Moments of the Dividend Payments and Related Problems in a Markov-Modulated Risk Model

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Abstract

In this paper we derive some results on the dividend payments prior to ruin in a Markov-modulated risk process in which the rate for the Poisson claim arrival process and the distribution of the claim sizes vary in time depending on the state of an underlying (external) Markov jump process $\{J(t);\ t \geq 0\}$. The main feature of the model is the flexibility in modeling the arrival process in the sense that periods with very frequent arrivals and periods with very few arrivals may alternate, and that the states of $\{J(t);\ t \geq 0\}$ could describe, for example, epidemic types in health insurance or weather conditions in car insurance. A system of integro-differential equations with boundary conditions satisfied by the nth moment of the present value of the total dividends prior to ruin, given the initial environment state, is derived and solved. We show that the probabilities that the surplus process attains a dividend barrier from the initial surplus without first falling below zero and the Laplace transforms of the time that the surplus process first hits a barrier without ruin occurring can be expressed in terms of the solution of the above-mentioned system of integro-differential equations. In the two-state model, explicit results are obtained when both claim amounts are exponentially distributed.

1. Introduction

Consider a risk model in continuous time. Denote by $\{J(t);\ t \geq 0\}$ the external environment process, which influences the frequencies of the claims and the distributions of the claims. As pointed out by Asmussen (1989), in health insurance, sojourns of $\{J(t);\ t \geq 0\}$ could be certain types of epidemics, or, in automobile insurance, these could be weather types (e.g., icy, foggy). The motivation for this particular type of generalization is partly the flexibility in the modeling of the arrival process, allowing one to model arrival streams that are more irregular than any renewal process, and partly that in some cases, one can interpret the model in a natural way in the sense that an underlying external environment may involve the insurance business (see Asmussen et al. 1995). Suppose that $\{J(t);\ t \geq 0\}$ is a homogeneous, irreducible, and recurrent Markov process with finite state space $E = \{1, 2, \ldots, m\}$. Denote the intensity matrix of $\{J(t);\ t \geq 0\}$ by $\mathbf{\Lambda} = (\alpha_{i,j})_{i,j=1}^{m}$, with $\alpha_{i,i} := -\alpha_i$ for $i \in E$.

Let $N(t)$ be the number of claims occurring in $(0, t]$. If $J(s) = i$ for all s in a small interval $(t, t + h]$, then the number of claims occurring in that interval, $N(t + h) - N(t)$, has a Poisson distribution with parameter λ_i (> 0). We assume further that given the process $\{J(t);\ t \geq 0\}$, the process $\{N(t);\ t \geq 0\}$ has independent increments. Then

$$\mathbb{P}[N(t + h) = n + 1 | N(t) = n, J(s) = i \quad \text{for } t < s \leq t + h] = \lambda_i h + o(h).$$

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The process $\{N(t); t \geq 0\}$ is called a Markov-modulated Poisson process, which is a special case of Cox processes. It also can be seen as a Poisson process with the parameter driven by an external environment process $\{J(t); t \geq 0\}$.

We also assume that, given $J(t) = i$, the claim amounts have distribution $F_i(x)$, with density function $f_i(x)$ and finite mean μ_i $(i \in E)$. Moreover, we assume that premiums are recieved continuously at a positive constant rate c. The corresponding surplus process $\{U(t); t \geq 0\}$ is given by

$$U(t) = u + ct - \sum_{n=1}^{N(t)} X_n, \qquad t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial surplus and X_n is the amount of the nth claim.

The definition of this risk model using an environmental Markov process $\{J(t); t \geq 0\}$ is first given in Asmussen (1989), where process J models the random environment of an insurance business. Models of this type have also been investigated by some authors, including Reinhard (1984), Bäuerle (1996), Schmidli (1997), Wu (1999), Snoussi (2002), Lu and Li (2005), and Lu (2006).

In this paper we consider the surplus process (1.1) modified by the payment of dividends. When the surplus exceeds a constant barrier b $(\geq u)$, dividends are paid continuously so the surplus stays at the level b until a new claim occurs. Let $U_b(t)$ be the surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy and define $T_b = \inf\{t \geq 0: U_b(t) < 0\}$ to be the time of ruin. Let $\delta > 0$ be the force of interest for valuation and define

$$D_{u,b} = \int_0^{T_b} e^{-\delta t}\, dD(t), \qquad 0 \leq u \leq b,$$

to be the present value of all dividends until time of ruin T_b given that the initial surplus is u, where $D(t)$ is the aggregate dividends paid by time t. Define the mean of $D_{u,b}$, given that the initial environment state is i, by

$$V_i(u; b) = \mathbb{E}[D_{u,b} | J(0) = i], \qquad 0 \leq u \leq b,\ i \in E.$$

Then the expected present value of the total dividend payments until ruin in the stationary case is given by

$$V(u; b) = \sum_{i=1}^{m} \zeta_i V_i(u; b), \qquad 0 \leq u \leq b,$$

where $\zeta = (\zeta_1, \ldots, \zeta_m)$ is the stationary initial distribution of process $\{J(t); t \geq 0\}$.

