On Approximating the Individual Risk Model

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Abstract

Error bounds are developed for Kornya-Presman–type approximations to the individual risk model. In particular, error bounds are given for the compound Poisson approximation, with the Poisson parameter equal to the expected number of claims.

The approximations considered are a modification by Hipp of an approximation originally developed by Kornya, as well as Kornya's original approximation. The error bounds are similar in concept to Hipp's original error bounds using concentration–functions, as refined by Čekanavičius, Roos, and others. Computation of the bounds, however, is considerably simplified. In particular, concentration function type bounds called *width-norm bounds* are calculated directly from the values of the approximation, avoiding completely the need for calculating a special auxiliary compound Poisson distribution. Two examples are given.

Depending on the portfolio parameters, the width-norm error bounds may or may not be sharper than the Hipp-Roos error bounds. In any event, it is recommended that the simpler width-norm calculation, possibly together with an increase in the order of the approximation, be considered in practical applications if the goal is to achieve a specified accuracy with the least amount of computation.

For large portfolios, where the maximum claim on any one policy is relatively small compared to the expected aggregate claims, the width-norm error bounds compare quite favorably with error bounds developed by De Pril and by Dhaene.

1. Introduction and Summary of Results

Approximating the individual risk model by a compound Poisson model is a fundamental technique in risk theory. The compound Poisson model has favorable analytical properties and can be computed using an effective recursive algorithm developed by Panjer (1981).

There is an extensive literature on error bounds for the compound Poisson approximation. Many of these error bounds increase directly with the size of the model. As a result, these bounds tend not to be very informative for large "real life" individual portfolios. A couple of techniques have been developed to sharpen the error bounds.

The first technique is to allow for negative "probabilities" for some of the claim amounts in the claim amount distribution. The first such model was developed by Kornya (1983). By allowing negative claim amount "probabilities," approximations with sharper error bounds can be developed, while still retaining certain desirable analytical properties and the recursive calculation method of the compound Poisson model. Of course, the approximations are then, in general, no longer necessarily true distributions.

The second technique, introduced to the actuarial literature by Hipp (1985), is to take advantage of stochastic convergence, which occurs as the portfolio increases in size. Hipp develops error bounds in terms of Lévy's concentration functions (see, e.g., Kolmogorov 1958). Concentration functions in general will tend to decrease with increasing portfolio size.

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The two techniques are combined by Hipp (1986). Hipp's approximations by concentration functions have been improved by Roos (2005), Čekanavičius (1997, 2003), and others. The use of Hipp's approach, however, has been limited because the error bounds are considered to be hard to compute in practical applications (De Pril and Dhaene 1994).

In this paper, we derive error bounds similar in concept to Hipp's concentration function bounds, but considerably simpler to compute in practice. Depending on the parameters of the portfolio, our bounds may or may not be sharper than the Hipp-Roos error bounds. Regardless, a suitable combination of the two techniques, using the simplified error bounds, often may achieve a specified level of accuracy with less computation than using the more complicated bounds.

For large portfolios, our bounds compare quite favorably with bounds developed by De Pril (1989) and De Pril and Dhaene (1992, 1994). See the remarks at the end of this section and the examples in Section 6.

We present our main results below. In Section 3, we develop the concept of a ''width-norm,'' which coincides with concentration functions for probability distributions. In Sections 4 and 5 we prove our main results. Since we are concerned primarily with practical applications, we consider only distributions and approximations concentrated on the nonnegative integers. In Section 6 we present some numerical examples.

To begin, consider a portfolio of individual insurance policies indexed by H. Suppose individual policy risks are independent. Let

$$P(z) = \prod_{j\in H} (p_j + q_j g_j(z)) = \sum_{k=0}^{\infty} f(k) z^k \tag{1.1}$$

be the probability generating function (pgf) encoding the portfolio's claim density function f over one time period. In (1.1), q_j is the probability that policy j has a claim, $p_j = 1 - q_j$, and g_j is the pgf of the claim amount. Assume that for all j, $0 < q_j < p_j$. Denote the distribution function of the portfolio claims by F. To simplify our formulas, we extend the domain of f, respectively F, to all integers by specifying that for $n < 0$ we have $f(n) = F(n) = 0$.

Writing

$$\ln[p_j + q_j g_j(z)] = \ln[1 - q_j(1 - g_j(z))] = -\sum_{k=1}^{\infty} \frac{q_j^k}{k} [1 - g_j(z)]^k \tag{1.2}$$

and, summing over H, we obtain

$$\ln P(z) = -\sum_{j\in H} \sum_{k=1}^{\infty} \frac{q_j^k}{k} [1 - g_j(z)]^k. \tag{1.3}$$

In practice, the q_j's are small quantities. Therefore $\ln P(z)$ often may be approximated by considering only the first K powers of q_j, for some small value of K. We then can approximate $P(z)$ as

$$P_K(z) = \exp\left(-\sum_{j\in H} \sum_{k=1}^{K} \frac{q_j^k}{k} [1 - g_j(z)]^k\right) = \sum_{n=0}^{\infty} f_K(n) z^n. \tag{1.4}$$

We measure the approximation error using the Kolmogorov distance

$$\|F - F_K\| = \sup_n |F(n) - F_K(n)|, \tag{1.5}$$

where $F_K(n) = \Sigma_{j=0}^n f_K(j)$.

Our main results are the following.

Theorem 1.1

We have

$$|F(n) - F_K(n)| \le \frac{1}{4}(e^\delta - 1)\left(\sup_{i\le n}|F_K(i) - F_K(i-l)| + 2\varepsilon_n\right) \tag{1.6}$$

and

$$\|F - F_K\| \le \frac{1}{4}(e^\delta - 1)\left(\sup_i |F_K(i) - F_K(i-l)| + 2\varepsilon\right), \tag{1.7}$$

where

$$\delta = \frac{1}{K+1}\sum_{j\in H}\frac{(2q_j)^{K+1}}{p_j - q_j}, \tag{1.8}$$

$$\varepsilon_n = \left|\sum_{j=0}^n \{f_K(j) : f_K(j) < 0\}\right|, \ \varepsilon = \lim_{n\to\infty}\varepsilon_n \tag{1.9}$$

and l is the maximum retention limit on any one policy.

The error bounds (1.6) and (1.7) can be made sharper, at the expense of more computation. Suppose that the portfolio H is the *disjoint* union

$$H = \bigcup_{l\in L} H_l \tag{1.10}$$

of subportfolios of H indexed by a set of positive integers L, where l is the maximum claim amount on any one policy in H_l. Then we have the following theorem

Theorem 1.2

We have

$$|F(n) - F_K(n)|$$
$$\le \frac{1}{4}\left(\sum_{l\in L}\delta_l \sup_{i\le n}|F_K(i) - F_K(i-l)| + (e^\delta - \delta - 1)\sup_{i\le n}|F_K(i) - F_K(i-l_M)| + 2(e^\delta - 1)\varepsilon_n\right) \tag{1.11}$$

and

$$\|F - F_K\|$$
$$\le \frac{1}{4}\left(\sum_{l\in L}\delta_l \sup_i |F_K(i) - F_K(i-l)| + (e^\delta - \delta - 1)\sup_i |F_K(i) - F_K(i-l_M)| + 2(e^\delta - 1)\varepsilon\right), \tag{1.12}$$

where

$$\delta_l = \frac{1}{K+1}\sum_{j\in H_l}\frac{(2q_j)^{K+1}}{p_j - q_j} \tag{1.13}$$

with $\delta = \Sigma\delta_l$, $l_M = \sup_{l\in L} l$, and ε, ε_n as in (1.9).

When, $K = 1$, then F_1 is the ''usual'' compound Poisson approximation of F. Noting that ε, $\varepsilon_n = 0$, a particular case of Theorem 1.1 is the following theorem.

