On the Class of Erlang Mixtures with Risk Theoretic Applications

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Abstract

A wide variety of distributions are shown to be of mixed-Erlang type. Useful computational formulas result for many quantities of interest in a risk-theoretic context when the claim size distribution is an Erlang mixture. In particular, the aggregate claims distribution and related quantities such as stop-loss moments are discussed, as well as ruin-theoretic quantities including infinite-time ruin probabilities and the distribution of the deficit at ruin. A very useful application of the results is the computation of finite-time ruin probabilities, with numerical examples given. Finally, extensions of the results to more general gamma mixtures are briefly examined.

1. Introduction and Background

One of the major problems in insurance risk theory is the evaluation of the aggregate claims distribution and ruin probabilities, together with related quantities such as stop-loss moments and the distribution of the deficit at ruin. Useful computational formulas are available for these quantities when the claim amount distribution is of mixed-Erlang type (e.g., Klugman, Panjer, and Willmot 2004).

The class of Erlang mixtures is preserved under a wide variety of risk-theoretic operations. In particular, the residual lifetime distribution, the equilibrium or integrated tail distribution, the aggregate claims distribution, and the conditional distribution of the deficit at ruin (given that ruin occurs) are all different mixtures of the same Erlangs (e.g., Willmot 2000; Willmot and Lin 2001). More generally, in the Sparre Andersen or renewal risk model with mixed-Erlang claim amounts, the ladder height distribution is a different mixture of the same Erlangs (Willmot 2007).

The mixed-Erlang structure considered in this paper involves the probability density function (pdf) of the Erlang-$j(E_j)$ random variable

$$\tau_j(y) = \frac{\beta(\beta y)^{j-1}e^{-\beta y}}{(j-1)!}, \qquad y > 0, \tag{1.1}$$

where $\beta > 0$ and $j \in \{1, 2, 3, \ldots\}$. The pdf

$$f(y) = \sum_{j=1}^{\infty} q_j \frac{\beta(\beta y)^{j-1}e^{-\beta y}}{(j-1)!} = \sum_{j=1}^{\infty} q_j\tau_j(y), \qquad y > 0, \tag{1.2}$$

where $\{q_1, q_2, \ldots\}$ is a discrete probability measure, is said to be that of an Erlang mixture or a mixture of Erlangs. The tail $\bar{F}(y) = 1 - F(y)$ satisfies (e.g., Willmot and Lin 2001, p. 12)

$$\bar{F}(y) = \int_y^{\infty} f(x)\,dx = e^{-\beta y}\sum_{k=0}^{\infty} \bar{Q}_k \frac{(\beta y)^k}{k!}, \qquad y > 0, \tag{1.3}$$

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where

$$\bar{Q}_k = \sum_{j=k+1}^{\infty} q_j, \qquad k = 0, 1, 2, \ldots . \tag{1.4}$$

The pdf (1.2) is an extremely useful probability model in risk theory and in other areas of applied probability for two reasons. First, the model is inherently mathematically tractable, and a wide variety of quantities of interest in aggregate claims and ruin theoretic analysis may be computed using series expansions. This is essentially because the Laplace transform of (1.2) is

$$\tilde{f}(s) = \int_0^{\infty} e^{-sy} f(y)\, dy = Q\left(\frac{\beta}{\beta + s}\right), \tag{1.5}$$

where

$$Q(z) = \sum_{j=1}^{\infty} q_j z^j \tag{1.6}$$

is the probability generating function (pgf) of $\{q_1, q_2, \ldots\}$. Thus, (1.2) may also be viewed as a compound distribution associated with a random sum of exponentially distributed random variables.

The second reason for its usefulness is its flexibility in terms of shape. In fact, any nonnegative continuous distribution may be approximated arbitrarily accurately (Tijms 1994, pp. 163–64) by a mixed-Erlang pdf of the form (1.2). Reliability properties of Erlang mixtures involving the force of mortality and the mean residual lifetime depend heavily on the mixing distribution $\{q_1, q_2, \ldots\}$ and are discussed by Esary, Marshall, and Proschan (1973).

Numerous special cases of (1.2) exist, many of which are discussed by Tijms (1994). In particular, note that if $q_1 = 1$, then $f(y)$ is an exponential pdf, whereas if $q_j = 1$, then $f(y)$ is the pdf of an Erlang-j random variable. A less-well-known example is the noncentral chi-squared random variable (Johnson, Kotz, and Kalakrishnan 1995, Chapter 29) with $2m$ degrees of freedom and noncentrality parameter λ, obtained from (1.2) with $\beta = \frac{1}{2}$ and $q_j = (\lambda/2)^{j-m} \exp(-\lambda/2)/(j - m)!$ for $j = m, m + 1, \ldots$ (with $q_j = 0$ otherwise).

In Section 2 we show that many other well-known pdfs may be expressed in the form (1.2). Mixtures of Erlangs with a finite number of different scale parameters, including finite mixtures of exponentials, are an example (see, e.g., Cheung 2007). Many finite sums of independent gamma random variables, including sums of independent Erlangs, have pdfs of the form (1.2). In particular, the generalized Erlang distribution of the sum of independent exponentials, a model used by Gerber and Shiu (2005) among others, has pdf of the form (1.2). Fitting of the closely related combination of exponentials is discussed by Dufresne (2006).

Evaluation of various risk theoretic quantities of interest is the subject of Section 3. Claim sizes are assumed to be of mixed-Erlang form, and special emphasis is placed on models considered in Section 2.

Some, but not all, of the distributions considered in Section 2 are of phase-type, and in these cases the approaches of Section 3 provide alternatives to the matrix-analytic approaches commonly employed in this context, as well as to root-finding techniques used in Laplace transform partial fraction expansion. For non–phase-type models, however, these alternatives are not generally available. An advantage of the mixed-Erlang representation is the availability of the numerical computational procedure given by Dickson and Willmot (2005) for the more technically complicated finite time probabilities in the classical Poisson model. This is the subject of Section 4, where numerical examples are provided.

Finally, extensions of the methodology to gamma mixtures are considered in Section 5, where in particular numerical evaluation of infinite time ruin probabilities using Shiu's approach (Shiu 1988; Willmot 1988) is discussed in the situation with claim sizes being finite sums or mixtures of independent gammas.

2. OTHER MIXTURES OF ERLANGS

In this section we show that various distributions have pdf of the form (1.2) by judicious use of a probabilistic (or equivalently mathematical) identity, namely, (2.1) below. In fact, it will not be difficult to see that any countable mixture of convolutions of Erlang pdfs may be put in the form (1.2), as long as the supremum of the set of scale parameters is finite. Several important special cases then will be considered.