The barrier strategy was initially proposed by de Finetti (1957) for a binomial model. More general barrier strategies have been studied in a number of papers and books, including Bühlmann (1970), Segerdahl (1970), Gerber (1972, 1979, 1981), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Lin, Willmot, and Drekic (2003), Dickson and Waters (2004), Li and Garrido (2004), Albrecher, Claramunt, and Marmol (2005), Albrecher and Hartinger (2006), Gerber, Lin, and Yang (2006), Gerber and Shiu (2006a), Li (2006), and Li and Dickson (2006). The main focus of these publications is on optimal dividend payouts and problems associated with time of ruin, under various barrier strategies and other economic conditions. For a Brownian motion risk process model, detailed analysis can be found in Gerber and Shiu (2004, 2006b).

2. INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

We now derive a system of integro-differential equations satisfied by $V_i(u; b)$, for $i \in E$ and $0 \leq u < b$. Considering a small time interval $[0, h]$, with h (> 0) being sufficiently small that $u + ch < b$, there are four possible events regarding the occurrence of the claim and the change of the environment:

1. No claim and no change of environment occur in $[0, h]$
2. A claim occurs in $[0, h]$ (it can either cause the ruin or not)

3. The environment changes in $[0, h]$ and
4. Two or more events occur in $[0, h]$.

Conditioning on the event occurring in the interval $[0, h]$, we have

$$V_i(u; b) = (1 - \alpha_i h - \lambda_i h)e^{-\delta h} V_i(u + ch; b) + \lambda_i h e^{-\delta h} \int_0^{u+ch} V_i(u + ch - x; b)\, dF_i(x)$$
$$+ h e^{-\delta h} \sum_{k=1,k\neq i}^{m} \alpha_{i,k} V_k(u + ch; b) + o(h), \qquad 0 \le u < b,\ i \in E,$$

where $o(h)/h \to 0$, when $h \to 0$. Since $e^{-\delta h} = 1 - \delta h + o(h)$, we then get

$$V_i(u; b) = [1 - (\alpha_i + \lambda_i + \delta)h] V_i(u + ch; b) + \lambda_i h \int_0^{u+ch} V_i(u + ch - x; b)\, dF_i(x)$$
$$+ h \sum_{k=1,k\neq i}^{m} \alpha_{i,k} V_k(u + ch; b) + o(h), \qquad 0 \le u < b,\ i \in E. \tag{2.1}$$

It follows from equation (2.1) that $V_i(u; b)$ is continuous in u. Moreover, we have, for $0 \le u < b$ and $i \in E$,

$$\frac{V_i(u + ch; b) - V_i(u; b)}{h} = (\lambda_i + \alpha_i + \delta) V_i(u + ch; b) - \lambda_i \int_0^{u+ch} V_i(u + ch - x; b)\, dF_i(x)$$
$$- \sum_{k=1,k\neq i}^{m} \alpha_{i,k} V_k(u + ch; b) + \frac{o(h)}{h}. \tag{2.2}$$

Letting $h \to 0$ in (2.2) and noting that $\alpha_{i,i} = -\alpha_i$, we get, for $0 \le u < b$ and $i \in E$, a system of integro-differential equations satisfied by $V_i(u; b)$:

$$cV_i'(u; b) = (\lambda_i + \delta) V_i(u; b) - \lambda_i \int_0^u V_i(u - x; b)\, dF_i(x) - \sum_{k=1}^{m} \alpha_{i,k} V_k(u; b). \tag{2.3}$$

For the case $u = b$, similarly, we condition on the event occurring in $[0, h]$ to obtain

$$V_i(b; b) = (1 - \alpha_i h - \lambda_i h)\left[e^{-\delta h} V_i(b; b) + c \int_0^h e^{-\delta s}\, ds\right]$$
$$+ \lambda_i h e^{-\delta h}\left[\int_0^b V_i(b - x; b)\, dF_i(x) + \int_0^h c e^{\delta s}\, ds\right]$$
$$+ h e^{-\delta h}\left[\sum_{k\neq i,k=1}^{m} \alpha_{i,k} V_k(b; b) + \int_0^h c e^{\delta s}\, ds\right] + o(h), \qquad i \in E.$$

Using the same arguments as above gives for $i \in E$

$$(\lambda_i + \delta) V_i(b; b) - \lambda_i \int_0^b V_i(b - x; b)\, dF_i(x) - \sum_{k=1}^{m} \alpha_{i,k} V_k(b; b) = c. \tag{2.4}$$

Setting $u = b$ in (2.3), using (2.4), and noting that $V_i(u; b)$ is continuous in u, we have that $V_i(u; b)$ satisfies conditions

$$V_i'(b-; b) = 1, \qquad i \in E.$$

Note that when $u \ge b$, dividend $u - b$ is paid immediately, so $V_i(u; b) = V_i(b; b) + u - b$ and $V_i'(b+; b) = 1$. Thus we have the following boundary conditions:

$$V_i'(u; b)\big|_{u=b} = 1, \qquad i \in E. \tag{2.5}$$

Now letting $v_i(u)$, $0 \le u < \infty$, $i \in E$, be the solutions of the integro-differential equations (2.3), we get, for $i \in E$,

$$cv_i'(u) = (\lambda_i + \delta)v_i(u) - \lambda_i \int_0^u v_i(u - x)\, dF_i(x) - \sum_{k=1}^{m} \alpha_{i,k} v_k(u). \tag{2.6}$$

