-sb_1} \times \frac{(s + \beta) m_2(b_1) - \frac{(\beta + \rho_{1,2})(\beta - R_{1,2})}{(\rho_{1,1} + R_{1,1})} \left(\left(m_2(0) - \frac{\beta - R_{1,1}}{\beta} \right) e^{\rho_{1,1}b_1} - \left(m_2(0) - \frac{\beta + \rho_{1,1}}{\beta} \right) e^{-R_{1,1}b_1} \right)}{(s - \rho_{1,2})(s + R_{1,2})}.$$

Define

$$A := \left(m_2(0) - \frac{\beta - R_{1,1}}{\beta} \right) e^{\rho_{1,1} b_1} - \left(m_2(0) - \frac{\beta + \rho_{1,1}}{\beta} \right) e^{-R_{1,1} b_1},$$

then

$$m_2^{(2)}(u) = \frac{I_{\{u \geq b_1\}}}{\rho_{1,2} + R_{1,2}} \left((\beta + \rho_{1,2}) \left(m_2(b_1) - \frac{\beta - R_{1,2}}{(\rho_{1,1} + R_{1,1})} A \right) e^{\rho_{1,2}(u-b_1)} - (\beta - R_{1,2}) \left(m_2(b_1) - \frac{\beta + \rho_{1,2}}{(\rho_{1,1} + R_{1,1})} A \right) e^{-R_{1,2}(u-b_1)} \right).$$

The constant $m_2(0)$ may be calculated by (2.3) and (2.4):

$$m_2(0) = \frac{1}{\beta} \frac{(\rho_{1,1} + R_{1,2})(\beta - R_{1,1})e^{\rho_{1,1} b_1} - (R_{1,2} - R_{1,1})(\beta + \rho_{1,1})e^{-R_{1,1} b_1}}{(\rho_{1,1} + R_{1,2})e^{\rho_{1,1} b_1} - (R_{1,2} - R_{1,1})e^{-R_{1,1} b_1}}.$$

Thus, we get

$$m_2(u) = \begin{cases} \frac{1}{\beta} \frac{(R_{1,2} - R_{1,1})e^{-R_{1,1} b_1}(\beta + \rho_{1,1})e^{\rho_{1,1} u} + (\rho_{1,1} + R_{1,2})e^{\rho_{1,1} b_1}(\beta - R_{1,1})e^{-R_{1,1} u}}{(\rho_{1,1} + R_{1,2})e^{\rho_{1,1} b_1} - (R_{1,2} - R_{1,1})e^{-R_{1,1} b_1}}, & u \leq b_1 \\ \frac{(\rho_{1,1} + R_{1,1})e^{(\rho_{1,1} - R_{1,1}) b_1}}{(\rho_{1,1} + R_{1,2})e^{\rho_{1,1} b_1} - (R_{1,2} - R_{1,1})e^{-R_{1,1} b_1}} \frac{\beta - R_{1,2}}{\beta} e^{-R_{1,2}(u-b_1)}, & u \geq b_1 \end{cases}, \quad (2.17)$$

which for the case $\alpha_1 = 0$ reproduces the result of Example 7.1 of Lin and Pavlova (2006).

In the general case, the corresponding Laplace transforms cannot be explicitly inverted. Another approach that can still lead to explicit results in such cases is outlined in Section 3.

2.2 Moment-Generating Function for the Discounted Dividends

Let $\delta > 0$ and define

$$M_k(u, y) = \mathbb{E} \left[\exp \left(y \int_0^{\tau_k(u)} e^{-\delta t} dD_k(t) \right) \middle| U_0 = u \right]$$

as the moment-generating function of the discounted dividend payments up to ruin. Then, using the differential approach again, one can condition on the occurrence of a claim in the next infinitesimal time step, use the Taylor expansion, and collect terms of order dt to arrive at a partial integro-differential equation (PIDE) for $M_k(u, y)$ in the usual way. Let $W_{k,m}(u, b)$ denote the m th moment of the discounted dividend payments until ruin in the model with k layers. As in Gerber and Shiu (2004), the representation

$$M_k(u, y) = 1 + \sum_{m=1}^{\infty} \frac{y^m}{m!} W_{k,m}(u)$$

in the PIDE for $M_k(u, y)$ then gives integro-differential equations for $W_{k,m}(u, b)$:

$$\left((c_i - a_i) \frac{\partial}{\partial u} - \lambda - \delta m \right) W_{k,m}(u) + \lambda \int_0^u W_{k,m}(u - z) dP(z) + m a_i W_{k,m-1}(u) = 0, \quad u \in [b_{i-1}, b_i] \quad (2.18)$$

for $m \in \mathbb{N}$ and $i = 1, \dots, k$, with boundary conditions

$$\lim_{u \rightarrow \infty} W_{k,m}(u) = \left(\frac{a_k}{\delta} \right)^m, \quad (2.19)$$

and asking for continuity of the solution across the layers implies

$$\lim_{u \rightarrow b_i^+} W_{k,m}(u) = \lim_{u \rightarrow b_i^-} W_{k,m}(u). \quad (2.20)$$

REMARK 2.3

In fact, the required continuity across layers of the solution above is again automatically implied by a renewal-type proof of (2.18); cf. Section 2.1. For the special case of two layers with $a_1 = 0$, $a_2 = c_1 = c_2$ (i.e., the horizontal barrier strategy), Dickson and Waters (2004) used a direct conditioning technique to derive equations for $W_{2,m}(u)$; however, in the general setting the above approach via the moment-generating function is more convenient.

Just like in Section 2.1, equation (2.18) can be reformulated as ordinary differential equations with constant coefficients in the case of Erlang-distributed claim sizes, allowing one to find the solution for each layer separately. Finally, the continuity conditions can be used to determine the involved constants through a system of linear equations. The idea is demonstrated in the following example.

Example 2.3

Consider the setting of Example 2.1 with exponential claims and define

$$W_{k,1}(u) = \sum_{i=1}^k W_{k,1}^{(i)}(u) I_{\{b_{i-1} \leq u < b_i\}}.$$

Multiplying (2.18) by the operator $((\partial/\partial u) + \beta)$, we get for $i = 1, \dots, k$

$$\alpha_i \beta + \left(\frac{\partial}{\partial u} + \beta \right) \left((c_i - \alpha_i) \frac{\partial}{\partial u} - \lambda - \delta \right) W_{k,1}^{(i)}(u) + \lambda \beta^n W_{k,1}^{(i)}(u) = 0, \quad u \in [b_{i-1}, b_i).$$

Thus, the solutions take the form

$$W_{k,1}^{(i)}(u) = \frac{\alpha_i}{\delta} + A_{k,1}^{(i)} e^{-R_{1,i}u} + B_{k,1}^{(i)} e^{\rho_{1,i}u}, \quad u \in [b_{i-1}, b_i), \quad (2.21)$$

for some real constants $A_{k,1}^{(i)}, B_{k,1}^{(i)}$. From (2.19), we get

$$A_1^{(k)} = 0.$$

The continuity conditions give, for $i = 1, 2, \dots, k - 1$,

$$A_{k,1}^{(i)} e^{-R_{1,i}b_i} + B_{k,1}^{(i)} e^{\rho_{1,i}b_i} - A_{k,1}^{(i+1)} e^{-R_{1,i+1}b_i} - B_{k,1}^{(i+1)} e^{\rho_{1,i+1}b_i} = \frac{\alpha_i + 1}{\delta} - \frac{\alpha_i}{\delta}.$$