Theorem 1.3

Suppose that F_1 is the compound Poisson approximation of F with Poisson parameter $\lambda = \Sigma_{j\in H}\, q_j$ and the pgf of the claim amount distribution given by $1/\lambda\ \Sigma_{j\in H}\, q_j g_j(z)$. Then

$$|F(n) - F_1(n)| \leq \frac{1}{4}\,(e^{\delta} - 1)\sup_{i\leq n}|F_1(i) - F_1(i - l)| \tag{1.14}$$

and

$$\|F - F_1\| \leq \frac{1}{4}\,(e^{\delta} - 1)\sup_{i}|F_1(i) - F_1(i - l)|, \tag{1.15}$$

where

$$\delta = 2\sum_{j\in H}\frac{q_j^2}{p_j - q_j} \tag{1.16}$$

and l is the maximum retention limit on any one policy in H.

The corresponding version of Theorem 1.2 follows

Theorem 1.4

We have

$$|F(n) - F_1(n)| \leq \frac{1}{4}\left(\sum_{l\in L}\delta_l\sup_{i\leq n}|F_1(i) - F_1(i - l)| + (e^{\delta} - \delta - 1)\sup_{i\leq n}|F_1(i) - F_1(i - l_M)|\right) \tag{1.17}$$

and

$$\|F - F_1\| \leq \frac{1}{4}\left(\sum_{l\in L}\delta_l\sup_{i}|F_1(i) - F_1(i - l)| + (e^{\delta} - \delta - 1)\sup_{i}|F_1(i) - F_1(i - l_M)|\right), \tag{1.18}$$

where

$$\delta_l = 2\sum_{j\in H_l}\frac{q_j^2}{p_j - q_j} \tag{1.19}$$

with $\delta = \Sigma\delta_l$ and $l_M = \sup_{l\in L} l$.

The approximation $P_K(z)$ is a modification by Hipp (1986) of the approximation

$$P_K^*(z) = \exp\left(\sum_{j\in H}\sum_{k=1}^{K}\frac{(-1)^k}{k}\frac{q_j^k}{p_j^k}\,[1 - g_j(z)^k]\right) \tag{1.20}$$

originally presented by Kornya (1983). Error bounds similar to the above can be derived for the corresponding approximation F_K^* of F.

Theorem 1.5

We have

$$|F(n) - F_K^*(n)| \leq \frac{K + 1}{2}\,(e^{2\gamma} - 1)\left(\sup_{i\leq n}|F_K^*(i) - F_K^*(i - l)| + 2\varepsilon_n\right) \tag{1.21}$$

and

$$\|F - F_K^*\| \leq \frac{K+1}{2}\,(e^{2\gamma} - 1)\left(\sup_i |F_K^*(i) - F_K^*(i-l)| + 2\varepsilon\right), \tag{1.22}$$

where

$$\gamma = \frac{1}{K+1} \sum_{j \in H} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1}, \tag{1.23}$$

l is the maximum retention limit on any one policy, and ε_n, ε are given by (1.9).

As in Theorem 1.2, the bounds in Theorem 1.5 can be sharpened by considering the portfolio decomposition (1.10).

Theorem 1.6

We have

$$|F(n) - F_K^*(n)|$$
$$\leq \frac{K+1}{2}\left(\sum_{l \in L} 2\gamma_l \sup_{i \leq n} |F_K^*(i) - F_K^*(i-l)| + (e^{2\gamma} - 2\gamma - 1)\sup_{i \leq n} |F_K^*(i) - F_K^*(i-l_M)| + 2(e^{2\gamma} - 1)\varepsilon_n\right) \tag{1.24}$$

and

$$\|F - F_K^*\|$$
$$\leq \frac{K+1}{2}\left(\sum_{l \in L} 2\gamma_l \sup_i |F_K^*(i) - F_K^*(i-l)| + (e^{2\gamma} - 2\gamma - 1)\sup_i |F_K^*(i) - F_K^*(i-l_M)| + 2(e^{2\gamma} - 1)\varepsilon_n\right), \tag{1.25}$$

where

$$\gamma_l = \frac{1}{K+1} \sum_{j \in H_l} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1} \tag{1.26}$$

with $\gamma = \Sigma\gamma_l$, $l_M = \sup_{l \in L} l$, and ε, ε_n as in (1.9).

Remark

The approximation f_K is calculated as follows: First, calculate, for each policy j, the polynomial $h_j(z) = -\sum_{k=1}^{K} q_j^k/k\ [1 - g_j(z)]^k$ and the policy's contribution to the δ component of the error bound. Next, sum the h_j over the portfolio to get

$$\ln P_K(z) = \sum_{j \in H} h_j(z)$$
$$= \sum_{n=0}^{\infty} b_n z^n,$$

and likewise sum δ the components of the error bound. In the same manner, determine each ε_n by summing the absolute values of any negative values of $f_K(j)$. Note, however, that in practical applications $f_K(j)$ in general will be nonnegative, with possibly an occasional exception. Therefore each ε_n will be either zero or at most an insignificant component of the overall error bound.

Then use a Panjer (1981) type recursion, based on the Newton-Girard formulas (see the next remark), to calculate

$$f_K(0) = e^{b_0},$$

$$f_K(n) = \frac{1}{n}\sum_{i=1}^{n} ib_i f_K(n - i) \text{ for } n > 0. \tag{1.27}$$

Note that grouping of policies by mortality or amount classes, or other criteria, is not really necessary (or necessarily recommended) for the calculation of f_K. Such groupings may, however, provide a convenient format for presenting examples. The calculation of the approximating distribution is similar in process to a *seriatim valuation* of the portfolio to obtain each h_j, $\ln P_K$, and the δ component of the error bounds, followed by a Panjer (1981) type recursive calculation to obtain f_K.

Once the f_K are calculated, they are then used to determine the ''sup'' component of the error bounds. In many cases the ''sup'' component of the error bounds may simply be evaluated by considering the largest values of f_K.

A similar procedure can be used for the approximation F_K^*.

REMARK

The Newton-Girard formulas are attributed to Newton (1707, pp. 57–63) and Girard (1629). For a modern exposition, see, for example, Séroul (2000, pp. 278–79). For an application of Newton-Girard in the actuarial literature, see White and Greville (1959).

REMARK

The individual risk model was generalized to a measure theoretic framework by Hipp (1986; see also Roos 2005). We briefly describe the model.

Formula (1.4) is a special case of approximating the sum $S_n = \Sigma_{i=1}^n X_i$ of independent nonnegative random variables, which are nonzero with a small probability, in the algebra S of finite signed measures. Algebra multiplication of signed measures v_1, v_2 in S is the convolution $v_1 {}^*v_2$ given by

$$v_1 {}^*v_2(E) = \int_R\int_R \chi_E(x + y)dv_1(x)dv_2(y),$$

where χ_E is the indicator function of the Borel set E. The distribution $L(X)$ of a random variable X is the measure defined by $L(X)(E) = \Pr[X \in E]$. The distribution function F_v of a finite signed measure v is defined by $F_v(t) = v((-\infty,t])$. The Kolmogorov norm on S is defined by $\|v\| = \sup_t|F_v(t)|$.

Let $\varepsilon_0 = \chi_{\{0\}}$ be the Dirac measure at 0. Let v^{*k} denote k-fold convolution, with $v^{*0} = \varepsilon_0$. Let $\exp v$ denote the finite signed measure

$$\exp v = \sum_{i=0}^{\infty} \frac{v^{*i}}{i!}.$$

Let each $q_i = \Pr[X_i > 0]$, and suppose that $0 < q_i < 0.5$. Let v_i be the distribution of the conditional random variable $[X_i|X_i > 0]$. Then the distribution $L(S_n)$ of S_n has the expansion

$$L(S_n) = \exp\left[-\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{q_i^k}{k}(\varepsilon_0 - v_i)^{*k}\right],$$

which converges in the Kolmogorov norm on S. Therefore $L(S_n)$ is approximated by the finite signed measures

$$H_K = \exp\left[-\sum_{i=1}^{n}\sum_{k=1}^{K}\frac{q_i^k}{k}(\varepsilon_0 - v_i)^{*k}\right].$$

The H_K are called Kornya-Presman signed measures, presumably because the exponential expansion of $L(S_n)$ was first considered by Kornya (1983) and Presman (1983). Note, however, that the approximations used by Kornya and Presman were slightly different.