The solutions of (2.6) are uniquely determined by the initial conditions $v_i(0)$, $i \in E$. Further for $j \in E$, let $v_{1,j}(u)$, $v_{2,j}(u)$, . . . , $v_{m,j}(u)$ be the particular solutions of (2.6) with initial conditions $v_{i,j}(0) = I(i = j)$, where $I(\cdot)$ denotes the indicator function. Then the general solutions of (2.6) are of the form

$$v_i(u) = \sum_{j=1}^{m} v_j(0) v_{i,j}(u), \qquad i \in E.$$

It follows that the solutions to the integro-differential equations (2.3) with the boundary conditions (2.5) can be expressed as

$$V_i(u; b) = \sum_{j=1}^{m} V_j(0; b) v_{i,j}(u), \qquad 0 \le u \le b,\ i \in E,$$

or in a matrix form,

$$\mathbf{V}(u; b) = \mathbf{v}(u)\mathbf{V}(0; b), \qquad 0 \le u \le b,$$

where $\mathbf{V}(u; b) = (V_1(u; b), \ldots, V_m(u; b))^T$ is an m-dimensional vector, and $\mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^{m}$ is an $m \times m$ matrix.

The value of vector $\mathbf{V}(0; b)$ can be obtained from the m boundary conditions $V_i'(u; b)\big|_{u=b} = 1$, $i \in E$, by solving the matrix equation

$$\mathbf{v}'(b)\mathbf{V}(0; b) = \mathbf{1}, \qquad 0 \le u \le b,$$

where $\mathbf{v}'(b) = (v_{i,j}'(b))_{i,j=1}^{m}$ is an $m \times m$ matrix with the (i, j)-element being the derivative of the function $v_{i,j}(u)$ evaluated at $u = b$, and $\mathbf{1} = (1, \ldots, 1)^T$ an m-dimensional vector. Then the explicit expression for $\mathbf{V}(u; b)$ is given by

$$\mathbf{V}(u; b) = \mathbf{v}(u)\mathbf{V}(0; b) = \mathbf{v}(u)[\mathbf{v}'(b)]^{-1}\mathbf{1}, \qquad 0 \le u \le b.$$

For a related discussion, see Ng (2006).

REMARK

When $m = 1$, the Markov-modulated risk model simplifies to the classical risk model; then the expected present value of total dividend payments until ruin, $V_1(u, b)$, simplifies to

$$V_1(u; b) = \frac{v(u)}{v'(b)}, \qquad 0 \le u \le b,$$

where $v(u)$ is the solution of the following integro-differential equation:

$$cv'(x) - (\lambda + \delta)v(x) + \lambda \int_0^x v(x - y)\, dF(y) = 0, \qquad x \ge 0.$$

See Bühlmann (1970, Section 6.4.8) and Gerber (1979, Section 10.1).

3. MOMENT-GENERATING FUNCTION OF $D_{u,b}$ AND HIGHER MOMENT

In this section we study the moment-generating function of $D_{u,b}$, through which we can analyze the higher moment of the present value of all dividend payments prior to ruin. Define the moment-generating function of $D_{u,b}$, given that the initial environment state is i, by

$$M_i(u, y; b) = \mathbb{E}[e^{yD_{u,b}}|J(0) = i], \qquad 0 \le u \le b, i \in E,$$

where y is such that $M_i(u, y; b)$ exists.

Similar to arguments as in Section 2, we condition on the events that can occur in the small time interval $[0, h]$:

$$\begin{aligned} M_i(u, y; b) = \mathbb{E}[e^{yD_{u,b}}|J(0) = i] &= (1 - \alpha_i h - \lambda_i h)M_i(u + ch, e^{-\delta h}y; b) \\ &+ \lambda_i h \left[\int_0^{u+ch} M_i(u + ch - x, e^{-\delta h}y; b)\, dF_i(x) + \bar{F}_i(u + ch)\right] \\ &+ h \sum_{k=1,k\neq i}^{m} \alpha_{i,k} M_k(u + ch, e^{-\delta h}y; b) + o(h), \qquad 0 \le u < b, i \in E, \end{aligned}$$

where $\bar{F}_i = 1 - F_i$ is the tail of the distribution function F_i.

Taylor's expansion gives

$$M_i(u + ch, e^{-\delta h}y; b) = M_i(u, y; b) + ch \frac{\partial M_i(u, y; b)}{\partial u} - \delta yh \frac{\partial M_i(u, y; b)}{\partial y} + o(h). \tag{3.1}$$

Substituting (3.1) into the expression of $M_i(u, y; b)$, dividing both sides by h, and letting $h \to 0$, we have, for $0 < u < b$ and $i \in E$,

$$c \frac{\partial M_i(u, y; b)}{\partial u} - \delta y \frac{\partial M_i(u, y; b)}{\partial y} - \lambda_i M_i(u, y; b) + \lambda_i \left[\int_0^{u} M_i(u - x, y; b)\, dF_i(x) + \bar{F}_i(u)\right] + \sum_{k=1}^{m} \alpha_{i,k} M_k(u, y; b) = 0. \tag{3.2}$$