Central to the subsequent discussion is the algebraic identity

$$\frac{\beta_1}{\beta_1 + s} = \frac{\beta_2}{\beta_2 + s}\left\{\frac{\dfrac{\beta_1}{\beta_2}}{1 - \left(1 - \dfrac{\beta_1}{\beta_2}\right)\dfrac{\beta_2}{\beta_2 + s}}\right\}, \tag{2.1}$$

where we will assume that $0 < \beta_1 \leq \beta_2 < \infty$. When viewed in terms of Laplace-Stieltjes transforms, (2.1) expresses the well-known result that a zero-truncated geometric sum of independent exponential random variables is again exponential (e.g., Taylor and Karlin 1998, pp. 74–75). Although stated somewhat differently, this result is also given by Steutel and van Harn (2004, pp. 128–29) in connection with infinite divisibility.

2.1 Sums of Exponentials

First, we consider the distribution of the sum of independent exponential random variables because of its popularity in insurance risk modeling. In particular, Gerber and Shiu (2005) considered this as a model for the interclaim time distribution in a renewal risk (or Sparre Andersen) model. Thus, let X_i have an exponential distribution with mean $1/\beta_i$ for $i = 1, 2, \ldots, n$, and let the X_i's be independent. The distribution of $X_1 + X_2 + \cdots + X_n$ is said to be of generalized Erlang form. In this subsection we shall assume that the β_i's are all distinct (this restriction is removed in Subsection 2.3). For references on this distribution see Gerber and Shiu (2005). We remark that the closely related combination of exponentials distribution is dense in the space of positive continuous distributions (Dufresne 2006), a property shared with the class of Erlang mixtures themselves, as was alluded to in the introduction. Hence, both of these classes are useful for modeling insurance claims.

The pdf is well known to be (e.g., Gerber and Shiu 2005, p. 53)

$$f(x) = \sum_{i=1}^{n} \eta_i \beta_i e^{-\beta_i x}, \qquad x > 0, \tag{2.2}$$

where

$$\eta_i = \prod_{k=1, k\neq i}^{n} \frac{\beta_k}{\beta_k - \beta_i}, \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

and the Laplace transform of (2.2) is

$$\tilde{f}(s) = \prod_{i=1}^{n}\left(\frac{\beta_i}{\beta_i + s}\right). \tag{2.4}$$

Clearly, (2.4) may be expressed in the form (1.5) with

$$Q(z) = z^n \prod_{i=1}^{n-1}\left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}\right\}, \tag{2.5}$$

where $z = \beta_n/(\beta_n + s)$ by applying (2.1).

Note that (2.5) yields

$$\frac{Q(z)}{z^n} = \prod_{i=1}^{n-1} \left\{ \frac{\beta_i/\beta_n}{1 - \left(1 - \dfrac{\beta_i}{\beta_n}\right) z} \right\} = \prod_{i=1}^{n-1} \left\{ \frac{\beta_i(\beta_n - \beta_i)^{-1}}{\beta_i(\beta_n - \beta_i)^{-1} + 1 - z} \right\},$$

and thus

$$\frac{Q(z)}{z^n} = \tilde{a}(1 - z), \tag{2.6}$$

where

$$\tilde{a}(s) = \int_0^\infty e^{-sx} a(x)\, dx = \prod_{i=1}^{n-1} \left\{ \frac{\beta_i(\beta_n - \beta_i)^{-1}}{\beta_i(\beta_n - \beta_i)^{-1} + s} \right\}. \tag{2.7}$$

Equations (2.6) and (2.7) express $Q(z)/z^n$ as a mixed Poisson pgf, with mixing pdf $a(x)$ that of the sum of independent exponentials. Obviously, (2.7) is of the same form as (2.4), but with β_i replaced by $\beta_i/(\beta_n - \beta_i)$ and n replaced by $n - 1$. Thus, by analogy with (2.2) and (2.3), one has

$$a(x) = \sum_{i=1}^{n-1} w_i \frac{\beta_i}{\beta_n - \beta_i} e^{-\beta_i x/(\beta_n - \beta_i)}, \qquad x > 0, \tag{2.8}$$

where

$$w_i = \prod_{k=1, k\neq i}^{n-1} \frac{\beta_k(\beta_n - \beta_k)^{-1}}{\beta_k(\beta_n - \beta_k)^{-1} - \beta_i(\beta_n - \beta_i)^{-1}}, \qquad i = 1, 2, \ldots, n - 1. \tag{2.9}$$

Substitution of (2.8) into (2.6) yields

$$\begin{aligned} \frac{Q(z)}{z^n} &= \sum_{i=1}^{n-1} w_i \int_0^\infty e^{-(1-z)x} \frac{\beta_i}{\beta_n - \beta_i} e^{-\beta_i x/(\beta_n - \beta_i)}\, dx \\ &= \sum_{i=1}^{n-1} w_i \frac{\beta_i(\beta_n - \beta_i)^{-1}}{\beta_i(\beta_n - \beta_i)^{-1} + 1 - z} \\ &= \sum_{i=1}^{n-1} w_i \frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}. \end{aligned}$$

Therefore,

$$\begin{aligned} Q(z) &= \sum_{i=1}^{n-1} w_i \frac{\beta_i}{\beta_n} \sum_{j=0}^{\infty} \left(1 - \frac{\beta_i}{\beta_n}\right)^j z^{j+n} \\ &= \sum_{j=n}^{\infty} \left\{ \sum_{i=1}^{n-1} w_i \frac{\beta_i}{\beta_n} \left(1 - \frac{\beta_i}{\beta_n}\right)^{j-n} \right\} z^j. \end{aligned}$$

To summarize, the coefficient of z^j in (2.5) is

$$q_j = \sum_{i=1}^{n-1} w_i \frac{\beta_i}{\beta_n} \left(1 - \frac{\beta_i}{\beta_n}\right)^{j-n}, \qquad j = n, n + 1, \ldots, \tag{2.10}$$

where w_i is given by (2.9). Thus,

$$\bar{Q}_j = \sum_{i=1}^{n-1} w_i \left(1 - \frac{\beta_i}{\beta_n}\right)^{j+1-n}, \qquad j = n, n + 1, \ldots. \tag{2.11}$$

□

We now consider a fairly general class of Erlang mixtures and show without loss of generality that it may be represented with only one scale parameter.