Moreover, comparing the coefficients of $e^{-\beta u}$ of these solutions in (2.18) gives, after some algebra,

$$A_{k,1}^{(i)} \frac{e^{-R_{1,i}b_i} \beta}{\beta - R_{1,i}} + B_{k,1}^{(i)} \frac{e^{\rho_{1,i}b_i} \beta}{\beta + \rho_{1,i}} - A_{k,1}^{(i+1)} \frac{e^{-R_{1,i+1}b_i} \beta}{\beta - R_{1,i+1}} - B_{k,1}^{(i+1)} \frac{e^{\rho_{1,i+1}b_i} \beta}{\beta + \rho_{1,i+1}} = \frac{\alpha_{i+1}}{\delta} - \frac{\alpha_i}{\delta},$$

$$i = 1, 2, \dots, k - 1$$

and

$$A_{k,1}^{(1)} \frac{\beta}{\beta - R_{1,1}} + B_{k,1}^{(1)} \frac{\beta}{\beta + \rho_{1,1}} = \frac{\alpha_1}{\delta}.$$

Hence, we have a system of $2k$ linear equations for the $2k$ coefficients $A_{k,1}^{(i)}, B_{k,1}^{(i)}$ ($i = 1, \dots, k$), leading to the solution (2.21) on each layer. This solution algorithm generalizes formula (6.14) of Gerber and Shiu (2006) obtained for $k = 2$.

For larger values of k it usually becomes cumbersome to calculate these closed-form solutions as a function of the model parameters (note that the symmetry of the matrix exploited in the explicit solution for the ruin probability in Example 2.1 is not present in the case of dividend payments). On the other hand, numerical solutions always can be obtained using this approach. For an illustration, see Section 3.3.

3. A RECURSIVE APPROACH

As already mentioned in the Introduction, a computational disadvantage of the above methods to derive explicit formulas is the fact that usually the obtained recursions among the different layers have to be solved with unknown constants that can be evaluated numerically only a posteriori, which makes the method rather infeasible for larger k or more involved quantities under study. In this section we will pursue another approach based on level crossings that can lead to explicit solutions also in more general situations. More concretely, we will set up a recursion with respect to the numbers of layers, meaning that the complete solution of the (classical) one-layer model is used to obtain the complete solution of the two-layer model, and so on. When feasible, this approach avoids the above problem of numerical evaluation of constants.

3.1 Preliminaries

Recollect first a few concepts developed for the one-layer model that will be needed later:

- The expected discounted payment of 1 at the time of recovery $\tau_0 = \inf\{t > \tau \geq 0 : U(t) = 0\}$ is given by

$$\Psi_\delta(u) := \mathbb{E}[e^{-\delta\tau_0} I_{\{\tau < \infty\}} | U(0) = u] = \mathbb{E}[e^{-\delta\tau + \rho U(\tau)} I_{\{\tau < \infty\}} | U(0) = u] \tag{2.22}$$

(cf. Gerber and Shiu 1998, eqs. (6.4), (6.10)). Clearly, $\Psi_\delta(u)$ is a discounted penalty function with $\varpi(x, y) = e^{-\rho y}$, where ρ is itself a function of δ ($\Psi_0(u)$ is just the ruin probability). For exponential claim sizes, $\Psi_\delta(u)$ has a simple form (see Gerber and Shiu 1998, eq. (6.37)).

- Let $B(u, 0, b)$ denote the Laplace transform of τ^+ (which can be interpreted as the expected present value of a payment of 1 at the time when the surplus reaches the level b for the first time provided that ruin has not occurred yet). From Gerber and Shiu (1998, eq. (6.25)) we have that

$$B(u, 0, b) := \mathbb{E}[e^{-\delta\tau^+(u,0,b)} | U(0) = u] = \frac{e^{\rho u} - \Psi_\delta(u)}{e^{\rho b} - \Psi_\delta(b)}.$$

- Let $f(x, y, t|u)$ denote the joint probability density of $U(\tau_-(u))$, $|U(\tau(u))|$ and $\tau(u)$. The joint density of the first two quantities is given by

$$f(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u) dt = \begin{cases} \frac{\lambda}{c - a} e^{-\rho x} p(x + y), & u = 0, \\ f(x, y|0) \frac{e^{\rho x} \Psi_\delta(u - x) - \Psi_\delta(u)}{1 - \Psi_\delta(0)}, & 0 < x \leq u, \\ f(x, y|0) \frac{e^{\rho u} - \Psi_\delta(u)}{1 - \Psi_\delta(0)}, & x > u; \end{cases}$$

see Gerber and Shiu (1998, eqs. (6.34)–(6.36)) (for the renewal model see Dickson and Drekic (2004), and for a more general semi-Markovian framework, see Albrecher and Boxma (2005)).

Gerber, Lin, and Yang (2006) recently used the quantity $B(u, 0, b)$ to derive a dividend-penalty identity for upwards skip-free strong Markov models. This nice identity allows one to construct constant dividend barriers that maximize the expected difference between the present value of the accumulated dividends and the discounted penalty at ruin.

The following observation is a crucial step in our derivations (see Zhou (2005) for the case $\delta = 0$).

Lemma 3.1

For $s_1 \leq u < s_2$,

$$\begin{aligned} & \mathbb{E}[e^{-\delta\tau^-(u,s_1,s_2)}H(U(\tau^-(u,s_1,s_2)-), |U(\tau^-(u,s_1,s_2))|)] \\ &= \mathbb{E}[e^{-\delta\tau^*(u,s_1,\infty)}H(U(\tau^*(u,s_1,\infty)-), |U(\tau^*(u,s_1,\infty))|)] \\ & \quad - \mathbb{E}[e^{-\delta\tau^+(u,s_1,s_2)}] \mathbb{E}[e^{-\delta\tau^*(s_2,s_1,\infty)}H(U(\tau^*(s_2,s_1,\infty)-), |U(\tau^*(s_2,s_1,\infty))|)], \end{aligned}$$

where H is any function such that the expectations exist.