For the case $u = b$,

$$\begin{aligned} M_i(b, y; b) &= (1 - \alpha_i h - \lambda_i h)e^{ych}M_i(b, e^{-\delta h}y; b) + \lambda_i h e^{ych} \left[\int_0^{b} M_i(b - x, e^{-\delta h}y; b)\, dF_i(x) + \bar{F}_i(b)\right] \\ &+ he^{ych} \sum_{k=1,k\neq i}^{m} \alpha_{i,k} M_k(b, e^{-\delta h}y; b) + o(h), \qquad i \in E. \end{aligned}$$

Using Taylor's expansion and noting that $\alpha_{i,i} = -\alpha_i$, we have, for $i \in E$,

$$\delta y \frac{\partial M_i(b, y; b)}{\partial y} + (\lambda_i - cy)M_i(b, y; b) = \lambda_i \left[\int_0^{b} M_i(b - x, y; b)\, dF_i(x) + \bar{F}_i(b)\right] + \sum_{k=1}^{m} \alpha_{i,k} M_k(b, y; b) = 0.$$

Comparing these equations with the corresponding equations in (3.2) for $u = b$, we have the boundary conditions

$$\left.\frac{\partial M_i(u, y; b)}{\partial u}\right|_{u=b} = yM_i(b, y; b), \qquad i \in E. \tag{3.3}$$

For $0 \le u \le b$ and $i \in E$, define

$$V_{i,n}(u; b) = \mathbb{E}[D_{u,b}^n|J(0) = i], \qquad n \in \mathbb{N},$$

to be the nth moment of $D_{u,b}$, with $V_{i,0}(u; b) = 1$ and $V_{i,1}(u; b) = V_i(u; b)$. Substituting $M_i(u, y; b) = 1 + \sum_{n=1}^{\infty}(y^n/n!)V_{i,n}(u; b)$ into (3.2) and comparing the coefficient of y^n yields the following integro-differential equations:

$$cV'_{i,n}(u; b) = (\lambda_i + n\delta)V_{i,n}(u; b) - \lambda_i \int_0^u V_{i,n}(u - x; b)\, dF_i(x) - \sum_{k=1}^{m} \alpha_{i,k}V_{k,n}(u; b), \qquad 0 \le u < b,\ i \in E. \tag{3.4}$$

It follows from (3.3) that

$$V'_{i,n}(b-; b) = nV_{i,n-1}(b; b), \qquad i \in E,\ n \in \mathbb{N}, \tag{3.5}$$

with $V_{i,0}(b; b) = 1$.

The way of solving the integro-differential equations (3.4) with boundary conditions (3.5) is similar to that of solving equations (2.3) with boundary conditions (2.5). For $j \in E$, let $v_{1,j}(u; n)$, $v_{2,j}(u; n)$, . . . , $v_{m,j}(u; n)$, with initial conditions $v_{i,j}(0; n) = I(i = j)$, be m particular solutions of integro-differential equations (2.6) with δ being replaced by $n\delta$.

For $k \in \mathbb{N}$, let $\mathbf{V}_k(u; b) = (V_{1,k}(u; b), \ldots, V_{m,k}(u; b))^T$, $\mathbf{v}_k(u) = (v_{i,j}(u; k))_{i,j=1}^m$, $\mathbf{v}'_k(b) = (v'_{i,j}(u; k)|_{u=b})_{i,j=1}^m$. Then for $0 \le u \le b$, we have

$$\mathbf{V}_n(u; b) = n\mathbf{v}_n(u)[\mathbf{v}'_n(b)]^{-1}\mathbf{V}_{n-1}(b; b),$$

where

$$\mathbf{V}_{n-1}(b; b) = (n - 1)!\mathbf{v}_{n-1}(b)[\mathbf{v}'_{n-1}(b)]^{-1} \cdots \mathbf{v}_1(b)[\mathbf{v}'_1(b)]^{-1}\mathbf{1},$$

and $\mathbf{1} = (1, \ldots, 1)^T$ is an m-dimensional vector.

4. TIME TO REACH THE DIVIDEND BARRIER

In this section we consider how long it takes for the surplus process to reach the dividend barrier b from the initial surplus u without ruin occurring. We define τ_b to be the first time that the surplus reaches b, and for $\delta > 0$, define

$$L_i(u; b) = \mathbb{E}[e^{-\delta\tau_b}I(\tau_b < T_b)|U(0) = u, J(0) = i], \qquad 0 \le u \le b,\ i \in E.$$

$L_i(u; b)$ can be interpreted as the expected present value of one dollar payable at the time of reaching the barrier b without ruin occurring, given that the initial environment state is i and the initial surplus is u. Alternatively, it can be viewed as the Laplace transform of the time to reach the dividend barrier b without ruin occurring with respect to the parameter δ.

Using the same arguments as in deriving (2.3), we can easily show that $L_i(u; b)$ satisfies the following integro-differential equations:

$$cL'_i(u; b) = (\lambda_i + \delta)L_i(u; b) - \lambda_i \int_0^u L_i(u - x; b)\, dF_i(x) - \sum_{k=1}^{m} \alpha_{i,k}L_k(u; b), \qquad 0 \le u < b,\ i \in E, \tag{4.1}$$

with boundary conditions

$$L_i(b; b) = 1, \qquad i \in E, \tag{4.2}$$

where equations (4.2) are obtained from the fact that $\tau_b = 0$ and $\mathbb{E}[e^{-\delta\tau_b}I(\tau_b < T_b)|U(0) = u, J(0) = i] = 1$ when $u = b$.