Remark

The error bounds in Theorems 1.1 to 1.6 are similar in concept to Hipp's (1986) error bounds using concentration functions. Hipp's error bounds have been refined by Roos (2005; see also Čekanavičius 1997, 2003). The calculation of the bounds is rather complex. As an example, Proposition 2 in Roos (2005) requires the calculation of an auxiliary compound Poisson distribution, to which concentration functions then are applied to derive the error bounds.

The bounds presented in this paper offer a more practical alternative. In particular, concentration function type bounds are calculated directly from the approximation F_K, respectively F_K^*, avoiding the need to calculate the auxiliary compound Poisson distribution. Depending on the parameters of the portfolio, the more practical bound may well also be sharper than the Hipp-Roos bound. In any event, a combination of increasing the approximation order to $K + 1$, and using the more practical bound calculation, may well result in a lower error bound, and may still require less computation than an approximation of order K with the more complex concentration function bound calculation. See the examples in Section 5.

The Hipp-Roos error bounds *are* considerably simpler to apply in the case where F is the distribution of claim *counts,* that is, if $g_j(z) \equiv z$. A special case of Proposition 1 in Roos (2005) is that, if F is the distribution of claim counts, then

$$\|F - P_\lambda\| \le \frac{\pi^2}{8} \sum_{j \in H} \frac{q_j^2}{p_j} \cdot p_{\bar{\lambda}}([\bar{\lambda}]), \tag{1.28}$$

where [ ] denotes the greatest integer function, P_λ is a Poisson distribution with $\lambda = \Sigma_{j\in H}\, q_j$, and $p_{\bar{\lambda}}$ is a Poisson density function with $\bar{\lambda} = 0.5\ \Sigma_{j\in H}\, p_j q_j$.

Remark

Another modification of P_K^* was considered by De Pril (1989) and De Pril and Dhaene (1994). De Pril adjusts P_K^* by a scalar

$$P_K^{**}(z) = \frac{F(0)}{F_K^*(0)} \cdot P_K^*(z) \tag{1.29}$$

to minimize certain theoretical error bounds. De Pril derives the error bound

$$\sum_{n=0}^{\infty} |f(n) - f_K^{**}(n)| \le e^\gamma - 1 \tag{1.30}$$

for the corresponding approximation f_K^{**} of f_K, where γ is given by (1.23). We may write the bound in (1.22) as

$$\begin{aligned} &\frac{K+1}{2}\,(e^{2\gamma} - 1)(\sup_i |F_K^*(i) - F_K^*(i-l)| + 2\varepsilon) \\ &= (e^\gamma - 1)\left(\frac{e^\gamma + 1}{2}\right)(K+1)(\sup_i |F_K^*(i) - F_K^*(i-l)| + 2\varepsilon) \\ &\approx (e^\gamma - 1)(K+1)(\sup_i |F_K^*(i) - F_K^*(i-l)|). \end{aligned} \tag{1.31}$$

For large portfolios, the impact on total claims of any one policy is often relatively minor. Therefore, for many large portfolios, and for small values of K, we often will have $\sup_i |F_K^*(i) - F_K^*(i - l)| < 1/(K + 1)$, and the bound (1.31) may well be sharper than the bound (1.30). See the examples in Section 6.

REMARK

De Pril and Dhaene (1992) develop error bounds for the compound Poisson approximation of the individual risk model. For the "usual" case where the Poisson parameter is $\lambda = \Sigma_{j\in H} q_j$, the error bound simplifies to

$$\sum_{j\in H} p_j - e^{-q_j} \le F(n) - F_1(n) \le \sum_{j\in H} 1 - e^{-q_j} - q_j e^{-q_j}. \tag{1.32}$$

By elementary calculus it is easily shown that, since $q_j > 0$,

$$0 < 1 - e^{-q_j} - q_j e^{-q_j} < e^{-q_j} - p_j.$$

Therefore, in terms of the Kolmogorov norm, (1.32) implies

$$\|F - F_1\| \le \sum_{j\in H} e^{-q_j} - p_j. \tag{1.33}$$

As in the previous remark, for large portfolios the factor $\sup_i|F_1(i) - F_1(i - l)|$ in (1.15) will be small, and the overall bound in (1.15) may well be sharper than the error bound in (1.33).

2. PRELIMINARIES

We use the letters i, j, k, l, m, n to denote integers. It is assumed that integers are nonnegative unless stated otherwise. We use the letters α, β, λ to denote complex numbers.

We denote by $C[\![z]\!]$ the algebra of formal power series over the complex numbers C(fpsC). We denote by $C_A[\![z]\!]$ the subalgebra of those fpsC's which are absolutely convergent at $z = 1$, and by $C_C[\![z]\!]$ the subalgebra of those fpsC's which are convergent at $z = 1$. Let $PGF[\![z]\!]$ be collection of probability generating functions (pgf's) of random variables on the nonnegative integers. We have the strict inclusions $PGF[\![z]\!] \subset C_A[\![z]\!] \subset C_C[\![z]\!] \subset C[\![z]\!]$.

We use the letters A, B, C, D, E to denote fpsC's. We use the corresponding indexed lowercase letters to denote series expansions. For example, $A \equiv A(z) = \Sigma_{k=0}^{\infty} a_k z^k$.

Let

$$a_k^S = \sum_{j=0}^{k} a_j. \tag{2.1}$$

To simplify some of our formulas, we also define a_k, a_k^S for negative values of k by specifying that

$$k < 0 \Rightarrow a_k = a_k^S = 0. \tag{2.2}$$

If $A \ne 0$, define the degree $d(A)$ of A to be largest k for which $a_k \ne 0$. If no such k exists, define $d(A) = \infty$.

Define $A^S(z) = A(z)/(1 - z)$. It is easily verified that

$$A^S(z) = \sum_{n=0}^{\infty} a_n^S z^n \tag{2.3}$$

and that the summation operator S satisfies

$$(AB)^S = A^S B = AB^S. \tag{2.4}$$

3. METRICS ON POWER SERIES

In this section we describe various distance measures on subspaces of $C[\![z]\!]$.

Let $|A|_1$ be the usual L_1 norm on $C_A[\![z]\!]$:

$$|A|_1 = \sum_{k=0}^{\infty} |a_k|. \tag{3.1}$$

We define the Kolmogorov norm $\|A\|$ on $C_C[\![z]\!]$ by

$$\|A\| = \sup_k |a_k^S|. \tag{3.2}$$

Remark

Note that if A, B in $PGF[\![z]\!]$ encode random variables with density f, respectively g, and distribution F, respectively G, then

$$|A - B|_1 = \sum_{j=0}^{\infty} |f(j) - g(j)| \tag{3.3}$$

is the usual *total variation distance* and

$$\|A - B\| = \sup_k |F(k) - G(k)| \tag{3.4}$$

is the usual *Kolmogorov distance* between F and G.

Define the w_k-width $w_k(a_n)$ of A at a_n by

$$w_k(a_n) = \sup_{i \le k} |a_n^S - a_{n-i}^S|. \tag{3.5}$$

The w_k-width $w_k(a_n)$ may be visualized by plotting the partial sums a_k^S in the complex plane. Then $w_k(a_n)$ is the radius of the smallest closed disk, with center a_n^S, which contains $a_{n-k}^S, \ldots, a_n^S$.

Define the w_k^l-width of A at a_n by

$$w_k^l(a_n) = \sup_{i \le l} w_k(a_{n-i}). \tag{3.6}$$

In particular, $w_k(a_n) \equiv w_k^0(a_n)$ and $w_k^\infty(a_n) = \sup_l w_k^l(a_n) = w_k^n(a_n) = w_k^{n+1}(a_n) = \cdots$.

Define the w_k-width of A by

$$w_k(A) = \sup_n w_k(a_n) = \sup_n w_k^l(a_n). \tag{3.7}$$

Note that (3.7) is independent of the choice of l, and that possibly $w_k(A) = \infty$.