PROOF

The result follows from

$$\begin{aligned} & \mathbb{E}[e^{-\delta\tau^-(u,s_1,s_2)}H(U(\tau^-(u,s_1,s_2)-), |U(\tau^-(u,s_1,s_2))|)] \\ &= \mathbb{E}[e^{-\delta\tau^*(u,s_1,s_2)}H(U(\tau^*(u,s_1,s_2)-), |U(\tau^*(u,s_1,s_2))|)I_{\{\tilde{U}(\tau^*(u,s_1,s_2)) < s_2\}}] \end{aligned}$$

and splitting $I_{\{\tilde{U} < a\}} = 1 - I_{\{\tilde{U} \geq a\}}$, using the strong Markov property at time $\tau^+(u, s_1, s_2)$ in the last term. \square

3.2 Time Value of ‘‘Upper Exit’’

Let $B_k(u, b) := B_k(u, 0, b) = \mathbb{E}[e^{-\delta\tau_k^+(u,0,b)}]$ denote the Laplace transform of the stopping time $\tau_k^+(u, 0, b)$ in the risk model with k layers. Recall that the notation $B_{k-1}(u, b)$ then corresponds to $B_k(u, b)$ with b_{k-1} shifted to infinity.

Lemma 3.2

For $k \in \mathbb{N}$ and $\delta > 0$, we have

(i)

$$B_k(u, b) = 1 \quad \text{if } u \geq b,$$

$$B_k(u, b) = 0 \quad \text{if } u < 0$$

(ii) For $0 \leq u < b_{k-1}$

$$B_k(u, b) = \begin{cases} B_{k-1}(u, b) & \text{if } b \leq b_{k-1}, \\ B_{k-1}(u, b_{k-1})B_k(b_{k-1}, b) & \text{if } b \geq b_{k-1} \end{cases}$$

(iii) For $b_{k-1} \leq u \leq b$

$$\begin{aligned} B_k(u, b) &= B_{1,k}(u - b_{k-1}, b - b_{k-1}) + \mathbb{E}[e^{-\delta\tau_{1,k}(u-b_{k-1})}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|, b)] \\ & \quad - B_{1,k}(u - b_{k-1}, b - b_{k-1})\mathbb{E}[e^{-\delta\tau_{1,k}(b-b_{k-1})}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(b - b_{k-1}))|, b)]. \end{aligned}$$

PROOF

Here (i) is true by definition. In case (ii), as the processes $U_k(t)$ and $U_{k-1}(t)$ coincide before the first exit of the interval $[0, b_{k-1})$, we have $\tau_k^+(u, 0, b) = \tau_{k-1}^+(u, 0, b)$, and hence $B_k(u, b) = B_{k-1}(u, b)$ for $0 < b \leq b_{k-1}$. For $b \geq b_{k-1}$, the strong Markov property of the surplus process at the stopping time $\tau_{k-1}^+(u, 0, b_{k-1})$ gives

$$\mathbb{E}[e^{-\delta\tau_k^+(u,0,b)}] = \mathbb{E}[e^{-\delta(\tau_{k-1}^+(u,0,b_{k-1}) + \tau_k^+(b_{k-1},0,b))}] = B_{k-1}(u, b_{k-1})B_k(b_{k-1}, b),$$

establishing (ii).

Now let $b_{k-1} \leq u < b$. Then

$$\begin{aligned}
\mathbb{E}[e^{-\delta\tau_k^*(u,0,b)}] &= \mathbb{E}[e^{-\delta(\tau_k^*(u,b_{k-1},b)+\tau_k^*(U_k(\tau_k^*(u,b_{k-1},b)),0,b))}] \\
&= \mathbb{E}[e^{-\delta(\tau_k^*(u,b_{k-1},b))}I_{\{U_k(\tau_k^*(u,b_{k-1},b))=b\}}] + \mathbb{E}[e^{-\delta(\tau_k^*(u,b_{k-1},b)+\tau_k^*(U_k(\tau_k^*(u,b_{k-1},b)),0,b))}I_{\{U_k(\tau_k^*(u,b_{k-1},b))<b\}}] \\
&= B_{1,k}(u - b_{k-1}, b - b_{k-1}) + \mathbb{E}[e^{-\delta\tau_{1,k}(u-b_{k-1})}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|, b)] \\
&\quad - B_{1,k}(u - b_{k-1}, b - b_{k-1})\mathbb{E}[e^{-\delta\tau_{1,k}(b-b_{k-1})}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(b - b_{k-1}))|, b)],
\end{aligned}$$

where the last term corrects for those trajectories that arrive at b before dropping below b_{k-1} (since $\tau_k^-(u, b_{k-1}, b)$ is replaced by the ruin time $\tau_{1,k}(u - b_{k-1})$). \square

From Lemma 3.2(ii) at $u = 0$, we get

$$B_k(0, b) = B_{k-1}(0, b_{k-1})B_k(b_{k-1}, b). \quad (3.2)$$

On the other hand, we have from Lemma 3.2(iii) at $u = b_{k-1}$ that

$$\begin{aligned}
B_k(b_{k-1}, b) &= B_{1,k}(0, b - b_{k-1}) + \mathbb{E}[e^{-\delta\tau_{1,k}(0)}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|, b)] \\
&\quad - B_{1,k}(0, b - b_{k-1})\mathbb{E}[e^{-\delta\tau_{1,k}(b-b_{k-1})}B_k(b_{k-1} - |U_{1,k}(\tau_{1,k}(b - b_{k-1}))|, b)].
\end{aligned}$$

Using Lemma 3.2(ii) inside the above expectations, we obtain

$$B_k(b_{k-1}, b) = B_{1,k}(0, b - b_{k-1}) + H_k(0) \frac{B_k(0, b)}{B_{k-1}(0, b)} - B_{1,k}(0, b - b_{k-1})H_k(b - b_{k-1}) \frac{B_k(0, b)}{B_{k-1}(0, b)}, \quad (3.3)$$

where

$$H_k(\varpi) = \mathbb{E}[e^{-\delta\tau_{1,k}(\varpi)}B_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(\varpi))|, b_{k-1})], \quad \varpi \geq 0.$$

This finally gives

$$B_k(0, b) = \frac{B_{1,k}(0, b - b_{k-1})B_{k-1}(0, b_{k-1})}{1 - H_k(0) + B_{1,k}(0, b - b_{k-1})H_k(b - b_{k-1})}, \quad (3.4)$$

providing a formula for $B_k(0, b)$ that solely depends on quantities from lower layers. Note that in the numerator we have the expression for a path from 0 to b_{k-1} and then going on to b , and the numerator gives the correction term for the possibility to fall again below b_{k-1} in between. This immediately gives a formula for $B_k(u, b)$ in terms of quantities from lower layers.

Proposition 3.1

For $k = 2, 3, \dots$, we have the following formulas.

(i) For $0 \leq u < b_{k-1} < b$:

$$B_k(u, b) = \frac{B_{1,k}(0, b - b_{k-1})B_{k-1}(u, b_{k-1})}{1 - H_k(0) + B_{1,k}(0, b - b_{k-1})H_k(b - b_{k-1})}.$$

(ii) For $b_{k-1} \leq u < b$:

$$B_k(u, b) = \frac{B_{1,k}(u - b_{k-1}, b - b_{k-1})(1 - H_k(0)) + H_k(u - b_{k-1})B_{1,k}(0, b - b_{k-1})}{1 - H_k(0) + B_{1,k}(0, b - b_{k-1})H_k(b - b_{k-1})}.$$

(iii) For $b < b_{k-1}$: $B_k(u, b) = B_{k-1}(u, b)$.