The solutions to equations (4.1) with boundary conditions (4.2) in a matrix form are

$$\mathbf{L}(u; b) = \mathbf{v}(u)\mathbf{L}(0; b), \qquad 0 \le u \le b,$$

where $\mathbf{L}(u; b) = (L_1(u; b), \ldots, L_m(u; b))^T$, $\mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^m$, and vector $\mathbf{L}(0; b)$ is the solution of

$$\mathbf{v}(b)\mathbf{L}(0; b) = \mathbf{1}, \qquad 0 \leq u \leq b.$$

Therefore the explicit expression for $\mathbf{L}(u; b)$ is given by

$$\mathbf{L}(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}\mathbf{1}, \qquad 0 \leq u \leq b.$$

Note that when $\delta = 0$, $L_i(u; b) = \mathbb{E}[e^{-\delta\tau_b}I(\tau_b < T_b)|U(0) = u, J(0) = i]$ simplifies to

$$\chi_i(u; b) = \mathbb{P}[\tau_b < T_b|U(0) = u, J(0) = i], \qquad 0 \leq u \leq b, i \in E.$$

Here $\chi_i(u; b)$ is the probability that the surplus process attains the given dividend barrier b from the initial surplus u without first falling below zero, given that the initial environment state is i. Since eventually either ruin occurs without the surplus process attaining b or the surplus attains level b, then $1 - \chi_i(u; b)$ is the probability that ruin occurs from the initial surplus u without the surplus process reaching level b prior to ruin, given that the initial environment state is i. Alternatively, $1 - \chi_i(u; b)$ is the probability of ruin in the presence of an absorbing barrier at b, given that the initial environment state is i.

5. THE DISTRIBUTIONS OF THE TOTAL DIVIDENDS PAYMENTS BEFORE RUIN

In this section, we consider the case $\delta = 0$. Hence $D_{u,b}$ simplifies to the total dividend payments up to the time of ruin $D(T_b)$. We aim at finding the distributions of $D(T_b)$ given the initial state.

When $\delta = 0$, it follows that $\mathbf{v}_k(u) = \mathbf{v}(u) = (v_{i,j}(u))_{i,j=1}^m$, for $k = 1, 2, 3, \ldots$. Then for $0 \leq u \leq b$,

$$\begin{aligned}\mathbf{V}_n(u; b) &= n!\mathbf{v}(u)[\mathbf{v}'(b)]^{-1}\{\mathbf{v}(b)[\mathbf{v}'(b)]^{-1}\}^{n-1}\mathbf{1}\\ &= n!\mathbf{v}(u)[\mathbf{v}(b)]^{-1}\{\mathbf{v}(b)[\mathbf{v}'(b)]^{-1}\}^{n}\mathbf{1}\\ &= n!\mathbf{p}(u; b)[\mathbf{W}(b; b)]^n\mathbf{1},\end{aligned}$$

where $\mathbf{p}(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1}$ and $\mathbf{W}(u; b) = \mathbf{v}(u)[\mathbf{v}'(b)]^{-1}$ are two $m \times m$ matrices. Define $\mathbf{M}(u, y; b) = (M_1(u, y; b), M_2(u, y; b), \ldots, M_m(u, y; b))^T$. Then

$$\begin{aligned}\mathbf{M}(u, y; b) &= 1 + \sum_{n=1}^{\infty}\frac{y^n}{n!}\mathbf{V}_n(u; b) = \left\{\mathbf{I} + \mathbf{p}(u; b)\sum_{n=1}^{\infty}y^n[\mathbf{W}(b; b)]^n\right\}\mathbf{1}\\ &= \{\mathbf{I} - \mathbf{p}(u; b) + \mathbf{p}(u; b)[\mathbf{I} - y\mathbf{W}(b; b)]^{-1}\}\mathbf{1}\\ &= \{\mathbf{I} - \mathbf{p}(u; b) + \mathbf{p}(u; b)[[\mathbf{W}(b; b)]^{-1} - y\mathbf{I}]^{-1}[\mathbf{W}(b; b)]^{-1}\}\mathbf{1}\\ &= 1 - \mathbf{p}(u; b)1 + \mathbf{p}(u; b)[[\mathbf{W}(b; b)]^{-1} - y\mathbf{I}]^{-1}[\mathbf{W}(b; b)]^{-1}\mathbf{1},\end{aligned}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix. Therefore

$$\begin{aligned}M_i(u, y; b) &= \mathbb{E}[e^{yD(T_b)}|U(0) = u, J(0) = i]\\ &= 1 - \chi_i(u; b) + \chi_i(u; b)\frac{\mathbf{p}_i(u; b)}{\chi_i(u; b)}[[\mathbf{W}(b; b)]^{-1} - y\mathbf{I}]^{-1}[\mathbf{W}(b; b)]^{-1}\mathbf{1},\end{aligned}$$

where $\mathbf{p}_i(u; b)$ is the ith row of the matrix $\mathbf{p}(u; b)$. Inverting the moment-generating function shows that the distribution of $D(T_b)$, given that the initial state is i, is a mixture of the degenerate distribution at 0 with weight $p_i = 1 - \chi_i(u; b)$ and a continuous distribution with weight $q_i = \chi_i(u; b)$ and pdf

$$h_i(x) = \boldsymbol{\alpha}_i e^{\mathbf{T}x}\mathbf{t}, \qquad i \in E, x > 0,$$

where $\boldsymbol{\alpha}_i = \mathbf{p}_i(u; b)/\chi_i(u; b)$ is an $1 \times m$ row vector, $\mathbf{T} = -[\mathbf{W}(b; b)]^{-1}$, and $t = -\mathbf{T}\mathbf{1}$.