2.2 Countable Scale and Shape Mixtures of Erlangs

Suppose that

$$f(y) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} \frac{\beta_i^k y^{k-1} e^{-\beta_i y}}{(k-1)!}, \qquad y > 0, \tag{2.12}$$

where $n \in \{2, 3, \ldots\}$, $p_{ik} \geq 0$ for all i and k, and $\Sigma_{i=1}^{n} \Sigma_{k=1}^{\infty} p_{ik} = 1$. For finite mixtures, $p_{ik} = 0$ for k sufficiently large, but the present formulation is chosen for ease of notation. In particular, if $p_{ik} = 0$ for $k \geq 2$, then (2.12) is a mixture of exponentials. Assume without loss of generality that $\beta_i < \beta_n$ for $i = 1, 2, \ldots, n-1$. Then the Laplace transform of (2.12) may be expressed as

$$\tilde{f}(s) = \int_0^\infty e^{-sy} f(y)\, dy = \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} \left(\frac{\beta_i}{\beta_i + s}\right)^k, \tag{2.13}$$

and using (2.1), (2.13) may be expressed in the form (1.5) with $\beta = \beta_n$ and (1.6) given by

$$Q(z) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} z^k \left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n) z}\right\}^k. \tag{2.14}$$

Thus, (2.12) may be expressed as (1.2) with q_j the coefficient of z^j in the pgf (2.14). For notational convenience, define the probabilities $h_m(\alpha, \phi)$ for $\alpha > 0$ and $\phi \in (0, 1]$ by

$$h_m(\alpha, \phi) = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha) m!} \phi^\alpha (1 - \phi)^m, \qquad m = 0, 1, 2, \ldots, \tag{2.15}$$

which for $\phi \in (0, 1)$ are of negative binomial type, and (adopting the notational convention that $0^0 = 1$ in this paper) for $\phi = 1$ (2.15) yields $h_0(\alpha, 1) = 1$ and $h_m(\alpha, 1) = 0$ for $m \neq 0$. Then

$$\sum_{m=0}^{\infty} h_m(\alpha, \phi) z^m = \left\{\frac{\phi}{1 - (1 - \phi) z}\right\}^\alpha, \tag{2.16}$$

and (2.14) may be expressed as

$$Q(z) = \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{\infty} h_m\left(k, \frac{\beta_i}{\beta_n}\right) z^{m+k}.$$

Let $j = m + k$ to obtain

$$\begin{aligned} Q(z) &= \sum_{i=1}^{n} \sum_{k=1}^{\infty} p_{ik} \sum_{j=k}^{\infty} h_{j-k}\left(k, \frac{\beta_i}{\beta_n}\right) z^j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{\infty} z^j \sum_{k=1}^{j} p_{ik} h_{j-k}\left(k, \frac{\beta_i}{\beta_n}\right). \end{aligned}$$

Thus,

$$q_j = \sum_{i=1}^{n} \sum_{k=1}^{j} p_{ik} h_{j-k}\left(k, \frac{\beta_i}{\beta_n}\right),$$

and using (2.15), it follows that

$$q_j = \sum_{i=1}^{n} \sum_{k=1}^{j} p_{ik} \binom{j-1}{k-1} \left(\frac{\beta_i}{\beta_n}\right)^k \left(1 - \frac{\beta_i}{\beta_n}\right)^{j-k}, \qquad j = 1, 2, \ldots. \tag{2.17}$$

It is worth mentioning that $\bar{Q}_j$ for $j = 1, 2, 3, \ldots$, are useful quantities in connection with the evaluation of infinite-time ruin probabilities where $\bar{Q}_j = \sum_{m=j+1}^{\infty} q_m$, but a simpler formula (particularly for finite mixtures) can be derived. First, for $\phi \in (0, 1]$, one has the identity

$$\frac{1 - z^k \left\{\frac{\phi}{1-(1-\phi)z}\right\}^k}{1-z} = \frac{1}{\phi}\sum_{m=0}^{k-1} z^m \left\{\frac{\phi}{1-(1-\phi)z}\right\}^{m+1}. \tag{2.18}$$

By Feller (1968, p. 265), $\sum_{j=0}^{\infty} \bar{Q}_j z^j = \{1 - Q(z)\}/(1 - z)$. Then from (2.14) and (2.18),

$$\frac{1-Q(z)}{1-z} = \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{k-1} z^m \left\{\frac{\beta_i/\beta_n}{1-(1-\beta_i/\beta_n)z}\right\}^{m+1}, \tag{2.19}$$

and thus, using (2.16),

$$\begin{aligned}\frac{1-Q(z)}{1-z} &= \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{k-1} \sum_{\ell=0}^{\infty} h_\ell\left(m+1, \frac{\beta_i}{\beta_n}\right) z^{\ell+m} \\ &= \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} h_\ell\left(m+1, \frac{\beta_i}{\beta_n}\right) z^{\ell+m} \sum_{k=m+1}^{\infty} p_{ik} \\ &= \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} h_{j-m}\left(m+1, \frac{\beta_i}{\beta_n}\right) z^{j} \sum_{k=m+1}^{\infty} p_{ik} \\ &= \sum_{j=0}^{\infty} z^j \sum_{m=0}^{j} \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} h_{j-m}\left(m+1, \frac{\beta_i}{\beta_n}\right) \sum_{k=m+1}^{\infty} p_{ik}.\end{aligned}$$

The coefficient of z^j is thus

$$\bar{Q}_j = \sum_{m=0}^{j} \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} h_{j-m}\left(m+1, \frac{\beta_i}{\beta_n}\right) \sum_{k=m+1}^{\infty} p_{ik}. \tag{2.20}$$

Hence, using (2.15), (2.20) becomes

$$\bar{Q}_j = \sum_{i=1}^{n} \sum_{m=0}^{j} \binom{j}{m} \left(\frac{\beta_i}{\beta_n}\right)^m \left(1 - \frac{\beta_i}{\beta_n}\right)^{j-m} \sum_{k=m+1}^{\infty} p_{ik}, \qquad j = 1, 2, \ldots. \tag{2.21}$$

It is instructive to note that (2.21) is a finite sum if (2.12) is a finite mixture. □

Unlike the situation in Subsection 2.1, in the following generalization the pdf is typically complicated unless an approach such as the present one is used.

2.3 Erlangian Sums of Gammas

Suppose that the exponential components of the sum in Subsection 2.1 are replaced by those of gamma type, and thus the Laplace transform (2.4) is replaced by

$$\tilde{f}(s) = \prod_{i=1}^{n} \left(\frac{\beta_i}{\beta_i + s}\right)^{\alpha_i}, \tag{2.22}$$

where we require that $m = \sum_{i=1}^{n} \alpha_i \in \{1, 2, \ldots\}$. Note that it is not necessary that the α_i's be positive integers, but their sum must be. Assume without loss of generality that $\beta_i < \beta_n$ for $i = 1, 2, \ldots, n - 1$. Application of (2.1) then allows (2.22) to be written as (1.5) with (1.6) given by

$$Q(z) = z^m \prod_{i=1}^{n-1} \left\{\frac{\beta_i/\beta_n}{1-(1-\beta_i/\beta_n)z}\right\}^{\alpha_i}, \tag{2.23}$$

and $\beta = \beta_n$. Thus, the pdf $f(y)$ with Laplace transform (2.22) is of the form (1.2), where $\beta = \beta_n$ and q_j is the coefficient of z^j in (2.23).