Define the total w-width of A by

$$w(A) = \sup_k w_k(A). \tag{3.8}$$

Again, note that possibly $w(A) = \infty$.

It is easily seen that if A is an fpsC, then

$$i \le j, k \le l \Rightarrow w_i^j(a_n) \le w_k^l(a_n) \tag{3.9}$$

and

$$w_1(A) \le w_2(A) \le \cdots \le w(A). \tag{3.10}$$

Let $W_k[\![z]\!]$ be the collection of fpsC's with finite w_k-width and let $W[\![z]\!]$ be the collection of fpsC's with finite w-width.

Denote the w_k^l-width of $\alpha A + \beta B$ at $\alpha a_n + \beta b_n$ by $w_k^l(\alpha a_n + \beta b_n)$.

Lemma 3.1

For all k, l,

$$w_k^l(\alpha a_n + \beta b_n) \le |\alpha| w_k^l(a_n) + |\beta| w_k^l(b_n), \tag{3.11}$$

and if A, B are in $W_k[\![z]\!]$, then

$$w_k(\alpha A + \beta B) \le |\alpha| w_k(A) + |\beta| w_k(B). \tag{3.12}$$

Furthermore, if A, B are in $W[\![z]\!]$, we have

$$w(\alpha A + \beta B) \le |\alpha| w(A) + |\beta| w(B). \tag{3.13}$$

PROOF

For arbitrary $i \le k$, $j \le l$ we have

$$\begin{aligned} |(\alpha a_{n-j} + \beta b_{n-j})^S - (\alpha a_{n-j-i} + \beta b_{n-j-i})^S| &\le |\alpha||a_{n-j}^S - a_{n-j-i}^S| + |\beta||b_{n-j}^S - b_{n-j-i}^S| \\ &\le |\alpha| w_k(a_{n-j}) + |\beta| w_k(b_{n-j}) \\ &\le |\alpha| w_k^l(a_n) + |\beta| w_k^l(b_n), \end{aligned} \tag{3.14}$$

and (3.11) follows by taking $\sup_{j\le l}\sup_{i\le k}$ in (3.14). Then (3.12) follows by taking $\sup_n$ in (3.11), and (3.13) follows by taking $\sup_k$ in (3.12).

Theorem 3.2

Regarding $C[\![z]\!]$ as a vector space over C, $W_k[\![z]\!]$, as well as $W[\![z]\!]$ are subspaces of $C[\![z]\!]$. For each $k > 0$, w_k is a norm on $W_k[\![z]\!]$ and w is a norm on $W[\![z]\!]$.

PROOF

By (3.12) and (3.13), if A, B are fpsC's with finite w_k-width, respectively finite w-width, then so is $\alpha A + \beta B$. Therefore $W_k[\![z]\!]$, $W[\![z]\!]$ are vector subspaces of $C[\![z]\!]$.

Suppose that $A \ne 0$. Let n be the smallest integer for which $a_n \ne 0$. Given $k > 0$ and remembering (2.2), we then have $0 < |a_n^S - a_{n-k}^S| \le w_k(a_n) \le w_k(A) \le w(A)$. Therefore

$$A \ne 0 \Rightarrow w_k(A) \ne 0,\ w(A) \ne 0. \tag{3.15}$$

Then (3.12), (3.13), and (3.15) show that w_k is a norm on $W_k[\![z]\!]$, and w is a norm on $W[\![z]\!]$.

For the following theorem, note that $|a_n|^S = \Sigma_{j=0}^n |a_j|$ via the mapping $T: a_i \to |a_i|$ followed by the summation operator S. By contrast, $|a_n^S| = |\Sigma_{j=0}^n a_j|$.

Theorem 3.3

We have

$$w_k^l(a_n) \le |a_n|^S, \tag{3.16}$$

$$|a_n^S| \le w_{n+1}^l(a_n) \le |a_n|^S, \tag{3.17}$$

$$\|A\| \le w(A) \le |A|_1. \tag{3.18}$$

PROOF

For $i \le k$, $j \le l$ we have

$$|a_{n-j}^S - a_{n-j-i}^S| = \left| \sum_{m=n-j-i+1}^{n-j} a_m \right| \le |a_n|^S. \tag{3.19}$$

Then (3.16) follows by taking $\sup_{j\le l}\sup_{i\le k}$ in (3.19).

Remembering (2.2), we have

$$|a_n^S| = |a_n^S - a_{-1}^S| \le w_{n+1}(a_n), \tag{3.20}$$

and (3.17) follows from (3.9) and (3.16). Finally, (3.18) follows by taking $\sup_n$ in (3.17).

In many applications, the coefficients a_n are real numbers and are nonnegative with possibly a few exceptions of negligible magnitude. In such cases the following lemma can be used to simplify computations, with only a slight loss in accuracy.

Lemma 3.4

Suppose that A is an fpsC with real coefficients. Let $A = T - U$, where $t_n = \max\{a_n,0\}$ and $u_n = -\min\{a_n,0\}$. Then

$$\begin{aligned} w_k^\infty(a_n) &\le w_k^\infty(t_n) + w_k^\infty(u_n) \\ &\le \sup_{i\le n}|a_i^S - a_{i-k}^S| + 2\varepsilon_n, \end{aligned} \tag{3.21}$$

where

$$\varepsilon_n = \left| \sum_{j=0}^{n} a_j : a_j < 0 \right|.$$

PROOF

Using (3.11), we have

$$w_k^\infty(a_n) \le w_k^\infty(t_n) + w_k^\infty(u_n). \tag{3.22}$$

Since the coefficients of u_n are nonnegative, we have

$$w_k^\infty(u_n) \le u_n^S = \varepsilon_n, \tag{3.23}$$

and since the coefficients of T are nonnegative, there is an $i \le n$ such that

$$w_k^\infty(t_n) = t_i^S - t_{i-k}^S. \tag{3.24}$$

Now

$$a_i^S - a_{i-k}^S = t_i^S - t_{i-k}^S + \sum_{i-k\le j\le i} \{a_j : a_j < 0\} \tag{3.25}$$

so that

$$\begin{aligned} t_i^S - t_{i-k}^S &= a_i^S - a_{i-k}^S - \sum_{i-k\le j\le i} \{a_j : a_j < 0\} \\ &\le |a_i^S - a_{i-k}^S| + \left| \sum_{i-k\le j\le i} \{a_j : a_j < 0\} \right| \\ &\le w_k^\infty(a_n) + \varepsilon_n, \end{aligned} \tag{3.26}$$

and the lemma now follows from (3.23) and (3.26).

4. PROOFS OF THEOREMS 1.1, 1.2, 1.3, AND 1.4

Lemma 4.1

Suppose B is a polynomial in $PGF[\![z]\!]$ and A is an fpsC. Let $C = 1 - B$, $D = CA$, and $E = BA$. Then

$$|d_n^S| \le w_{d(B)}(a_n), \tag{4.1}$$

and for arbitrary k, l we have

$$w_k^l(e_n) \le w_k^{l+d(B)}(a_n) \tag{4.2}$$

and

$$w_k^l(d_n) \le 2w_k^{l+d(B)}(a_n). \tag{4.3}$$

PROOF

Using $D^S = (CA)^S = CA^S = (1 - B)A^S$, we have

$$\begin{aligned}
|d_n^S| &= \left| a_n^S - \sum_{j=0}^{d(B)} a_{n-j}^S b_j \right| \\
&= \left| \sum_{j=0}^{d(B)} (a_n^S - a_{n-j}^S) b_j \right| \\
&\le \sum_{j=0}^{d(B)} |a_n^S - a_{n-j}^S| b_j \\
&\le \sum_{j=0}^{d(B)} w_{d(B)}(a_n) b_j \\
&= w_{d(B)}(a_n),
\end{aligned}$$

establishing (4.1). Note that for arbitrary $i \le k$, $j \le l$, we have

$$\begin{aligned}
|e_{n-j}^S - e_{n-j-i}^S| &= \left| \sum_{m=0}^{d(B)} b_m a_{n-j-m}^S - \sum_{m=0}^{d(B)} b_m a_{n-j-i-m}^S \right| \\
&\le \sum_{m=0}^{d(B)} b_m |a_{n-j-m}^S - a_{n-j-i-m}^S| \\
&\le \sum_{m=0}^{d(B)} b_m\, w_k^{l+d(B)}(a_n) \\
&= w_k^{l+d(B)}(a_n).
\end{aligned}$$

Therefore (4.2) follows by noting that

$$w_k^l(e_n) = \sup_{j \le l} \sup_{i \le k} |e_{n-j}^S - e_{n-j-i}^S|.$$

Finally, using (3.12) and (4.2), we have

$$\begin{aligned}
w_k^l(d_n) &= w_k^l(a_n - e_n) \\
&\le w_k^l(a_n) + w_k^l(e_n) \\
&\le w_k^l(a_n) + w_k^{l+d(B)}(a_n) \\
&\le 2w_k^{l+d(B)}(a_n),
\end{aligned}$$

establishing (4.3).