REMARKS

1. As seen in Section 4, the weight $p_i = 1 - \chi_i(u; b)$ is the probability that the process does not reach barrier b from the initial surplus u before ruin, given that the initial state is i.
2. $h_i(x)$ is a phase-type density function with representation $(\boldsymbol{\alpha}_i, \mathbf{T})$.
3. When $m = 1$, the model simplifies to the classical risk model, and the distribution of the total dividend payments before ruin, $D(T_b)$, is a mixture of the degenerate distribution at 0 and the exponential distribution with mean $V(b; b)$. The weights of the mixture are $p = 1 - \chi(u; b)$ and $q = \chi(u; b)$ with $\chi(u; b)$ being the probability that the surplus hits barrier b from the initial surplus u without ruin occurring for the classical risk model. See Dickson and Waters (2004).
4. Define

$$W_{i,j}(u; b) = \mathbb{E}[D_{u,b} I(J(T_b) = j) | J(0) = i], \qquad 0 \le u \le b,\ i, j \in E,$$

to be the expected present value of the dividend payment before ruin if ruin is caused by a claim in state j given the initial state is i. Then

$$V_i(u; b) = \sum_{j=1}^{m} W_{i,j}(u; b), \qquad 0 \le u \le b,\ i \in E.$$

Define

$$\chi_{i,j}(u; b) = \mathbb{P}[\tau_b < T_b,\ J(\tau_b) = j | J(0) = i], \qquad 0 \le u \le b,\ i, j \in E,$$

to be the probability that the surplus process hits barrier b at the state j before ruin given the initial state is i. Then

$$\chi_i(u; b) = \sum_{j=1}^{m} \chi_{i,j}(u; b), \qquad 0 \le u \le b,\ i \in E.$$

Further it can be shown that

$$\mathbf{W}(u; b) = \mathbf{v}(u)[\mathbf{v}'(b)]^{-1} = (W_{i,j}(u; b))_{i,j=1}^{m},$$

$$\mathbf{p}(u; b) = \mathbf{v}(u)[\mathbf{v}(b)]^{-1} = (\chi_{i,j}(u; b))_{i,j=1}^{m}.$$

6. LAPLACE TRANSFORMS

We now apply Laplace transforms to find the particular solutions $v_{i,j}(u)$ of the system of equations (2.6). Let $\hat{v}_{i,j}$ and $\hat{f}_i$ be the Laplace transforms of $v_{i,j}$ and f_i, respectively:

$$\hat{v}_{i,j}(s) = \int_0^\infty e^{-su} v_{i,j}(u)\, du, \qquad \hat{f}_i(s) = \int_0^\infty e^{-sx} f_i(x)\, dx, \qquad i, j \in E.$$

Taking Laplace transforms on both sides of (2.6) yields

$$\left[s - \frac{\lambda_i + \delta}{c} + \frac{\lambda_i}{c}\hat{f}_i(s)\right]\hat{v}_{i,j}(s) + \frac{1}{c}\sum_{k=1}^{m} \alpha_{i,k}\hat{v}_{k,j}(s) = v_{i,j}(0), \qquad i, j \in E, \tag{6.1}$$

with $v_{i,j}(0) = I(i = j)$. For simplicity, define $S_i(s) = s - (\lambda_i + \delta)/c + \lambda_i \hat{f}_i(s)/c$, for $i \in E$. Then the matrix form of (5.1) is given by

$$\mathbf{A}(s)\hat{\mathbf{v}}(s) = \mathbf{I},$$

where

$$\mathbf{A}(s) = \begin{pmatrix} S_1(s) & & \\ & \ddots & \\ & & S_m(s) \end{pmatrix} + \frac{1}{c}\,\mathbf{\Lambda},$$

$\hat{\mathbf{v}}(s) = (\hat{v}_{i,j}(s))_{i,j=1}^{m}$, $\mathbf{I}$ is the $m \times m$ identity matrix, and $\mathbf{\Lambda} = (\alpha_{i,j})_{i,j=1}^{m}$ is the intensity matrix.

Then $\hat{\mathbf{v}}(s)$ can be solved as

$$\hat{\mathbf{v}}(s) = [\mathbf{A}(s)]^{-1}.$$

Note that when the claim sizes are rationally distributed, each element of $\mathbf{A}(s)$ is a rational function, as is each element of $[\mathbf{A}(s)]^{-1}$; therefore $v_{i,j}(u)$ can be obtained by inverting $\hat{v}_{i,j}(s)$ through partial fractions. This can be shown by the example in the next section.

7. ILLUSTRATIONS FOR A TWO-STATE MODEL

In this section we derive explicit expressions for $v_{1,1}(u)$, $v_{2,1}(u)$, $v_{1,2}(u)$, and $v_{2,2}(u)$ under some special claim size distributions when $m = 2$, that is, $\{J(t); t \geq 0\}$ is a two-state Markov process, which reflects the random environmental effects due to "normal" versus "abnormal," or "high risk" versus "low risk" conditions.