Note that $q_j = 0$ from (2.23) for $j < m$ because $Q(z)/z^m$ is a product of negative binomial pgfs. Also,

$$q_m = \prod_{i=1}^{n-1} \left(\frac{\beta_i}{\beta_n}\right)^{\alpha_i}. \tag{2.24}$$

The coefficients $\{q_j; j = m + 1, m + 2, \ldots\}$ may be computed recursively, beginning with (2.24). It follows from (2.23) that

$$\ln\{z^{-m}Q(z)\} = \ln(q_m) - \sum_{i=1}^{n-1} \alpha_i \ln\left\{1 - \left(1 - \frac{\beta_i}{\beta_n}\right) z\right\}$$

and thus

$$z \frac{d}{dz} \{z^{-m}Q(z)\} = \{z^{-m}Q(z)\} \sum_{i=1}^{n-1} \alpha_i \sum_{k=1}^{\infty} \left(1 - \frac{\beta_i}{\beta_n}\right)^k z^k.$$

Equating coefficients of z^ℓ on both sides of this equation yields

$$\ell q_{\ell+m} = \sum_{k=1}^{\ell} \left\{\sum_{i=1}^{n-1} \alpha_i \left(1 - \frac{\beta_i}{\beta_n}\right)^k\right\} q_{\ell+m-k}$$

for $\ell = 0, 1, 2, \ldots$, and division by ℓ followed by replacement of ℓ by $j = \ell + m$ yields

$$q_j = \frac{1}{j - m} \sum_{k=1}^{j-m} \left\{\sum_{i=1}^{n-1} \alpha_i \left(1 - \frac{\beta_i}{\beta_n}\right)^k\right\} q_{j-k}, \qquad j = m + 1, m + 2, \ldots. \tag{2.25}$$

Equation (2.25), which may be used to compute $\{q_{m+1}, q_{m+2}, \ldots\}$ recursively, is essentially Panjer's compound Poisson recursion applied to convolutions of negative binomials (e.g., Klugman, Panjer, and Willmot 2004, pp. 100–101).

Numerical evaluation of $q_j; j = m + 1, m + 2, \ldots$ is not always necessary, however. For example, if $n = 2$, then (2.23) becomes

$$Q(z) = z^m \left\{\frac{\beta_1/\beta_2}{1 - (1 - \beta_1/\beta_2)z}\right\}^{\alpha_1}, \tag{2.26}$$

and using (2.15) and (2.16) yields

$$q_j = \frac{\Gamma(\alpha_1 + j - m)}{\Gamma(\alpha_1)(j - m)!} \left(\frac{\beta_1}{\beta_2}\right)^{\alpha_1} \left(1 - \frac{\beta_1}{\beta_2}\right)^{j-m}, \qquad j = m, m + 1, \ldots, \tag{2.27}$$

with $m = \alpha_1 + \alpha_2$.

3. Compound Distributions and Related Quantities

Many quantities of interest in risk theory, under the assumption of mixed-Erlang claim amounts, involve the compound distribution function $G(x) = 1 - \bar{G}(x)$ with Laplace-Stieltjes transform

$$\tilde{g}(s) = \int_0^\infty e^{-sx} \, dG(x) = P\{\tilde{f}(s)\}, \tag{3.1}$$

where $\tilde{f}(s)$ is given by (1.5) and $P(z) = \Sigma_{n=0}^\infty p_n z^n$ is a pgf. Substitution of (1.5) into (3.1) yields

$$\tilde{g}(s) = P\left\{Q\left(\frac{\beta}{\beta + s}\right)\right\} = C\left(\frac{\beta}{\beta + s}\right), \tag{3.2}$$

where

$$C(z) = \sum_{n=0}^{\infty} c_n z^n = P\{Q(z)\}, \tag{3.3}$$

is a compound pgf. To be more precise, numerical evaluation of $G(x)$ and related quantities is often possible as long as $\{c_n;\ n = 0, 1, 2, \ldots\}$ may be obtained numerically. Fortunately, this is relatively straightforward for a wide variety of choices of $P(z)$ using recursive and other techniques (e.g., Klugman, Panjer, and Willmot 2004).

It is clear from (3.2) that $G(x)$ has a (possible) mass point $c_0 = p_0$ at 0, and continuous pdf

$$g(x) = \sum_{n=1}^{\infty} c_n \tau_n(x), \qquad x > 0, \tag{3.4}$$

that is, on the positive real line. Thus, $g(x)$ is also of mixed Erlang form.

For aggregate claims and stop-loss analysis (as well as for applications), it is convenient to note that (e.g., Willmot 2007)

$$g(x + y) = \beta^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{j+k+1} \tau_{j+1}(x) \tau_{k+1}(y). \tag{3.5}$$

Stop-loss moments of any order (even fractional) are easily obtainable using (3.5). For $\alpha \geq 0$ and $x \geq 0$,

$$\begin{aligned}
\int_x^{\infty} (y - x)^{\alpha}\, dG(y) &= \int_0^{\infty} y^{\alpha} g(x + y)\, dy \\
&= \beta^{-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{j+k+1} \tau_{j+1}(x) \int_0^{\infty} y^{\alpha} \tau_{k+1}(y)\, dy \\
&= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} c_{j+k+1} \tau_{j+1}(x) \frac{\Gamma(\alpha + k + 1)}{\beta^{\alpha+1} \Gamma(k + 1)}.
\end{aligned}$$

Therefore,

$$\int_x^{\infty} (y - x)^{\alpha}\, dG(y) = e^{-\beta x} \sum_{j=0}^{\infty} \gamma_{j,\alpha} \frac{(\beta x)^j}{j!}, \tag{3.6}$$

where

$$\gamma_{j,\alpha} = \beta^{-\alpha} \sum_{k=1}^{\infty} c_{j+k} \frac{\Gamma(\alpha + k)}{\Gamma(k)}. \tag{3.7}$$

For $\alpha = 0$, $\gamma_{j,0} = \sum_{k=1}^{\infty} c_{j+k} = \bar{C}_j$, and (3.6) becomes

$$\bar{G}(x) = e^{-\beta x} \sum_{j=0}^{\infty} \bar{C}_j \frac{(\beta x)^j}{j!}, \qquad x \geq 0. \tag{3.8}$$

In the classical continuous-time risk model, the number of claims from an insurance portfolio is assumed to follow a Poisson process N_t with mean λ. The individual claim sizes $X_1, X_2, \ldots$, independent of N_t, are positive, independent, and identical random variables with common distribution function $F(x) = Pr(X \leq x)$ and moments $p_k = \int_0^{\infty} x^k\, dF(x)$. The aggregate claims process is $\{S_t;\ t \geq 0\}$, where $S_t = X_1 + X_2 + \cdots + X_{N_t}$ (with $S_t = 0$ if $N_t = 0$). Obviously the aggregate claims process S_t is a compound Poisson process. The insurer's surplus process is $\{U_t;\ t \geq 0\}$, where $U_t = u + ct - S_t$, where $u \geq 0$ is the initial surplus, $c = \lambda p_1(1 + \theta)$ the premium rate per unit time, and $\theta \geq 0$ the relative security loading. Define $T = \inf\{t;\ U_t < 0\}$ to be the first time that the surplus becomes negative and is called the time of ruin. The probability $\psi(u) = Pr\{T < \infty | U_0 = u\}$ is called the probability of (ultimate) ruin. The finite-time ruin probabilities are $\psi(u, t) = Pr\{T < t | U_0 = u\}$ so that $\psi(u) = \lim_{t\to\infty} \psi(u, t)$.