Lemma 4.2

Suppose that B is a polynomial in $PGF[\![z]\!]$ and A is an fpsC with nonnegative real coefficients. Let $C = (1 - B)^2 A$. Then

$$|c_n^S| \leq w_{d(B)}^{d(B)}(a_n). \tag{4.4}$$

PROOF

Let $D = BA$, and note that D also has nonnegative real coefficients. We may write $C = (1 - B)A - (1 - B)D$. Therefore, using (3.9) and (4.2), we have

$$\begin{aligned}
|c_n^S| &= \left| \sum_{j=0}^{d(B)} (a_n^S - a_{n-j}^S)b_j - \sum_{j=0}^{d(B)} (d_n^S - d_{n-j}^S)b_j \right| \\
&\leq \max\left\{ \sum_{j=0}^{d(B)} |a_n^S - a_{n-j}^S|b_j, \sum_{j=0}^{d(B)} |d_n^S - d_{n-j}^S|b_j \right\} \\
&\leq \max\left\{ \sum_{j=0}^{d(B)} w_{d(B)}\ (a_n)b_j, \sum_{j=0}^{d(B)} w_{d(\mathbf{B})}\ (d_n)b_j \right\} \\
&= \max\{w_{d(B)}(a_n), w_{d(B)}(d_n)\} \\
&\leq \max\{w_{d(B)}(a_n), w_{d(B)}^{d(B)}(a_n)\} \\
&= w_{d(B)}^{d(B)}(a_n)
\end{aligned} \tag{4.5}$$

as required.

Lemma 4.3

Suppose that A is an fpsC with real coefficients. Suppose $M(z_1, \ldots, z_m) = \alpha z_1^{j_1} \cdot \ldots \cdot z_m^{j_m}$ is a monomial in $z_1, \ldots, z_m$ with at least one $j_i \geq 2$. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_m$ are complex numbers and that B_1, $B_2, \ldots, B_m$ are polynomials in $PGF[\![z]\!]$. Suppose that we have $d(B_i) \leq l$ for $i = 1, \ldots, m$. Let $C_i = 1 - B_i$ and let $D = M(\lambda_1 C_1, \ldots, \lambda_m C_m)A$. Then, if the coefficients of A are nonnegative, we have

$$|d_n^S| \leq \frac{1}{4}\,|M\ (2\lambda_1, \ldots, 2\lambda_m)| \cdot w_l^\infty(a_n), \tag{4.6}$$

and in any event we have

$$|d_n^S| \leq \frac{1}{4}\,|M(2\lambda_1, \ldots, 2\lambda_m)| \cdot \left(\sup_{i \leq n} |a_i^S - a_{i-l}^S| + 2\varepsilon_n \right), \tag{4.7}$$

where

$$\varepsilon_n = \sum_{j=0}^{n} \{|a_j| : a_j < 0\}. \tag{4.8}$$

PROOF

First we establish (4.6). By hypothesis we have, for some i, $M(z_1, \ldots, z_m) = z_i^2 N(z_1, \ldots, z_m)$ for a monomial N in the z_i. Then $D = \lambda_i^2(1 - B_i)^2\, E$, where $E = N(\lambda_1 C_1, \ldots, \lambda_m C_m)A$. Let $d = j_1 d(B_1) + \cdots + (j_i - 2)d(B_i) + \cdots + j_m d(B_m)$. Suppose, first, that the coefficients of A are nonnegative. Then, using (4.4), repeatedly using (4.3), together with (3.9) and (3.11), we have

$$\begin{aligned}
|d_n^S| &\leq |\lambda_i|^2\, w_{d(B_i)}^{d(B_i)}\ (e_n) \\
&\leq |\lambda_i|^2 |\, N(2\lambda_1, \ldots, 2\lambda_m)|\, w_{d(B_i)}^{d(B_i)+d}(a_n) \\
&\leq \frac{1}{4}\,|M(2\lambda_1, \ldots, 2\lambda_m)| \cdot w_l^\infty(a_n),
\end{aligned}$$

establishing (4.6).

Now suppose that A has arbitrary real coefficients. Write $A = T - U$ where $t_n = \max\{a_n, 0\}$ and $u_n = -\min\{a_n, 0\}$. Let $V = M(\lambda_1 C_1, \ldots, \lambda_m C_m)T$ and $W = M(\lambda_1 C_1, \ldots, \lambda_m C_m)U$. Then, using (3.11) and (4.6), we have

$$\begin{aligned} |d_n^S| &= |v_n^S - w_n^S| \\ &\le |v_n^S| + |w_n^S| \\ &\le \frac{1}{4}\,|M(2\,\lambda_1, \ldots, 2\lambda_n)| \cdot (w_l^\infty(t_n) + w_l^\infty(u_n)), \end{aligned} \tag{4.9}$$

and the result follows from (3.21).

Definition 4.4

Suppose that $B_1, B_2, \ldots, B_m$ are polynomials in $PGF[\![z]\!]$ such that $d(B_i) \le l$ for each i. Let $C_i = 1 - B_i$. Suppose $\varphi(z_1, z_2, \ldots, z_m)$ is a complex valued function of the complex variables $z_1, z_2, \ldots, z_m$ and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be a sequence of complex numbers such that

a. $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_m$ are in the domain of φ
b. φ has absolutely convergent expansions

$$\varphi(\lambda_1, \lambda_2, \ldots, \lambda_m) = \sum_{k=1}^{\infty} M_k(\lambda_1, \lambda_2, \ldots, \lambda_m)$$

and

$$\varphi(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_m) = \sum_{k=1}^{\infty} M_k(2\lambda_1, 2\lambda_2, \ldots, 2\lambda_m),$$

where the M_k are nonconstant monomials in $z_1, z_2, \ldots$. Define

$$|\varphi|(\lambda_1, \ldots, \lambda_m) = \sum_{k=1}^{\infty} |M_k(\lambda_1, \ldots, \lambda_m)| \tag{4.10}$$

and define the fpsC $U_\varphi(z)$ by

$$U_\varphi(z) = \sum_{k=1}^{\infty} M_k(\lambda_1 C_1\,(z), \lambda_2 C_2(z), \ldots, \lambda_m C_m(z)). \tag{4.11}$$

Lemma 4.5

$U_\varphi(z)$ is a well-defined analytic function on the disk $|z| \le 1$.

PROOF

Note that for $|z| \le 1$ we have $|C_i(z)| = |1 - B_i(z)| \le 1 + |B_i(z)| \le 2$. Therefore, for each k, $|M_k(\lambda_1 C_1(z), \lambda_2 C_2(z), \ldots, \lambda_m C_m(z))| \le |M_k(2\lambda_1, \ldots, 2\lambda_m)|$. Furthermore, each $M_k(\lambda_1 C_1(z), \lambda_2 C_2(z), \ldots, \lambda_m C_m(z))$ is a polynomial in z, and therefore analytic on $|z| \le 1$. The lemma now follows from hypothesis b, using the Weierstrass M-test.