We now consider the case where the claim size distributions f_1 and f_2 are exponentially distributed and their Laplace transformations are of the form

$$\hat{f}_1(s) = \frac{\beta_1}{s + \beta_1}, \qquad \hat{f}_2(s) = \frac{\beta_2}{s + \beta_2},$$

where $\beta_1 > 0$, $\beta_2 > 0$. In this case,

$$S_i(s) = s - \frac{\lambda_i + \delta}{c} + \frac{\lambda_i \beta_i}{c(s + \beta_i)}, \qquad i = 1, 2,$$

and matrix $\mathbf{A}(s)$ has the form

$$\mathbf{A}(s) = \begin{pmatrix} S_1(s) - \alpha_1/c & \alpha_1/c \\ \alpha_2/c & S_2(s) - \alpha_2/c \end{pmatrix}.$$

Then the Laplace transforms $\hat{v}_{i,j}(s)$ can be solved as

Table 1

***V*(*u*; *b*) and *SD*(*u*; *b*) for *b* = 10, . . . , 80 and *u* = 10, . . . , 50**

u/*b*	10	20	30	40	50	60	70	80
10	15.870 16.207	24.897 32.289	32.243 41.904	35.701 45.050	35.903 48.528	34.284 42.447	31.898 39.839	29.308 37.136
20		37.273 33.848	48.399 44.302	53.657 47.130	53.971 46.241	51.529 44.039	47.935 41.482	44.037 38.894
30			59.758 44.411	66.384 46.745	66.789 45.482	63.752 43.265	59.289 40.913	54.457 38.605
40				76.899 46.481	77.400 44.818	73.853 42.570	68.653 40.415	63.036 38.380
50					87.381 44.624	83.324 42.301	77.401 40.318	71.030 38.534

Figure 1
$L(u; b)$ for u = 10, . . . , 50 (from Bottom to Top)

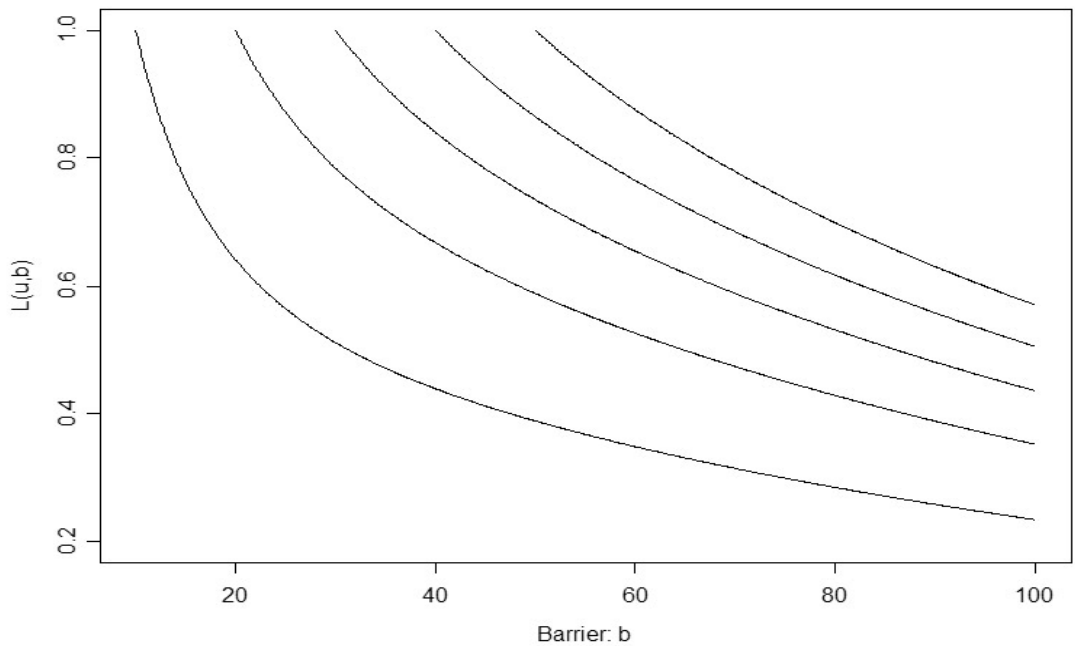
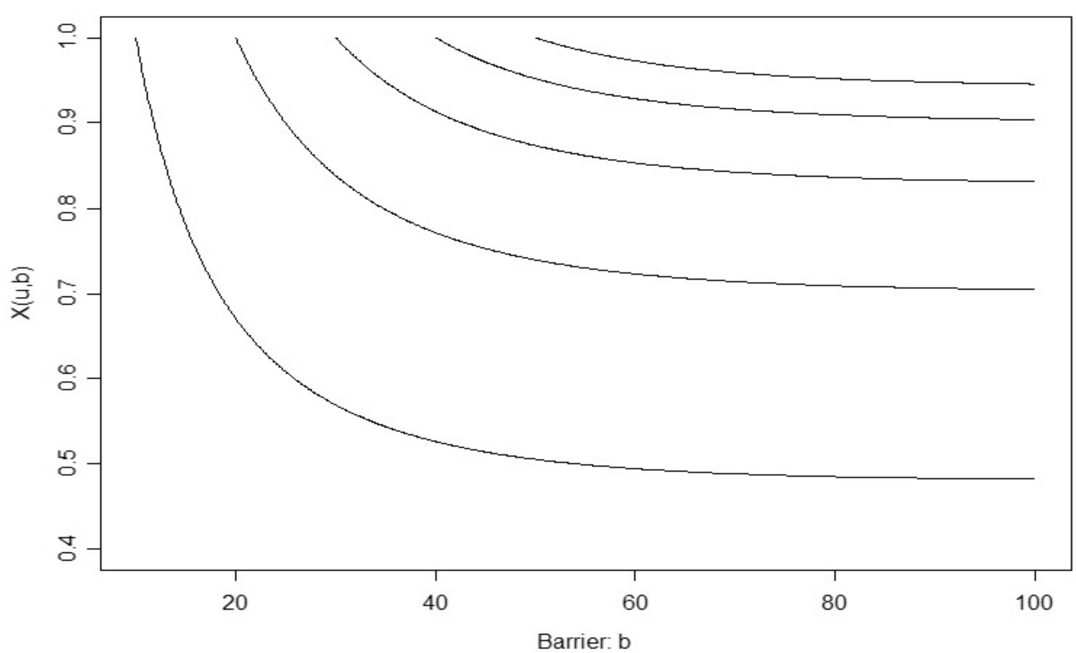