For the analysis of ruin and related quantities, it is convenient to introduce the discrete distribution

$$q_j^* = \frac{\bar{Q}_{j-1}}{\sum_{k=0}^{\infty} \bar{Q}_k}, \qquad j = 1, 2, \ldots, \tag{3.9}$$

where $\bar{Q}_k$ is given by (1.4). By Feller (1968, p. 265), it follows that $\Sigma_{k=0}^{\infty} \bar{Q}_k = \Sigma_{k=1}^{\infty} kq_k = Q'(1)$. Define the compound geometric distribution $\{c_n^*;\, n = 0, 1, 2, \ldots\}$ by its pgf (with $\theta > 0$)

$$\sum_{n=0}^{\infty} c_n^* z^n = \left\{1 - \frac{1}{\theta}\left(\sum_{j=1}^{\infty} q_j^* z^j - 1\right)\right\}^{-1}. \tag{3.10}$$

Next, define the functions

$$\phi_m(x) = \sum_{n=0}^{\infty} c_n^* \frac{x^{n+m}}{(n+m)!}, \qquad m = 0, 1, 2, \ldots. \tag{3.11}$$

Then the conditional distribution of the deficit at ruin, given that ruin occurs beginning with initial reserve u, has mixed-Erlang pdf

$$g_u(y) = \sum_{j=1}^{\infty} \bar{q}_j(u)\tau_j(u), \tag{3.12}$$

where the distribution $\{\bar{q}_j(u);\, j = 1, 2, \ldots\}$ is given by

$$\bar{q}_j(u) = \frac{\sum_{k=j}^{\infty} q_k^* \phi_{k-j}(\beta u)}{\sum_{k=1}^{\infty} q_k^* \sum_{m=0}^{k-1} \phi_m(\beta u)}, \qquad j = 1, 2, 3, \ldots. \tag{3.13}$$

See Willmot and Lin (2001, Section 10.3) for details.

For the evaluation of infnite-time ruin probabilities, it is convenient to define the tail probabilities associated with (3.9):

$$\bar{Q}_j^* = \sum_{k=j+1}^{\infty} q_k^*, \qquad j = 0, 1, 2, \ldots. \tag{3.14}$$

Then the tail $\bar{C}_n^* = \Sigma_{k=n+1}^{\infty} c_k^*$, $n = 0, 1, 2, \ldots$ of the compound geometric distribution with pgf (3.10) can be computed recursively from

$$\bar{C}_n^* = \frac{1}{1+\theta} \sum_{k=1}^{n} q_k^* \bar{C}_{n-k}^* + \frac{\bar{Q}_n^*}{1+\theta}, \qquad n = 1, 2, \ldots, \tag{3.15}$$

beginning with $\bar{C}_0^* = 1/(1+\theta)$; and the ultimate ruin probabilities are given by (e.g., Klugman, Panjer, and Willmot 2004, pp. 243–44)

$$\psi(u) = e^{-\beta u} \sum_{n=0}^{\infty} \bar{C}_n^* \frac{(\beta u)^n}{n!}, \qquad u \geq 0. \tag{3.16}$$

The class of distributions given by (2.12) now is considered in the present context.

3.1 Infinite Ruin for Scale and Shape Mixtures of Erlangs

Suppose that the claim size pdf is given by (2.12), and hence the mixing pgf is given by (2.14). One has from (2.19) with $z = 1$ that

$$\sum_{k=0}^{\infty} \bar{Q}_k = \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} kp_{ik}, \tag{3.17}$$

and hence using (2.21), (3.9) becomes, for $j = 1, 2, \ldots,$

$$q_j^* = \frac{\sum_{i=1}^{n} \sum_{m=0}^{j-1} \binom{j-1}{m} \left(\frac{\beta_i}{\beta_n}\right)^m \left(1 - \frac{\beta_i}{\beta_n}\right)^{j-1-m} \sum_{k=m+1}^{\infty} p_{ik}}{\sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} k p_{ik}}. \tag{3.18}$$

If (2.12) is a finite mixture, then all the sums in (3.18) are finite. The pgf of (3.18) is thus, using (3.9),

$$Q^*(z) = \sum_{j=1}^{\infty} q_j^* z^j = \frac{z \sum_{j=1}^{\infty} \bar{Q}_{j-1} z^{j-1}}{\sum_{k=0}^{\infty} \bar{Q}_k} = \frac{z}{\sum_{k=0}^{\infty} \bar{Q}_k} \frac{1 - Q(z)}{1 - z},$$

and from (2.19) it follows that

$$Q^*(z) = \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} p_{ik} \sum_{x=1}^{k} z^x \left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}\right\}^x \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k. \tag{3.19}$$

Clearly, $k = \Sigma_{x=1}^{k} (1)$, implying from (3.17) that $\Sigma_{k=0}^{\infty} \bar{Q}_k = \Sigma_{i=1}^{n} (\beta_n/\beta_i) \Sigma_{k=1}^{\infty} p_{ik} \Sigma_{x=1}^{k} (1)$, which when substituted into (3.19) implies that

$$\sum_{j=0}^{\infty} \bar{Q}_j^* z^j = \frac{1 - Q^*(z)}{1 - z} = \sum_{i=1}^{n} \frac{\beta_n}{\beta_i} \sum_{k=1}^{\infty} p_{ik} \sum_{x=1}^{k} \frac{1 - z^x \left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}\right\}^x}{1 - z} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k.$$

It then follows from (2.18) with k replaced by x that

$$\sum_{j=0}^{\infty} \bar{Q}_j^* z^j = \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{k=1}^{\infty} p_{ik} \sum_{x=1}^{k} \sum_{m=0}^{x-1} z^m \left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}\right\}^{m+1} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k,$$

and using (2.16),

$$\begin{aligned}
\sum_{j=0}^{\infty} \bar{Q}_j^* z^j &= \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{k-1} \sum_{x=m+1}^{k} z^m \left\{\frac{\beta_i/\beta_n}{1 - (1 - \beta_i/\beta_n)z}\right\}^{m+1} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k \\
&= \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{k=1}^{\infty} p_{ik} \sum_{m=0}^{k-1} (k - m) \sum_{\ell=0}^{\infty} h_\ell \left(m + 1, \frac{\beta_i}{\beta_n}\right) z^{\ell+m} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k \\
&= \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} h_\ell \left(m + 1, \frac{\beta_i}{\beta_n}\right) z^{\ell+m} \sum_{k=m+1}^{\infty} (k - m) p_{ik} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k.
\end{aligned}$$