PROOF

Assertion (i) follows from Lemma 3.2(ii), (3.2) and (3.4). Assertion (ii) follows from substitution of assertion (i) inside the expectations in Lemma 3.2(iii). Finally, assertion (iii) is just Lemma 3.2(ii). \square

We now work out an explicit expression for $B_k(u, b)$ for exponential claim sizes. To that end, we introduce the following notation:

$$\begin{aligned} Z_i(u) &= (\beta + \rho_{1,i})e^{\rho_{1,i}u} - (\beta - R_{1,i})e^{-R_{1,i}u}, \\ \bar{Z}_i(u) &= e^{\rho_{1,i}u} - e^{-R_{1,i}u}, \\ \hat{Z}_i(u) &= e^{\rho_{1,i}u}/(\beta + \rho_{1,i}) - e^{-R_{1,i}u}/(\beta - R_{1,i}). \end{aligned}$$

Proposition 3.2

Let $k \in \mathbb{N}$ be the number of layers, $P(u) = 1 - e^{-\beta u}$, and $c_i = c$ for $i = 1, \dots, k$. Consider the following recursions:

$$\begin{aligned} N_1(u) &= Z_1(u), \quad N_i(u) = N_{i-1}(b_{i-1})Z_i(u - b_{i-1}) - (\beta - R_{1,i})(\beta + \rho_{1,i})M_i\bar{Z}_i(u - b_{i-1}) \quad \text{for } i \geq 2 \\ M_2 &= \bar{Z}_1(b_1), \quad M_{i+1} = N_{i-1}(b_{i-1})\bar{Z}_i(b_i - b_{i-1}) - (\beta - R_{1,i})(\beta + \rho_{1,i})M_i\hat{Z}_i(b_i - b_{i-1}) \quad \text{for } i \geq 2. \end{aligned}$$

Let $b \geq b_{k-1}$. Then, for $i = 1, \dots, k$ and $b_{i-1} \leq u < b_i$ and $u < b$, we have

$$B_k(u, b) = \frac{N_i(u)}{N_k(b)} \prod_{j=i+1}^k (\rho_{1,j} + R_{1,j}), \quad (3.5)$$

where the convention $\prod_{j=k+1}^k \cdot = 1$ is used.

From the previous considerations it is clear that this determines the expression $B_k(u, b)$ for all the other combinations of u and b too.

PROOF

In a first step, we show by induction that for all $k \geq 1$ the auxiliary quantity M_k fulfills

$$e^{\beta b_k} M_{k+1} = \sum_{i=1}^k \int_{b_{i-1}}^{b_i} N_i(y) e^{\beta y} dy \prod_{j=i+1}^k (\rho_{1,j} + R_{1,j}). \quad (3.6)$$

Trivially, we have $e^{\beta b_1} M_2 = e^{\beta b_1} (e^{\rho_{1,1} b_1} - e^{-R_{1,1} b_1}) = \int_0^{b_1} Z_1(y) e^{\beta y} dy$, which is (3.6) for $k = 1$. Assume that (3.6) holds for $k - 1$, then we have for k

$$\begin{aligned} e^{\beta b_k} M_{k+1} &= e^{\beta b_k} (\bar{Z}_k(b_k - b_{k-1})N_{k-1}(b_{k-1}) - (\beta - R_{1,k})(\beta + \rho_{1,k})M_k\hat{Z}_k(b_k - b_{k-1})) \\ &= e^{\beta b_k} \bar{Z}_k(b_k - b_{k-1})N_{k-1}(b_{k-1}) - (\rho_{1,k} + R_{1,k})e^{\beta b_{k-1}} M_k \\ &\quad - e^{\beta b_k} (\beta - R_{1,k})(\beta + \rho_{1,k})M_k\hat{Z}_k(b_k - b_{k-1}) + (\rho_{1,k} + R_{1,k})e^{\beta b_{k-1}} M_k \\ &= \int_{b_{k-1}}^{b_k} N_k(y) e^{\beta y} dy + (\rho_{1,k} + R_{1,k})e^{\beta b_{k-1}} M_k \\ &= \sum_{i=1}^k \int_{b_{i-1}}^{b_i} N_i(y) e^{\beta y} dy \prod_{j=i+1}^k (\rho_{1,j} + R_{1,j}), \end{aligned}$$

where the induction hypothesis was used in the last line.

Now we can prove (3.5) by another induction. First, we know from Gerber and Shiu (1998) that

$$B_1(u, b) = \frac{N_1(u)}{N_1(b)} = \frac{Z_1(u)}{Z_1(b)},$$

which coincides with (3.5) for $k = 1$. Assume then that (3.5) holds for k . We start by calculating $H_{k+1}(0)$. Observe that $H_i(v) = H_i(0)e^{-R_{1,i}v}$ (see Gerber and Shiu 1998, eq. (5.42)), and furthermore

$$H_i(0) = (\beta - R_{1,i}) \int_0^{b_{i-1}} B_{i-1}(b_{i-1} - y, b_{i-1}) e^{-\beta y} dy,$$

for all $i \in \mathbb{N}$. Thus by the validity of (3.5) and (3.4) for k ,

$$H_{k+1}(0) = (\beta - R_{1,k+1}) e^{-\beta b_k} \int_0^{b_k} B_k(y, b_k) e^{\beta y} dy = \frac{(\beta - R_{1,k+1}) M_{k+1}}{N_k(b_k)}.$$

The denominator in Proposition 3.1(i) and (ii) for $k + 1$ layers is now given by

$$\begin{aligned} 1 - H_{k+1}(0) + B_{1,k+1}(0, b - b_k) H_{k+1}(b - b_k) \\ = \frac{(N_k(b_k) - (\beta - R_{1,k+1}) M_{k+1}) Z_{k+1}(b - b_k) + (\rho_{1,k+1} + R_{1,k+1})(\beta - R_{1,k+1}) M_{k+1} e^{-R_{1,k+1}(b-b_k)}}{N_k(b_k) Z_{k+1}(b - b_k)} \\ = \frac{N_{k+1}(b)}{N_k(b_k) Z_{k+1}(b - b_k)}, \end{aligned}$$

since

$$Z_{k+1}(b - b_k) - (\rho_{1,k+1} + R_{1,k+1}) e^{-R_{1,k+1}(b-b_k)} = (\beta + \rho_{1,k+1}) \bar{Z}_{k+1}(b - b_k).$$

Now, Proposition 3.1(ii) and a little algebra gives for $u \geq b_k$

$$B_{k+1}(u, b) = \frac{N_k(b_k) Z_{k+1}(u - b_k) - (\beta - R_{1,k+1})(\beta + \rho_{1,k+1}) M_k \bar{Z}_{k+1}(u - b_k)}{N_{k+1}(b)} = \frac{N_{k+1}(u)}{N_{k+1}(b)}.$$