Theorem 4.6

Suppose A is an fpsC, and let $D = U_\varphi(z)A(z)$. Then

$$|d_n^S| \le \frac{1}{4}\,|\varphi|(2\lambda_1, \ldots, 2\lambda_m) \cdot \left(\sup_{i \le n}|a_i^S - a_{i-l}^S| + 2\varepsilon_n\right), \tag{4.12}$$

where ε_n is given by (4.8).

Proof

Note that the values of a_n^S, d_n^S will not be affected if we truncate terms of degree greater than n from A. Therefore, to establish (4.12), we may assume without loss of generality that A is a polynomial with $d(A) \le n$. Thus, by Lemma 4.5,

$$U_\varphi(z)A(z) = \sum_{k=1}^{\infty} M_k(\lambda_1 C_1(z), \lambda_2 C_2(z), \ldots, \lambda_m C_m(z))A(z) \tag{4.13}$$

converges on $|z| \le 1$. Let M_k $(\lambda_1 C_1(z), \lambda_2 C_2(z), \ldots)A(z) = \sum_{n=0}^{\infty} d_{n,k} z^n$. Equating coefficients of z^n in (4.13) we have

$$d_n^S = \sum_{k=1}^{\infty} d_{n,k}^S. \tag{4.14}$$

Now it follows from (4.14) and Lemma 4.3 that

$$\begin{aligned} |d_n^S| &\le \sum_{k=1}^{\infty} |d_{n,k}^S| \\ &\le \frac{1}{4} \sum_{k=1}^{\infty} |M_k(2\lambda_1, \ldots, 2\lambda_m)| \left(\sup_{i \le n} |a_i^S - a_{i-l}^S| + 2\varepsilon_n \right) \\ &= \frac{1}{4} |\varphi|(2\lambda_1, \ldots, 2\lambda_m) \cdot \left(\sup_{i \le n} |a_i^S - a_{i-l}^S| + 2\varepsilon_n \right) \end{aligned} \tag{4.15}$$

as required.

Proof of Theorems 1.1, 1.2, 1.3, and 1.4

Let m be the number of policies in the portfolio H and let $\{z_j : j \in H\} = \{z_1, z_2, \ldots, z_m\}$ be a set of complex variables indexed by H. Let $H = \cup_{l \in L} H_l$ be as in (1.10). Define

$$\varphi_l(z_1, z_2, \ldots) = - \sum_{j \in H_l} \sum_{k=K+1}^{\infty} \frac{z_j^k}{k} \tag{4.16}$$

and

$$\varphi(z_1, z_2, \ldots) = \exp\left[\sum_{l \in L} \varphi_l(z_1, z_2, \ldots) \right] - \sum_{l \in L} \varphi_l(z_1, z_2, \ldots) - 1. \tag{4.17}$$

Let $P(z)$, $P_K(z)$, and $g_j(z)$ be as in (1.1) and (1.4). Since $0 < q_j < 0.5$, φ, φ_l satisfy conditions a and b of Definition 4.4 with $\lambda_j \equiv q_j$ and $B_j \equiv g_j$. Let U_φ, U_{φ_l} be as in Definition 4.4, and note that, by (1.3) and (1.4),

$$P(z) - P_K(z) = \left(\sum_{l \in L} U_{\varphi_l}(z) + U_\varphi(z) \right) P_K(z). \tag{4.18}$$

Then, by Theorem 4.6, we have

$$|F(n) - F_K(n)| \le \frac{1}{4} \left(\begin{aligned} &\sum_{l \in L} |\varphi_l|(2q_1, 2q_2, \ldots, 2q_m) \sup_{i \le n} |F_K(i) - F_K(i-l)| + \cdots \\ &+ |\varphi|(2q_1, 2q_2, \ldots, 2q_m) \sup_{i \le n} |F_K(i) - F_K(i - l_M)| + 2\varepsilon_n \end{aligned} \right), \tag{4.19}$$

where $l_M = \sup_{l \in L} l$. Now it is easily shown that, with δ_l, δ given by (1.13), we have

$$|\varphi_l|(2q_1, 2q_2, \ldots, 2q_l) \le \delta_l \tag{4.20}$$

and, since $e^x - x - 1$ is an increasing function of x for $x \ge 0$, that

$$|\varphi|(2q_1, 2q_2, \ldots, 2q_m) \le e^\delta - \delta - 1. \tag{4.21}$$

Theorem 1.2 then follows from (4.19), (4.20), and (4.21). Theorem 1.1 follows immediately from Theorem 1.2, and Theorems 1.3 and 1.4 are then obtained by letting $K = 1$.

5. PROOFS OF THEOREMS 1.5 AND 1.6

Lemma 5.1

Suppose that B is a polynomial in $PGF[\![z]\!]$ and A is an fpsC. Let

$$D(z) = \lfloor 1 - B(z)^k \rfloor A(z). \tag{5.1}$$

Then

$$|d_n^S| \le k w_{d(B)}^{\infty}(a_n), \tag{5.2}$$

and for arbitrary l, m we have

$$w_l^m(d_n) \le 2 w_l^{m+kd(B)}(a_n). \tag{5.3}$$

PROOF

Let $E_j(z) = B(z)^j[1 - B(z)]A(z) = \sum_{n=0}^{\infty} e_{n,j} z^n$. Then

$$\begin{aligned} D &= (1 - B)(1 + B + \cdots + B^{k-1})A \\ &= E_0 + E_1 + \cdots + E_{k-1}. \end{aligned} \tag{5.4}$$

Now using (4.1) and (4.2) we have

$$\begin{aligned} |d_n^S| &\le \sum_{j=0}^{k-1} |e_{n,j}^S| \\ &\le \sum_{j=0}^{k-1} w_{d(B)}^{jd(B)}(a_n) \\ &\le \sum_{j=0}^{k-1} w_{d(B)}^{\infty}(a_n) \\ &= k w_{d(B)}^{\infty}, \end{aligned} \tag{5.5}$$

establishing (5.2). Noting that $B(z)^k$ is in $PGF[\![z]\!]$, (5.3) follows immediately from (4.3).

Lemma 5.2

Suppose that B is a polynomial in $PGF[\![z]\!]$ and A is an fpsC. Suppose that q is a real number, $p = 1 - q$, and $0 < q < p$. Let

$$D(z) = \sum_{k=K+1}^{\infty} \frac{(-1)^k}{k} \frac{q^k}{p^k} [1 - B(z)^k] A(z). \tag{5.6}$$

Then $D(z)$ is well defined,

$$|d_n^S| \le \frac{p}{p-q}\left(\frac{q}{p}\right)^{K+1} w_{d(B)}^{\infty}(a_n), \tag{5.7}$$

and, for arbitrary l, m, we have

$$w_l^m(d_n) \le \frac{2}{K+1}\frac{p}{p-q}\left(\frac{q}{p}\right)^{K+1} w_l^{\infty}(a_n). \tag{5.8}$$

Proof

Without loss of generality, we assume that $A(z)$ is a polynomial of degree $\le n$. Let $D_k(z) = (-1)^k/k\ q^k/p^k\ [1 - B(z)^k]A(z)$. Then each $D_k(z)$ is analytic, and for $|z| \le 1$ we have, since $B(z)^k$ is in $PGF[\![z]\!]$,

$$\begin{aligned} |D_k(z)| &\le \frac{1}{k}\frac{q^k}{p^k}(1 + |B(z)^k|)|A(z)| \\ &\le |a_n|^S \frac{2q^k}{kp^k}. \end{aligned} \tag{5.9}$$

Therefore, by the Weierstrass M-test, $D(z)$ is analytic on $|z| \le 1$ and therefore is a well-defined fpsC. By Lemma 5.1 we then have

$$\begin{aligned} |d_n^S| &\le \sum_{k=K+1}^{\infty} \frac{1}{k}\frac{q^k}{p^k} k w_{d(B)}^{\infty}(a_n) \\ &= \frac{p}{p-q}\left(\frac{q}{p}\right)^{K+1} w_{d(B)}^{\infty}(a_n), \end{aligned} \tag{5.10}$$

establishing (5.7). Similarly, by (5.3), we have

$$\begin{aligned} w_l^m(d_n) &\le \sum_{k=K+1}^{\infty} \frac{q^k}{p^k} \cdot 2w_l^{m+kd(B)}(a_n) \\ &\le \frac{2}{K+1}\frac{p}{p-q}\left(\frac{q}{p}\right)^{K+1} w_l^{\infty}(a_n), \end{aligned} \tag{5.11}$$

establishing (5.8).