Replacement of ℓ by $j = \ell + m$ yields

$$\begin{aligned}
\sum_{j=0}^{\infty} \bar{Q}_j^* z^j &= \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{m=0}^{\infty} \sum_{j=m}^{\infty} h_{j-m} \left(m + 1, \frac{\beta_i}{\beta_n}\right) z^j \sum_{k=m+1}^{\infty} (k - m) p_{ik} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k \\
&= \sum_{j=0}^{\infty} z^j \sum_{i=1}^{n} \left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{m=0}^{j} h_{j-m} \left(m + 1, \frac{\beta_i}{\beta_n}\right) \sum_{k=m+1}^{\infty} (k - m) p_{ik} \Bigg/ \sum_{k=0}^{\infty} \bar{Q}_k,
\end{aligned}$$

and equating coefficients of z^j yields

$$\bar{Q}_j^* = \sum_{i=1}^{n}\left(\frac{\beta_n}{\beta_i}\right)^2 \sum_{m=0}^{j} h_{j-m}\left(m+1, \frac{\beta_i}{\beta_n}\right) \sum_{k=m+1}^{\infty}(k-m)p_{ik} \Big/ \sum_{k=0}^{\infty}\bar{Q}_k.$$

Application of (2.15) and (3.17) yields

$$\bar{Q}_j^* = \frac{\sum_{i=1}^{n}\sum_{m=0}^{j}\binom{j}{m}\left(\frac{\beta_i}{\beta_n}\right)^{m-1}\left(1-\frac{\beta_i}{\beta_n}\right)^{j-m}\sum_{k=m+1}^{\infty}(k-m)p_{ik}}{\sum_{i=1}^{n}\frac{\beta_n}{\beta_i}\sum_{k=1}^{\infty}kp_{ik}}, \qquad j = 1, 2, 3, \ldots, \tag{3.20}$$

which involves only finite sums if (2.12) is a finite mixture. □

As mentioned, many mixed-Erlang distributions are of phase-type, and in these cases the infinite series expansions derived above provide computational alternatives to matrix-analytic or partial fraction Laplace transform approaches. For other Erlang mixtures that do not have a rational Laplace transform (i.e., a ratio of polynomials), the above series expansions appear to be the only known technique in general to obtain exact numerical solutions.

An advantage of the present mixed-Erlang approach described here is the availability of computational procedures for the numerical evaluation of finite-time ruin probabilities, which is discussed in the next section.

4. FINITE-TIME RUIN PROBABILITIES

A computational algorithm for the numerical evaluation of finite-time ruin probabilities in the case of mixed-Erlang claim amounts is given by Dickson and Willmot (2005). Thus, representation of distributions in mixed-Erlang form as in this paper allows for the implementation of the algorithm in these situations.

The algorithm itself requires the evaluation of the coefficients $\{q_j^{*k}; j = 1, 2, 3, \ldots\}$ of the k-fold convolution of the distribution $\{q_j; j = 1, 2, 3, \ldots\}$ defined by (1.6). Thus,

$$\{Q(z)\}^k = \sum_{j=1}^{\infty} q_j^{*k} z^j, \qquad k = 1, 2, 3, \ldots, \tag{4.1}$$

and q_j^{*k} is the coefficient of z^j in $\{Q(z)\}^k$. A recursive algorithm for these coefficients is given by Knuth (1981, p. 507), which is valid even for nonintegral k. For some choices of $f(y)$ this is not necessary, however, as in the following.

4.1 A Gamma-Erlang Mixture

Suppose that $f(y)$ has Laplace transform

$$\tilde{f}(s) = p\left(\frac{\beta_1}{\beta_1+s}\right)^{\alpha}\left(\frac{\beta_2}{\beta_2+s}\right)^{i+m-\alpha} + (1-p)\left(\frac{\beta_2}{\beta_2+s}\right)^{m}, \tag{4.2}$$

where $0 < p < 1$, $0 < \beta_1 < \beta_2 < \infty$, $i \in \{0, 1, 2, \ldots\}$, $m \in \{1, 2, 3, \ldots\}$, and $0 < \alpha \le i + m$. If $\alpha = i + m$, then $f(y)$ is a mixture of two Erlangs, and if $\alpha = m = 1$ with $i = 0$, then $f(y)$ is a mixture of two exponentials. By applying (2.1) to (4.2) it follows that $\tilde{f}(s) = Q\,(\beta_2/(\beta_2 + s))$, where

$$Q(z) = z^m\left\{1 - p + pz^i\left(\frac{\beta_1/\beta_2}{1-(1-\beta_1/\beta_2)z}\right)^{\alpha}\right\}. \tag{4.3}$$

Therefore,

$$\{Q(z)\}^k = z^{mk}\left\{1 - p + pz^i\left(\frac{\beta_1/\beta_2}{1-(1-\beta_1/\beta_2)z}\right)^\alpha\right\}^k, \tag{4.4}$$

and $\{Q(z)\}^k$ is a shifted (by mk) compound binomial pgf with shifted (by i) negative binomial secondary pgf. Furthermore, if $\alpha = 1$ and $i = 0$ as in the mixed exponential case, then (4.4) may be reexpressed as

$$\{Q(z)\}^k = z^{mk}\left\{\frac{1-p}{1-pV(z)}\right\}^k, \tag{4.5}$$

where

$$V(z) = \frac{1-\phi}{1-\phi z}, \tag{4.6}$$

with

$$\phi = \frac{(1-p)(\beta_2-\beta_1)}{\beta_1+(1-p)(\beta_2-\beta_1)}. \tag{4.7}$$

Hence, when $\alpha = 1$, $\{Q(z)\}^k$ is a shifted compound negative binomial pgf with geometric secondary pgf. The compound negative binomial recursion is known to be numerically stable (e.g., Klugman, Panjer, and Willmot 2004, p. 166). □

Similarly, if $\tilde{f}(s)$ is given by (2.22), then from (2.23)

$$\{Q(z)\}^k = z^{mk}\prod_{i=1}^{n-1}\left\{\frac{\beta_i/\beta_n}{1-(1-\beta_i/\beta_n)z}\right\}^{k\alpha_i}, \tag{4.8}$$

which is again a shifted convolution of negative binomials, but with m replaced by mk and α_i by $k\alpha_i$ in (2.23), and with similar modifications to the recursive scheme in (2.24) and (2.25).