Figure 2
Probability $\chi(u; b)$ for u = 10, . . . , 50 (from Bottom to Top)



$$\begin{pmatrix} \hat{v}_{1,1}(s) & \hat{v}_{1,2}(s) \\ \hat{v}_{2,1}(s) & \hat{v}_{2,2}(s) \end{pmatrix} = \frac{1}{|\mathbf{A}(s)|} \begin{pmatrix} S_2(s) - \alpha_2/c & -\alpha_1/c \\ -\alpha_2/c & S_1(s) - \alpha_1/c \end{pmatrix},$$

where $|\mathbf{A}(s)| = [S_1(s) - \alpha_1/c][S_2(s) - \alpha_2/c] - \alpha_1\alpha_2/c^2$ is the determinant of $\mathbf{A}(s)$.

By multiplying both the numerator and denominator of $\hat{v}_{i,j}(s)$ by $(s + \beta_1)(s + \beta_2)$, it is easy to see that the common denominator of the formulas is a polynomial of degree four, which has four distinct zeros, say, R_1, R_2, R_3 and R_4. Then inverting the above Laplace transforms gives

$$v_{i,j}(u) = \sum_{k=1}^{4} r_{i,j,k} e^{R_k u}, \qquad i, j = 1, 2,$$

where the coefficients, $r_{i,j,k}$, are given, for $k = 1, 2, 3, 4$, by

$$\begin{pmatrix} r_{1,1,k} & r_{1,2,k} \\ r_{2,1,k} & r_{2,2,k} \end{pmatrix} = \frac{(R_k + \beta_1)(R_k + \beta_2)}{\Pi_{l=1,l\neq k}^{4}(R_k - R_l)} \begin{pmatrix} S_2(R_k) - \alpha_2/c & -\alpha_1/c \\ -\alpha_2/c & S_1(R_k) - \alpha_1/c \end{pmatrix}.$$

To illustrate the results numerically, set $c = 103.5$, $\lambda_1 = 100$, $\lambda_2 = 40$, $\alpha_1 = 1/4$, $\alpha_2 = 3/4$, $\beta_1 = 1$, $\beta_2 = 0.5$, and $\delta = 0.1$. Then we get $\zeta_1 = 3/4$, $\zeta_2 = 1/4$, and $R_1 = -0.138$, $R_2 = -0.066$, $R_3 = 0.010$, and $R_4 = 0.059$. The first row of each pair of rows in Table 1 gives the expected present value of the total divident payments until ruin $V(u; b)$ in the stationary case, given by $\zeta_1 V_1(u; b) + \zeta_2 V_2(u; b)$, and the second of each pair gives the standard deviation of the present value of the total dividend payments until ruin in the stationary case, given by $SD(u; b) = \sqrt{[\zeta_1 V_{1,2}(u; b) + \zeta_2 V_{2,2}(u; b)] - V(u; b)^2}$.

Figure 1 shows the expected present value of one dollar payable at time τ_b in the stationary case for the above example, given by $L(u; b) = \zeta_1 L_1(u; b) + \zeta_2 L_2(u; b)$, as a function of barrier b when the initial surplus u is 10, . . . , 50, respectively. It can be observed that $L(u; b)$ is increasing in u and decreasing in b, as expected.

Further, by setting $\delta = 0$ we are able to find $\chi_i(u; b)$, the probability that the surplus process attains the given dividend barrier b from the initial surplus u without first falling below zero, given the initial state i, for this example. Figure 2 shows the probabilities in the stationary case, given by $\chi(u; b) = \zeta_1\chi_1(u; b) + \zeta_2\chi_2(u; b)$, as a function of barrier b when the initial surplus u is 10, . . . , 50, respectively. The curves for $\chi(u; b)$ have the same patterns as those for $L(u; b)$: increasing in u and decreasing in b.

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