Finally, Proposition 3.1(i) and (3.5) for $i = 1, \dots, k$ gives for $b_{i-1} \leq u < b_i$

$$B_{k+1}(u, b) = \frac{N_i(u) (\rho_{1,k+1} + R_{1,k+1}) \prod_{j=i+1}^k (\rho_{1,j} + R_{1,j})}{\frac{N_k(b_k) Z_{k+1}(b - b_k)}{N_{k+1}(b)}} = \frac{N_i(u) \prod_{j=i+1}^{k+1} (\rho_{1,j} + R_{1,j})}{N_{k+1}(b)}.$$

□

Example 3.1

For $k = 2$ and exponential claim amounts we obtain the explicit expression $B_2(u, b) = A_1/A_2$ with

$$A_1 = (\rho_{1,2} + R_{1,2}) ((\beta + \rho_{1,1}) e^{\rho_{1,1} u} - (\beta - R_{1,1}) e^{-R_{1,1} u}),$$

for $0 \leq u < b_1$,

$$\begin{aligned} A_1 = & ((\beta + \rho_{1,1}) e^{\rho_{1,1} b_1} - (\beta - R_{1,1}) e^{-R_{1,1} b_1}) ((\beta + \rho_{1,2}) e^{\rho_{1,2}(u-b_1)} - (\beta - R_{1,2}) e^{-R_{1,2}(u-b_1)}) \\ & - (\beta - R_{1,2})(\beta + \rho_{1,2}) (e^{\rho_{1,1} b_1} - e^{-R_{1,1} b_1}) (e^{\rho_{1,2}(u-b_1)} - e^{-R_{1,2}(u-b_1)}), \end{aligned}$$

for $u \geq b_1$, and

$$\begin{aligned} A_2 = & ((\beta + \rho_{1,1}) e^{\rho_{1,1} b_1} - (\beta - R_{1,1}) e^{-R_{1,1} b_1}) ((\beta + \rho_{1,2}) e^{\rho_{1,2}(b-b_1)} - (\beta - R_{1,2}) e^{-R_{1,2}(b-b_1)}) \\ & - (\beta - R_{1,2})(\beta + \rho_{1,2}) (e^{\rho_{1,1} b_1} - e^{-R_{1,1} b_1}) (e^{\rho_{1,2}(b-b_1)} - e^{-R_{1,2}(b-b_1)}). \end{aligned}$$

For illustrative purposes, we give the corresponding explicit formula for the four-layer model with parameters (2.13) and $\delta = 0.01$ (with $b = 20$):

$$B_4(u, 20) = \begin{cases} 0, & u < 0 \\ 0.483e^{0.024u} - 0.329e^{-0.302u}, & 0 \leq u \leq 5 \\ 0.475e^{0.030u} - 0.290e^{-0.253u}, & 5 \leq u \leq 10 \\ 0.413e^{0.0417u} - 0.0498e^{-0.2u}, & 10 \leq u \leq 15 \\ 0.537e^{-0.145u} + 0.276e^{0.063u}, & 15 \leq u \leq 20 \\ 1, & u \geq 20 \end{cases}$$

Figure 3 depicts $B_k(u, 20)$ as a function of u for $k = 1, \dots, 4$ for this parameter setting (where, again, for each k the higher layers are shifted to infinity).

3.3 The Expected Discounted Dividends

We now show how to use the quantities of the previous section for the calculation of the expected discounted dividends $W_{k,1}$ in the k -layer model. In the sequel $W_{(1,k),1}$ denotes the expected discounted dividends in a one-layer model with parameters from the k th layer. Let $0 \leq u < b_{k-1}$ and condition on the event of either reaching b_{k-1} or getting ruined:

$$W_{k,1}(u) = \mathbb{E} \left[\int_0^{\tau_k^*(u, 0, b_{k-1})} e^{-\delta t} dD_k(t) \right] + \mathbb{E} \left[\int_{\tau_k^*(u, 0, b_{k-1})}^{\tau_k(u)} e^{-\delta t} dD_k(t) \right] := I_1 + I_2. \tag{3.7}$$

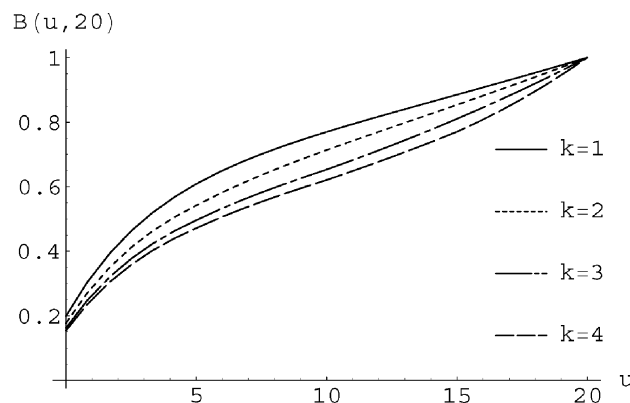
The second term gives only a nonzero contribution in the case $\tau_k^*(u, 0, b_{k-1}) = \tau_k^+(u, 0, b_{k-1})$, which (using $\tau_k^+(u, 0, b_{k-1}) = \tau_{k-1}^+(u, 0, b_{k-1})$) leads to

$$I_2 = \mathbb{E}[e^{-\delta \tau_{k-1}^+(u, 0, b_{k-1})}] W_{k-1}(b_{k-1}) = B_{k-1}(u, b_{k-1})W_{k,1}(b_{k-1}).$$

In the first term in (3.7) we replace $\tau_k^*(u, 0, b_{k-1}) = \tau_{k-1}^*(u, 0, b_{k-1})$ by the ruin time $\tau_{k-1}(u)$ and correct, for all trajectories of the process that exceed b_{k-1} , for the dividend contribution after this event to obtain

$$\begin{aligned} I_1 &= \mathbb{E} \left[\int_0^{\tau_{k-1}^*(u, 0, b_{k-1})} e^{-\delta t} dD_{k-1}(t) \right] \\ &= \mathbb{E} \left[\int_0^{\tau_{k-1}(u)} e^{-\delta t} dD_{k-1}(t) \right] - \mathbb{E}[e^{-\delta \tau_{k-1}^+(u, 0, b_{k-1})}] \mathbb{E} \left[\int_{\tau_{k-1}^+(u, 0, b_{k-1})}^{\tau_{k-1}(b_{k-1})} e^{-\delta t} dD_{k-1}(t) \right] \\ &= W_{k-1,1}(u) - B_{k-1}(u, b_{k-1})W_{k-1,1}(b_{k-1}). \end{aligned}$$

Figure 3
 $B_k(u, 20)$ for Parameter Set (2.13) and $k = 1, \dots, 4$



Hence

$$W_{k,1}(u) = W_{k-1,1}(u) + B_{k-1}(u, b_{k-1})(W_{k,1}(b_{k-1}) - W_{k-1,1}(b_{k-1})).$$

Letting $u = 0$ in the previous equation, we arrive at

$$W_{k,1}(u) = W_{k-1,1}(u) + \frac{B_{k-1}(u, b_{k-1})}{B_{k-1}(0, b_{k-1})} (W_{k,1}(0) - W_{k-1,1}(0)). \quad (3.8)$$