To illustrate the methodology, finite-time ruin probabilities using the Dickson and Willmot (2005) approach are calculated for three different claim size distributions (all expressed in mixed-Erlang form as described in the paper) and are given in Tables 1–6, as we will describe.

The first claim size distribution is the mixture of two exponentials with

$$\tilde{f}(s) = \left(\frac{1}{3}\right)\left(\frac{1}{1+2s}\right) + \left(\frac{2}{3}\right)\left(\frac{2}{2+s}\right), \tag{4.9}$$

mean 1, and standard deviation 1.414214. The second is the sum of two exponentials with Laplace transform

Table 1
Mixture of Two Exponentials (u = 1)

t	$\psi(1, t)$		
	Exact	Discretization	De Vylder
2	0.3111800	0.3112170	0.3009786
4	0.4338971	0.4339348	0.4253761
6	0.5034873	0.5035235	0.4957896
8	0.5495076	0.5495422	0.5423328
10	0.5827376	0.5827708	0.5759528
20	0.6706329	0.6706617	0.6649948
40	0.7358065	0.7358317	0.7311475
∞	0.8425516	0.8425731	0.8396948

Table 2
Mixture of Two Exponentials (u = 10)

t	$\psi(10, t)$		
	Exact	Discretization	De Vylder
2	0.0086734	0.0086753	0.0083237
4	0.0224369	0.0224415	0.0221687
6	0.0383385	0.0383460	0.0382706
8	0.0548488	0.0548591	0.0549854
10	0.0711839	0.0711968	0.0714963
20	0.1422078	0.1422298	0.1429800
40	0.2347051	0.2347350	0.2356187
∞	0.4913739	0.4914185	0.4919539

$$\tilde{f}(s) = \left(\frac{3}{3 + 2s}\right)\left(\frac{3}{3 + s}\right), \tag{4.10}$$

mean 1, and standard deviation 0.745356. The third is the sum of two gammas with Laplace transform

$$\tilde{f}(s) = \left(\frac{3}{3 + 4s}\right)^{1/2}\left(\frac{3}{3 + 2s}\right)^{1/2}, \tag{4.11}$$

mean 1, and standard deviation 1.054093.

The Poisson claim rate is assumed to be 1 and the premium loading 10% in all cases. The initial surplus is assumed to be $u = 1$ and $u = 10$. The times range from $t = 2$ to $t = 40$, and the infinite-time ruin probabilities are given with $t = \infty$. The finite-time ruin probabilities are denoted $\psi(u, t)$.

The first column is labeled ''Exact'' and gives the values calculated by the Dickson and Willmot (2005) algorithm, with infinite-time ruin probabilities calculated using (3.16). The second column is labeled ''Discretization'' and gives the values obtained using the Dickson and Waters (1991) recursive approximation technique with a scaling factor of 20.

The third column labeled ''DeVylder'' gives the values obtained by De Vylder's (1978) approximation. De Vylder proposed an approximation to the infinite-time ruin probability by replacing the original surplus process with an approximating process having exponentially distributed claim sizes and four newly defined parameters. The parameters of the approximating process are chosen by matching the first three moments of the original and approximating surplus processes. Thus, the new parameters are

$$\beta_D = \frac{3p_2}{p_3}, \qquad \lambda_D = \frac{9\lambda p_2^3}{2p_3^2}, \qquad \theta_D = \frac{2p_1p_3\theta}{3p_2^2}, \qquad c_D = \frac{\lambda_D(1 + \theta_D)}{\beta_D}. \tag{4.12}$$

The approximating surplus process has individual claim size distribution given by $F_D(x) = 1 - e^{-\beta_D x}$, Poisson parameter λ_D, and security loading θ_D. Hence,

Table 3
Sum of Two Exponentials (u = 1)

t	$\psi(1, t)$		
	Exact	Discretization	De Vylder
2	0.3619122	0.3619378	0.3600330
4	0.4804148	0.4804380	0.4792420
6	0.5437340	0.5437558	0.5428670
8	0.5844607	0.5844818	0.5837587
10	0.6133853	0.6134060	0.6127853
20	0.6880854	0.6881055	0.6876941
40	0.7415004	0.7415212	0.7412163
∞	0.8143244	0.8143512	0.8141437

Table 4
Sum of Two Exponentials (u = 10)

t	$\psi(10, t)$		
	Exact	Discretization	De Vylder
2	0.0002544	0.0002547	0.0002764
4	0.0018053	0.0018072	0.0018631
6	0.0050869	0.0050916	0.0051656
8	0.0098598	0.0098679	0.0099435
10	0.0157082	0.0157198	0.0157868
20	0.0505434	0.0505710	0.0505637
40	0.1102909	0.1103369	0.1102490
∞	0.2821805	0.2822690	0.2821176

$$\psi(u) \simeq \psi_D(u) = \frac{1}{1 + \theta_D} \exp\left\{-\frac{\theta_D \beta_D}{1 + \theta_D} u\right\}. \tag{4.13}$$

Furthermore, Dickson, Hughes, and Zhang (2005) show that the pdf of the time of ruin is

$$\frac{\partial}{\partial t} \psi(u, t) = \lambda_D e^{-\beta_D u - (\lambda_D + c_D \beta_D)t} \left\{ I_0\left(\sqrt{4\beta_D \lambda_D t(c_D t + u)}\right) - \frac{c_D t}{c_D t + u} I_2\left(\sqrt{4\beta_D \lambda_D t(c_D t + u)}\right) \right\}, \tag{4.14}$$

where

$$I_v(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+v}}{n!(n + v)!}$$

is the modified Bessel function of order v (an earlier formula was given by Seal 1978). Therefore, finite-time ruin probabilities can be approximated by integrating the pdf (4.14) with respect to time t.

Some remarks are in order at this point. First, the ‘‘Exact’’ values for the first distribution are in agreement with those of Garcia (2005), where only values of $t \le 10$ were given. Second, the values in the second column labeled ‘‘Discretization’’ are very accurate approximations, with relative error decreasing slightly with t but increasing with u. De Vylder's approximation, while somewhat less accurate in general, exhibits similar behavior insofar as both t and u are concerned in most cases. Finally, the ordering of the infinite-time ruin probabilities by claim size distribution are consistent with the ‘‘danger’’ implied by the standard deviation, but the order of the finite-time ruin probabilities varies as a function of t.