Lemma 5.3

Suppose that H is an individual portfolio given by (1.1). Suppose that H has the decomposition given by (1.10). Let $A(z)$ be an fpsC. Let

$$D(z) = \sum_{j\in H}\sum_{k=K+1}^{\infty} \frac{(-1)^k}{k}\left(\frac{q_j}{p_j}\right)^k [1 - g_j(z)^k], \tag{5.12}$$

and, for $m \ge 1$, let

$$\begin{aligned} E_m(z) &= D(z)^m A(z) \\ &= \sum_{n=0}^{\infty} e_{m,n} z^n. \end{aligned} \tag{5.13}$$

Then

$$|e_{1,n}^S| \le \sum_{l}\sum_{j\in H_l} \frac{p_j}{p_j - q_j}\left(\frac{q_j}{p_j}\right)^{K+1} w_l^{\infty}(a_n). \tag{5.14}$$

Furthermore, if $l_M = \sup_{l\in L} l$ then we have

$$w_{l_M}^{\infty}(e_{1,n}) \le \frac{2}{K+1} \sum_{j\in H} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1} w_{l_M}^{\infty}(a_n), \tag{5.15}$$

and for $m \ge 1$ we have

$$|e_{m,n}^S| \le \left(\frac{2}{K+1}\right)^{m-1} \left(\sum_{j\in H} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1}\right)^m w_{l_M}^{\infty}(a_n). \tag{5.16}$$

PROOF

Summing the inequality (5.7) over H establishes (5.14). Also, (5.15) follows from (3.11) and (5.8).

We now establish (5.16) by induction on m. For $m = 1$, (5.16) follows immediately from (5.14). Suppose then that $m > 1$ and that the assertion is true for all integers less than m. Then by the induction hypothesis, noting that $E_m(z) = D(z)^{m-1}E_1(z)$, we have

$$|e_{m,n}^S| \le \left(\frac{2}{K+1}\right)^{m-2} \left(\sum_{j\in H} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1}\right)^{m-1} w_{l_M}^{\infty}(e_{1,n}), \tag{5.17}$$

and the assertion follows for m by (5.15) and (5.17).

Lemma 5.4

Suppose that $D(z)$ is as in (5.12). Let $A(z)$ be an fpsC, and let

$$E(z) = [\exp D(z)]A(z) \tag{5.18}$$

Then $E(z)$ is well defined, and for arbitrary n we have

$$|e_n^S - a_n^S| \le \frac{K+1}{2} \left(\sum_{l\in L} 2\gamma_l w_l^{\infty}(a_n) + (e^{2\gamma} - 2\gamma - 1) w_{l_M}^{\infty}(a_n)\right), \tag{5.19}$$

where

$$\gamma_l = \frac{1}{K+1} \sum_{j\in H_l} \frac{p_j}{p_j - q_j} \left(\frac{q_j}{p_j}\right)^{K+1} \tag{5.20}$$

with $\gamma = \Sigma_{l\in L}\, \gamma_l$ and $l_M = \sup_{l\in L} l$.

PROOF

Without loss of generality, we may assume that $A(z)$ is a polynomial of degree $\le n$. Then, by (5.14) and (5.16), and by the Weierstrass M-test, the fpsC

$$E(z) - A(z) = \sum_{m=1}^{\infty} \frac{D(z)^m\, A(z)}{m!} \tag{5.21}$$

is well defined. From (5.14), (5.16), and (5.21) we then have

$$\begin{aligned} |e_n^S - a_n^S| &\le \frac{K+1}{2} \sum_{l\in L} 2\gamma_l w_l^{\infty}(a_n) + \frac{K+1}{2} \sum_{m=2}^{\infty} \frac{(2\gamma)^m}{m!} w_{l_M}^{\infty}(a_n) \\ &= \frac{K+1}{2} \left(\sum_{l\in L} 2\gamma_l w_l^{\infty}(a_n) + (e^{2\gamma} - 2\gamma - 1) w_{l_M}^{\infty}(a_n)\right) \end{aligned} \tag{5.22}$$

as required.

Proof of Theorems 1.5 and 1.6

Theorem 1.6 follows from Lemma 3.4 and Lemma 5.4, by noting that

$$P(z) = [\exp D(z)]P_K^*(z), \tag{5.23}$$

where $D(z)$ is given by (5.12). Theorem 1.5 is then a special case of Theorem 1.6.

6. Examples

Example 1

Consider a portfolio H of individual lives, each insured for the same face amount, and with a mortality rate of $q = 3.00 \times 10^{-3}$. The size of the portfolio is, respectively, $N = 10{,}000$, 30,000, 50,000, 70,000, and 90,000. We consider a first-order approximation $K = 1$. While this example is highly simplified, the orders of magnitude might represent small to medium-large portfolios of relatively similar term policies with an overall average mortality rate of \$3.00 per thousand insured.

We take the common face amount to be one unit. Then $F \equiv B(N, 0.003)$ is a binomial distribution with parameters N and 0.003. Note that $F_1 \equiv P_\lambda$ is a Poisson distribution with parameter $\lambda = 0.003N$ equal to the expected number of claims and $F_1^* \equiv P_\lambda^*$ is a Poisson distribution with parameter $\lambda^* = 0.003N/0.997$. Values of $F(n)$, $F_1(n)$, and $F_1^*(n)$ are directly available from standard sources and are displayed, for $n = \lambda \equiv \mu_F$, in Table 1. The approximation F_K^{**}, also displayed in Table 1, is obtained by $F_K^{**} = [pe^{-q}]^N F_K^*$.

The approximation F_K appears to be remarkably accurate, and the actual error $|F(\lambda) - F_K(\lambda)|$ is less than 10^{-7}.

Since F is also the distribution of the number of claims, the Hipp-Roos error bounds are quite straightforward to compute using (1.28). To compute the width-norm error bounds, note that (1.14) simplifies to

$$\|F - F_K\| \leq \frac{1}{4}(e^\delta - 1)p_\lambda([\lambda]),$$

where p_λ is the Poisson density function with parameter λ and δ is given by (1.16). Both bounds are *broadly* informative, with the Hipp-Roos bounds ranging from percentiles for $N = 10{,}000$ to 3.45 percentiles for $N = 90{,}000$, and the width-norm bounds ranging from 0.36 percentiles for $N = 10{,}000$ to 2.49 percentiles for $N = 90{,}000$.

Table 1

Example 1: Comparison of Approximations and Error Bounds

Number of Policies	10,000	30,000	50,000	70,000	90,000
q	0.003	0.003	0.003	0.003	0.003
$\lambda \equiv \mu_F$	30	90	150	210	270
$F(\lambda)$	0.5484	0.5280	0.5217	0.5183	0.5162
$F_K(\lambda)$	0.5484	0.5280	0.5217	0.5183	0.5162
$\lvert F(\lambda) - F_K(\lambda)\rvert$			Less than 10^{-7}		
Hipp-Roos error bound (1.28)	0.0114	0.0199	0.0257	0.0304	0.0345
Width-norm error bound (1.14)	0.0036	0.0076	0.0120	0.0176	0.0249
De Pril–Dhaene error bound (1.33)	0.0450	0.1349	0.2248	0.3147	0.4046
$F_K^*(\lambda)$	0.5418	0.5166	0.5070	0.5010	0.4965
$\lvert F(\lambda) - F_K^*(\lambda)\rvert$	0.0066	0.0114	0.0147	0.0174	0.0197
Width-norm error bound (1.21)	0.0069	0.0132	0.0187	0.0244	0.0307
$F_K^{**}(\lambda)$	0.5668	0.5916	0.6355	0.6873	0.7456
$\lvert F(\lambda) - F_K^{**}(\lambda)\rvert$	0.0250	0.0750	0.1285	0.1864	0.2491
De Pril error bound (1.30)	0.0465	0.1459	0.2549	0.3742	0.5048

The error bound (1.33) developed by De Pril and by Dhaene is considerably less sharp than either (1.14) or (1.28). It is a broadly informative 4.5 percentiles for $N = 10{,}000$ but not significantly informative for larger portfolios.