Above the upper threshold, $u \geq b_{k-1}$, we have

$$\begin{aligned} W_{k,1}(u) &= W_{(1,k),1}(u - b_{k-1}) + \mathbb{E}[e^{-\delta\tau_{1,k}(u-b_{k-1})}W_{k,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|)] \\ &= W_{(1,k),1}(u - b_{k-1}) + \mathbb{E}[e^{\delta\tau_{1,k}(u-b_{k-1})}W_{k-1,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|)] \\ &\quad + \mathbb{E}[e^{\delta\tau_{1,k}(u-b_{k-1})}B_{k-1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|, b_{k-1})] \frac{W_{k,1}(0) - W_{k-1,1}(0)}{B_{k-1}(0, b_{k-1})}, \end{aligned} \quad (3.9)$$

where (3.8) was used in the last step. Now, evaluating (3.9) and (3.8) at $u = b_{k-1}$, we get by continuity

$$\frac{W_{k,1}(0) - W_{k-1,1}(0)}{B_{k-1}(0, b_{k-1})} = \frac{\mathbb{E}[e^{-\delta\tau_{1,k}(0)}W_{k-1,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|)] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})}{1 - H_k(0)}$$

and subsequently

$$W_{k,1}(u) = \begin{cases} W_{k-1,1}(u) + \frac{B_{k-1}(u, b_{k-1})}{1 - H_k(0)} (\mathbb{E}[e^{-\delta\tau_{1,k}(0)}W_{k-1,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|)] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})), & \text{for } 0 \leq u < b_{k-1} \\ W_{(1,k),1}(u - b_{k-1}) + \mathbb{E}[e^{-\delta\tau_{1,k}(u-b_{k-1})}W_{k-1,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(u - b_{k-1}))|)] \\ \quad + \frac{H_k(u - b_{k-1})}{1 - H_k(0)} (\mathbb{E}[e^{-\delta\tau_{1,k}(0)}W_{k-1,1}(b_{k-1} - |U_{1,k}(\tau_{1,k}(0))|)] + W_{(1,k),1}(0) - W_{k-1,1}(b_{k-1})), & \text{for } u \geq b_{k-1}. \end{cases}$$

The latter formula gives, for all values of u , $W_{k,1}(u)$ explicitly as a function of $W_{k-1,1}(u)$ only, that is, we can recursively determine the solution “bottom-up” (whereas in the differential approach of Section 2 it was necessary to simultaneously go through all the layers to obtain the solution through a system of linear equations). Hence this approach leads to analytic solution formulas whenever $B_j(u, b)$ ($j = 1, \dots, k$) is analytically available. From Proposition 3.2 we know that this is, for instance, the case for exponential claim sizes.

Example 3.2

For $\text{Exp}(\beta)$ claim sizes, one has

$$W_{1,1}(u) = \frac{\alpha_1}{\delta} (1 - \mathbb{E}[e^{-\delta\tau_1(u)}]) = \frac{\alpha_1}{\delta} \left(1 - \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u} \right)$$

(where the formula for the Laplace transform of τ_1 can, for instance, be found in Gerber and Shiu [1998]). Moreover, together with Gerber and Shiu (1998, eq. (5.42)),

$$\mathbb{E}[e^{-\delta\tau_{1,2}(u-b_1)}W_{1,1}(b_1 - |U_{1,2}(\tau_{1,2}(u - b_1))|)] = \frac{\alpha_1}{\delta} \frac{\beta - R_{1,2}}{\beta} e^{-R_{1,2}(u-b_1)}(1 - e^{-R_{1,1}b_1})$$

(here one used the fact the $\tau_{1,2}$ and $U_{1,2}(\tau_{1,2})$ are independent for exponential claims). In the two-layer case this leads to

$$W_{2,1}(u) = \begin{cases} \frac{a_1}{\delta} \left(1 - \frac{\beta - R_{1,1}}{\beta} e^{-R_{1,1}u} \right) + \frac{(\beta + \rho_{1,1})e^{\rho_{1,1}u} - (\beta - R_{1,1})e^{-R_{1,1}u}}{(R_{1,2} + \rho_{1,1})e^{\rho_{1,1}b_1} - (R_{1,2} - R_{1,1})e^{-R_{1,1}b_1}} \\ \left(\frac{a_2 - a_1 R_{1,2}}{\delta} + \frac{a_1 R_{1,2} - R_{1,1}}{\delta} e^{-R_{1,1}b_1} \right), & \text{for } 0 \leq u < b_1 \\ \frac{a_2}{\delta} - \frac{1}{\delta} \frac{(a_2 - a_1)(\rho_{1,1}e^{\rho_{1,1}b_1} + R_{1,1}e^{-R_{1,1}b_1}) + a_1(\rho_{1,1} + R_{1,1})e^{(\rho_{1,1} - R_{1,1})b_1}}{(R_{1,2} + \rho_{1,1})e^{\rho_{1,1}b_1} - (R_{1,2} - R_{1,1})e^{-R_{1,1}b_1}} \\ \frac{(\beta - R_{1,2})}{\beta} e^{-R_{1,2}(u-b_1)}, & \text{for } u \geq b_1, \end{cases}$$

which for the special case $a_1 = 0$ coincides with formula (6.15) of Gerber and Shiu (2006).

Coming back to the example of the four-layer model with parameters (2.13) and $\delta = 0.01$, we obtain for this case

$$W_{2,1}(u) = \begin{cases} e^{-0.302u}(-15.841 + 23.238e^{0.326u}), & 0 \leq u < 5 \\ 10 - 10.483e^{-0.505u} - 8.316e^{-0.253u} + 13.609e^{0.031u}, & 5 \leq u < 10 \\ 20 - 195.233e^{-0.669u} - 0.638e^{-0.2u} + 5.299e^{0.0417u}, & 10 \leq u < 15 \\ 30 - 10885.564e^{-0.751u}, & u \geq 15 \end{cases}$$

Figure 4 depicts $W_{k,1}(u)$ as a function of u for $k = 2, \dots, 4$ (again, for each k the higher layers are shifted to infinity)

3.4 The Discounted Penalty Function

In this section we consider the recursive approach for the discounted penalty function defined in (2.1). For $0 \leq u \leq b_{k-1}$, the same reasoning as in Section 3.3 yields

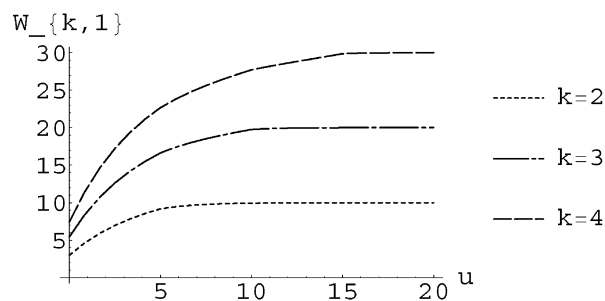
$$\begin{aligned} m_k(u) &= \mathbb{E}[e^{-\delta\tau_{\#}(u,0,b_{k-1})}]m_k(b_{k-1}) + m_{k-1}(u) - B_{k-1}(u, b_{k-1})m_{k-1}(b_{k-1}) \\ &= m_{k-1}(u) + B_{k-1}(u, b_{k-1})(m_k(b_{k-1}) - m_{k-1}(b_{k-1})). \end{aligned}$$