Note that the approximation F_K^* is significantly less accurate than F_K, with an *actual* error of 0.66 percentiles for $N = 10{,}000$, which increases with portfolio size to an actual error of 1.97 percentiles for $N = 90{,}000$. The width-norm error bound (1.21) is *broadly* informative, ranging from 0.69 percentiles for $N = 10{,}000$ to 3.07 percentiles for $N = 90{,}000$. Also interesting is that the width-norm error *bound* for the approximation F_K^* is slightly sharper than the Hipp-Roos error bound for the approximation F_K, even though, as stated above, the *approximation* F_K is itself considerably sharper than the approximation F_K^*.

Rescaling makes F_K^{**} *even less* accurate, with an actual error of 2.50 percentiles for $N = 10{,}000$ increasing with portfolio size to 24.91 percentile for $N = 90{,}000$. The error bound (1.30) is a moderately informative 4.65 percentiles for $N = 10{,}000$ and is not significantly informative for larger portfolios.

Example 2

Consider the portfolio H of 50,000 lives displayed in Table 2, each insured for various face amounts in units of $50,000.

While this example is also highly simplified, the orders of magnitude might represent a medium portfolio of ''select'' term policies with a retention limit of $500,000 ($300,000 at age 65), which is approximately 2% of expected claims of about 26.5 million.

We approximate the aggregate claims distribution of the portfolio using (1.4), alternately (1.20) and (1.29), with $K = 2$. The coefficients of $\ln P_K(z) = \sum_{k=0}^{\infty} b_k z^k$ and $\ln P_K^*(z) = \sum_{k=0}^{\infty} b_k^* z^k$ are displayed in Table 3.

From Table 3 and the Newton-Girard formulas (1.27), we obtain the approximations F_K and F_K^*. Values of F_K^{**} are then obtained by rescaling F_K^* by the scalar

$$\frac{F(0)}{F_K^*(0)} \approx 0.99332.$$

Values of F_K, F_K^*, and F_K^{**}at key percentiles are displayed in Table 4. For comparison, Table 4 also displays the ''exact'' values of F, which coincide, up to four decimals, with values of F_K. (Values of F, exact to a specified number of decimals, can be calculated by suitably increasing the order K of the approximation.)

Table 2

Example 2: Number of Lives by Age and by Amounts Insured

	Age				
Amount in Units	**25**	**35**	**45**	**55**	**65**
1	1500	3750	4500	3750	1600
2	1150	2875	3450	2875	1300
3	1000	2500	3000	2500	1100
4	450	1125	1350	1125	500
5	250	625	750	625	300
6	200	500	600	500	200
7	150	375	450	375	0
8	100	250	300	250	0
9	100	250	300	250	0
10	100	250	300	250	0
Mortality rate per 1000	0.90	0.95	2.22	5.34	15.15

Note: Expected claims in thousands: 26,472.

Table 3

Example 2: Coefficients of $P_K(z) = \sum_{k=0}^{\infty} b_k z^k$ and ln $P_K^*(z) = \sum_{k=0}^{\infty} b_k^* z^k$

k	b_k	b_k^*	k	b_k	b_k^*
0	−192.97166	−192.97152	10	2.29184	2.29071
1	59.66845	59.67472	11	0.00000	0.00000
2	46.62317	46.62192	12	−0.03187	−0.03266
3	40.29162	40.29592	13	0.00000	0.00000
4	18.00768	18.00452	14	−0.00669	−0.00675
5	10.45739	10.45856	15	0.00000	0.00000
6	7.57992	7.57638	16	−0.00446	−0.00450
7	3.50612	3.50618	17	0.00000	0.00000
8	2.25998	2.25805	18	−0.00446	−0.00450
9	2.33741	2.33746	19	0.00000	0.00000
			20	−0.00446	−0.00450

Since, in fact, f_K and f_K^* turn out to be unimodal, we can calculate the width-norm component of the error bounds (1.7), (1.12), (1.22), and (1.25) using the 10 largest values of f_K, f_K^*. These largest values are listed in Table 5, along with the relevant values of δ_l and γ_l.

From Table 5 we obtain, for the width-norm error bound (1.7), the value

$$\|F - F_K\| \leq 0.00118,$$

for the width-norm error bound (1.22) the value

$$\|F - F_K^*\| \leq 0.00178,$$

for the width-norm error bound (1.12) the value

$$\|F - F_K\| \leq 0.00032,$$

and for the width-norm error bound (1.25) the value

$$\|F - F_K^*\| \leq 0.00045.$$

Likewise, from Table 5, we can calculate the De Pril error bound (1.30) as

$$\|F - F_K^{**}\| \leq 0.00689.$$

To calculate the Hipp-Roos error bound for the approximation F_K, we need to first calculate an auxiliary compound Poisson distribution as described in Proposition 1 in Roos (2005). Based on this auxiliary distribution, we can then calculate the Hipp-Roos error bound using equation (3) of Proposition 2 of Roos (2005). The resulting value of the Hipp-Roos error bound is

Table 4

Example 2: Comparison of Approximations

Claims in Thousands	F_K or F	F_K^*	F_K^{**}	Claims in Thousands	F_K or F	F_K^*	F_K^{**}
25,800	0.3972	0.3976	0.3950	28,350	0.7956	0.7959	0.7906
25,850	0.4056	0.4060	0.4033	28,400	0.8015	0.8018	0.7964
26,400	0.4997	0.5001	0.4968	29,450	0.8999	0.9001	0.8941
26,450	0.5083	0.5087	0.5053	29,500	0.9035	0.9037	0.8976
26,950	0.5930	0.5934	0.5894	30,300	0.9482	0.9483	0.9420
27,000	0.6012	0.6016	0.5976	30,350	0.9503	0.9504	0.9441
27,600	0.6954	0.6958	0.6911	32,050	0.9900	0.9901	0.9835
27,650	0.7027	0.7031	0.6984	32,100	0.9905	0.9906	0.9840

Table 5
Example 2: Calculation of Width-Norm and De Pril Error Bounds

		Largest Values of		Sum of First l Largest Values of		From (1.13)	From (1.26)
l	Number of Units	$f_K(n)$	$f_K^*(n)$	$f_K(n)$	$f_K^*(n)$	δ_l	γ_l
1	527	0.008600	0.008601	0.008600	0.008601	0.01698	0.00218
2	526	0.008599	0.008600	0.017199	0.017201	0.01372	0.00176
3	528	0.008597	0.008598	0.025796	0.025799	0.01164	0.00150
4	525	0.008594	0.008596	0.034391	0.034395	0.00529	0.00068
5	529	0.008590	0.008590	0.042980	0.042985	0.00315	0.00041
6	524	0.008586	0.008587	0.051566	0.051572	0.00214	0.00027
7	530	0.008579	0.008579	0.060144	0.060151	0.00017	0.00002
8	523	0.008573	0.008575	0.068717	0.068726	0.00011	0.00001
9	531	0.008564	0.008564	0.077281	0.077289	0.00011	0.00001
10	522	0.008556	0.008558	0.085837	0.085847	0.00011	0.00001
Total		0.085837	0.085847			0.05342	0.00687

Error bounds:

Width-norm error bound (1.7):	0.00118
Width-norm error bound (1.22):	0.00178
Width-norm error bound (1.12)	0.00032
Width-norm error bound (1.25):	0.00045
De Pril error bound (1.30):	0.00689

$$\|F - F_K\| \leq 0.00217.$$

Note that the width-norm error bounds compare favorably, *in this instance,* to the Hipp-Roos error bound. Both bounds compare favorably to the error bound (1.30).

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