Evaluating this formula at $u = 0$, one obtains

$$m_k(u) = m_{k-1}(u) + \frac{B_{k-1}(u, b_{k-1})}{B_{k-1}(0, b_{k-1})} (m_k(0) - m_{k-1}(0)), \quad 0 \leq u \leq b_{k-1}. \quad (3.10)$$

For $u \geq b_{k-1}$ we distinguish whether ruin occurs directly from the k th layer or some lower layer is reached first:

Figure 4
 $W_{k,1}(u)$ for Parameter Set (2.13) and $k = 2, 3, 4$



$$\frac{m_3(0) - m_2(0)}{B_2(0, b_2)} B_2(u, b_2) = \frac{(N_1(b_1) - (\beta - R_{1,1})M_2)}{(N_1(b_1) - (\beta - R_{1,2})M_2)} \frac{(R_{1,2} - R_{1,3})e^{-R_{1,2}(b_2-b_1)}}{(N_2(b_2) - (\beta - R_{1,3})M_3)} \frac{N_i(u)}{\beta} \prod_{j=i+1}^2 (\rho_{1,j} + R_{1,j}),$$

and thus, for $u \geq b_2$, we have

$$m_3(u) = \frac{\prod_{j=1}^2 (N_j(b_j) - (\beta - R_{1,j})M_{j+1})}{\prod_{j=1}^2 (N_j(b_j) - (\beta - R_{1,j+1})M_{j+1})} e^{\sum_{j=1}^2 (R_{1,j+1} - R_{1,j})b_j} m_{1,3}(u).$$

For notational convenience, define $N_0(u) = 1$ and $M_1 = 0$, then for $k \geq 3$, we can derive

$$\begin{aligned} \frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0, b_{k-1})} B_{k-1}(u, b_{k-1}) \\ = \frac{\prod_{j=1}^{k-2} (N_j(b_j) - (\beta - R_{1,j})M_{j+1})}{\prod_{j=1}^{k-1} (N_j(b_j) - (\beta - R_{1,j+1})M_j)} (R_{1,k-1} - R_{1,k}) e^{-R_{1,k-1}(b_{k-1}-b_{k-2})} \frac{N_i(u)}{\beta} \prod_{j=i+1}^{k-1} (\rho_{1,j} + R_{1,j}). \end{aligned}$$

Using

$$N_j(b_j) - (\beta - R_{1,j})M_{j+1} = (R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}(N_{j-1}(b_{j-1}) - (\beta - R_{1,j})M_j),$$

we get for $b_{i-1} \leq u < b_i$, ($i = 1, \dots, k-1$)

$$\begin{aligned} Y_k^{(i)} &:= \frac{m_k(0) - m_{k-1}(0)}{B_{k-1}(0, b_{k-1})} \frac{B_{k-1}(u, b_{k-1})}{N_i(u)} \\ &= e^{\sum_{j=i}^{k-2} (R_{1,j+1} - R_{1,j})b_j} \frac{\prod_{j=1}^{k-2} (R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}}{(N_{k-1}(b_{k-1}) - (\beta - R_{1,k})M_k)} \frac{(R_{1,k-1} - R_{1,k})e^{-R_{1,k-1}(b_{k-1}-b_{k-2})}}{(N_{k-2}(b_{k-2}) - (\beta - R_{1,k-1})M_{k-1})} \frac{1}{\beta} \prod_{j=i+1}^{k-1} (\rho_{1,j} + R_{1,j}). \end{aligned}$$

Finally, with the notation of Section 2.1, for $i = 1, \dots, k$, we arrive at

$$m_k^{(i)}(u) = \frac{\prod_{j=1}^{i-1} (R_{1,j} + \rho_{1,j})e^{\rho_{1,j}(b_j-b_{j-1})}}{(N_{i-1}(b_{i-1}) - (\beta - R_{1,i})M_i)} e^{\sum_{j=1}^{i-1} (R_{1,j+1} - R_{1,j})b_j} m_{1,i}(u) + \left(\sum_{j=i+1}^k Y_j^{(i)} \right) N_i(u).$$

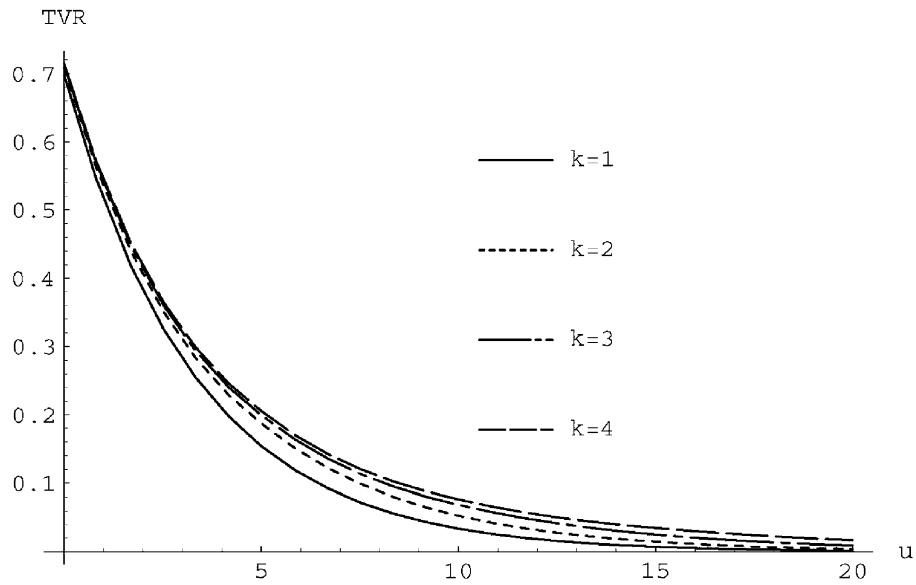
Observe that this provides an explicit formula for $m_k(u)$ in terms of the model parameters even for $\delta > 0$ (the quantities N, M are only abbreviations for available functions of the model parameters), whereas in the differential approach of Section 2 the case $\delta > 0$ led to a system of equations that did not give rise to an explicit solution.

Finally, we again illustrate the tractability of the approach by giving the time value of ruin for the four-layer case with exponential claims and parameter setting (2.13) and $\delta = 0.01$:

$$\mathbb{E}(e^{-0.01\tau_4(u)}) = \begin{cases} 0.662e^{-0.302u} + 0.0528e^{0.024u}, & 0 \leq u < 5 \\ 0.654e^{-0.253u} + 0.0183e^{0.030u}, & 5 \leq u < 10 \\ 0.508e^{-0.2u} + 0.005e^{0.042u}, & 10 \leq u < 15 \\ 0.307e^{-0.145u}, & 15 \leq u < \infty. \end{cases}$$

Figure 5 depicts $\mathbb{E}(e^{-0.01\tau_4(u)})$ as a function of u for $k = 1, \dots, 4$ (for each k the higher layers are shifted to infinity). The formula for arbitrary δ also can be obtained with the aid of a symbolic computation program such as Mathematica, but for larger values of k this can lead to very lengthy expressions. Taking derivatives of this general expression with respect to δ and evaluating the result at

Figure 5
 $\mathbb{E}(e^{-0.01\tau_4(u)})$ for Parameter Set (2.13) and $k = 1, \dots, 4$

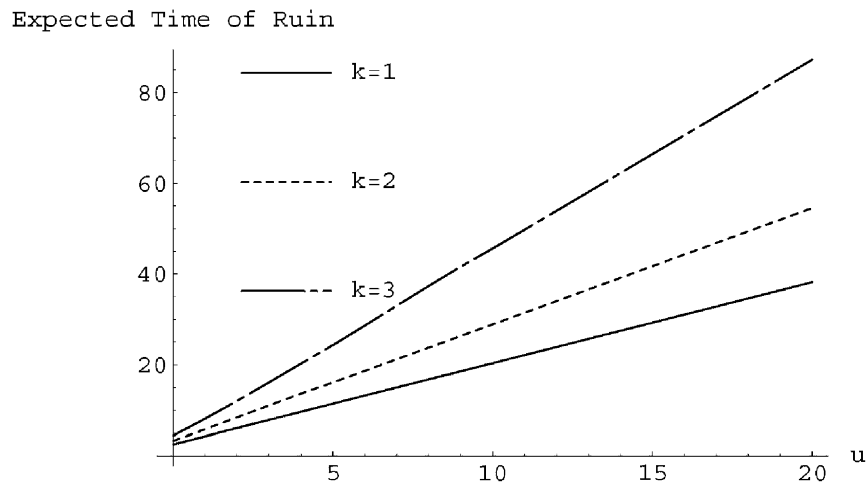


$\delta = 0$ gives the moments of the time of ruin, given that ruin occurs. Here we give the corresponding result for the first moment in the three-layer model (i.e., b_3 is shifted to infinity):

$$\mathbb{E}(\tau_3(u) | \tau_3(u) < \infty) = \begin{cases} \frac{600.062 - 32.897u}{7.676 + e^{0.280u}} - 2.5u + 73.594, & 0 \leq u \leq 5 \\ -3.343u + \frac{118.856u - 2463.073}{20.122 + e^{0.231u}} + 121.277, & 5 \leq u < 10 \\ 4.167u + 3.952, & 10 \leq u < \infty. \end{cases}$$

Figure 6 depicts $\mathbb{E}(\tau_k(u) | \tau_k(u) < \infty)$ as a function of u for $k = 1, 2, 3$.

Figure 6
 $\mathbb{E}(\tau_k(u) | \tau_k(u) < \infty)$ for Parameter Set (2.13) and $k = 1, 2, 3$



6. ACKNOWLEDGMENT

We would like to thank Prof. Sheldon Lin and two anonymous referees for helpful comments that improved the presentation of the manuscript.

REFERENCES

- ALBRECHER, H., AND O. BOXMA. 2005. On the Discounted Penalty Function in a Markov-Dependent Risk Model. *Insurance: Mathematics and Economics* 37(3): 650–72.
- ALBRECHER, H., M. M. CLARAMUNT, AND M. MÁRMOL. 2005. On the Distribution of Dividend Payments in a Sparre Andersen Model with Generalized Erlang(n) Interclaim Times. *Insurance: Mathematics & Economics* 37(2): 324–34.
- ALBRECHER, H., J. HARTINGER, AND R. TICHY. 2005. On the Distribution of Discounted Dividend Payments and the Discounted Penalty Function in a Risk Model with Linear Dividend Barrier. *Scandinavian Actuarial Journal*, 103–26.
- ALBRECHER, H., AND S. THONHAUSER. 2006. Discussion of “Optimal Dividend Strategies in the Compound Poisson Model.” *North American Actuarial Journal* 10(3): 68–71.
- ASMUSSEN, S. 2000. *Ruin Probability*. Singapore: World Scientific.
- ASMUSSEN, S., AND H. NIELSEN. 1995. Ruin Probabilities via Local Adjustment Coefficients. *Journal of Applied Probability* 32(3): 736–55.
- AZCUE, P., AND N. MULER. 2005. Optimal Reinsurance and Dividend Distribution Policies in the Cramér-Lundberg Model *Mathematical Finance* 15(2): 261–308.
- DICKSON, D., AND S. DREKIC. 2004. The Joint Distribution of the Surplus Prior to Ruin and the Deficit at Ruin in Some Sparre Andersen Models. *Insurance Mathematics and Economics* 34(1): 97–107.
- DICKSON, D., AND H. WATERS. 2004. Some Optimal Dividends Problems. *ASTIN Bulletin* 34(1): 49–74.
- GERBER, HANS U. 1969. Entscheidungskriterien für den zusammengesetzten Poisson-Prozess. *Mitteilungen der Schweizerischen Aktuarvereinigung* (1): 185–227.
- GERBER, HANS U., X. S. LIN, AND H. YANG. 2006. A Note on the Dividends-Penalty Identity and the Optimal Dividend Barrier. *ASTIN Bulletin* 36(2): 489–503.
- GERBER, HANS U., AND ELIAS S. W. SHIU. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2: 48–78.
- . 2004. Optimal Dividends: Analysis with Brownian Motion. *North American Actuarial Journal* 8: 1–20.
- . 2006. On Optimal Dividend Strategies in the Compound Poisson Model. *North American Actuarial Journal* 10(2): 76–93.
- KEREKESHA, D. P. 2004. An Exact Solution of the Risk Equation with a Step Current Reserve Function. *Theory of Probability and Mathematical Statistics* 69: 61–66.
- LIN, X. S., AND K. P. PAVLOVA. 2006. The Compound Poisson Risk Model with a Threshold Dividend Strategy. *Insurance: Mathematics and Economics* 38: 57–80.
- LIN, X. S., AND K. P. SENDOVA. 2006. The Compound Poisson Risk Model with Multiple Thresholds. Preprint.
- SCHMIDL, H. 2007. *Stochastic Control in Insurance* New York: Springer.
- SCHOCK PETERSEN, S. 1990. Calculation of Ruin Probabilities When the Premium Depends on the Current Reserve. *Scandinavian Actuarial Journal*, 147–59.
- ZHOU, X. 2004. When Does Surplus Reach a Certain Level before Ruin? *Insurance: Mathematics & Economics* 35(3): 553–61.
- . 2005. On a Classical Risk Model with a Constant Dividend Barrier. *North American Actuarial Journal* 9(4): 95–108.
- . Classical Risk Model with a Multi-layer Premium Rate. Preprint.

Discussions on this paper can be submitted until October 1, 2007. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.