5. Extensions of the Approach

In this section we briefly consider applications of the ideas to models involving gamma distributions that do not involve the simple Erlang structure. In particular, for aggregate claims and related analysis,

Table 5
Sum of Two Gammas (u = 1)

t	$\psi(1, t)$		
	Exact	Discretization	De Vylder
2	0.3490723	0.3491022	0.3462764
4	0.4700497	0.4700791	0.4675126
6	0.5357338	0.5357619	0.5334223
8	0.5783352	0.5783622	0.5761922
10	0.6087590	0.6087852	0.6067453
20	0.6880537	0.6880774	0.6864057
40	0.7456915	0.7457135	0.7443300
∞	0.8317360	0.8317594	0.8308223

Table 6
Sum of Two Gammas (u = 10)

t	$\psi(10, t)$ Exact	Discretization	De Vylder
2	0.0020146	0.0020156	0.0019054
4	0.0076404	0.0076439	0.0075048
6	0.0159734	0.0159801	0.0158644
8	0.0259205	0.0259306	0.0258558
10	0.0366772	0.0366906	0.0366586
20	0.0899268	0.0899530	0.0900604
40	0.1677368	0.1677757	0.1679500
∞	0.3838102	0.3838763	0.3839840

mathematically tractable formulas for the k-fold convolution of the claim size distribution are of interest. This is also of interest in connection with the evaluation of infinite time ruin probabilities via Shiu's expansion (Shiu 1988). This approach yields (e.g., Willmot 1988)

$$\psi(u) = 1 - \frac{\theta}{1+\theta} e^{\lambda u/c} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-\lambda/c)^k}{k!} r_k(u) \right\}, \qquad u > 0, \tag{5.1}$$

where λ is the Poisson claim rate, c the premium rate, θ the relative premium loading, and

$$r_k(u) = \int_0^u (u-y)^k e^{-\lambda y/c} \, dF^{*k}(y), \tag{5.2}$$

with $F^{*k}(y)$ the distribution function of the k-fold convolution of F. In the case with gamma claim amounts,

$$f(y) = \frac{\beta(\beta y)^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}, \qquad y > 0, \tag{5.3}$$

one has $r_k(u) = \sigma_k(u, k\alpha, \beta)$, where

$$\sigma_k(u, \alpha, \beta) = \frac{\beta^\alpha u^{\alpha+k} k!}{\Gamma(\alpha+k+1)} \frac{M(k+1, \alpha+k+1, u(\beta+\lambda/c))}{e^{u(\beta+\lambda/c)}}, \tag{5.4}$$

with the confluent hypergeometric function $M(a, b, z)$ given by

$$M(a, b, z) = 1 + \sum_{i=1}^{\infty} \left\{ \sum_{j=0}^{i-1} \frac{a+j}{b+j} \right\} \frac{z^i}{i!}. \tag{5.5}$$

We now consider generalizations of the gamma assumption in (5.3).

5.1 Sums of Gammas

Suppose that $f(y)$ has Laplace transform given by (2.22), but with $\Sigma_{i=1}^n \alpha_i$ not necessarily an integer. Then let m be the greatest integer less than or equal to $\Sigma_{i=1}^n \alpha_i$, and $\rho = \Sigma_{i=1}^n \alpha_i - m$. Then as in (2.23),

$$\tilde{f}(s) = \left(\frac{\beta_n}{\beta_n + s}\right)^\rho Q\left(\frac{\beta_n}{\beta_n + s}\right), \tag{5.6}$$

where $Q(z)$ is given by (2.23). Therefore, with $\{Q(z)\}^k = \sum_{j=1}^{\infty} q_j^{*k} z^j$, one has

$$\{\tilde{f}(s)\}^k = \sum_{j=1}^{\infty} q_j^{*k} \left(\frac{\beta_n}{\beta_n + s}\right)^{j+k\rho}, \tag{5.7}$$

and thus from (5.4),

$$r_k(u) = \sum_{j=1}^{\infty} q_j^{*k} \sigma_k(u, j + k\rho, \beta_n), \tag{5.8}$$

where σ_k is given by (5.4). □

5.2 Mixture of Two Gammas

For the mixture of two gammas with Laplace transform

$$\tilde{f}(s) = p\left(\frac{\beta_1}{\beta_1 + s}\right)^{\alpha_1} + (1 - p)\left(\frac{\beta_2}{\beta_2 + s}\right)^{\alpha_2}, \tag{5.9}$$

where $0 < \beta_1 \leq \beta_2 < \infty$, it follows from (2.1) that

$$\tilde{f}(s) = p\left(\frac{\beta_2}{\beta_2 + s}\right)^{\alpha_1} Q\left(\frac{\beta_2}{\beta_2 + s}\right) + (1 - p)\left(\frac{\beta_2}{\beta_2 + s}\right)^{\alpha_2}, \tag{5.10}$$

where

$$Q(z) = \left\{\frac{\beta_1/\beta_2}{1 - (1 - \beta_1/\beta_2)z}\right\}^{\alpha_1}, \tag{5.11}$$

a negative binomial pgf. Therefore, from (5.10),

$$\{\tilde{f}(s)\}^k = \sum_{i=0}^{k} \binom{k}{i} p^i (1 - p)^{k-i} \left(\frac{\beta_2}{\beta_2 + s}\right)^{i\alpha_1 + (k-i)\alpha_2} \left\{Q\left(\frac{\beta_2}{\beta_2 + s}\right)\right\}^i, \tag{5.12}$$

and using (5.11) and (2.16),

$$\{\tilde{f}(s)\}^k = \sum_{i=0}^{k} \binom{k}{i} p^i (1 - p)^{k-i} \sum_{j=0}^{\infty} h_j\left(i\alpha_1, \frac{\beta_1}{\beta_2}\right)\left(\frac{\beta_2}{\beta_2 + s}\right)^{j + i\alpha_1 + (k-i)\alpha_2}, \tag{5.13}$$

assuming further that $h_0(\alpha, \phi) = 1$ and $h_j(\alpha, \phi) = 0$ for $j \neq 0$ if either $\alpha = 0$ or $\phi = 1$. Thus, from (5.4),

$$r_k(u) = \sum_{i=0}^{k} \binom{k}{i} p^i (1 - p)^{k-i} \sum_{j=0}^{\infty} h_j\left(i\alpha_1, \frac{\beta_1}{\beta_2}\right) \sigma_k(u, j + i\alpha_1 + (k - i)\alpha_2, \beta_2). \tag{5.14}$$

Mixtures of more than two gammas may be done in a similar manner using a multinomial (rather than a binomial) expansion. The details are straightforward but tedious and thus are omitted. □

6. ACKNOWLEDGMENTS

The authors wish to thank Chi Kin Eric Cheung for helpful comments and suggestions. Support from the Munich Reinsurance Company is gratefully acknowledged, as is support from a grant from the Natural Sciences and Engineering Research Council of Canada.

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DISCUSSION

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The authors are to be congratulated for an interesting paper, in particular for the way they show that the algorithm derived in Dickson and Willmot (2005) can be used to compute exact values of finite time ruin probabilities for a wide range of individual claim amount distributions.

We are pleased that the authors chose to compare approximations calculated by the algorithm of Dickson and Waters (1991) with their exact values, and that these approximations performed well. In Tables 1 to 3, we have calculated approximations based on a scaling factor of 100 instead of the scaling

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Table 1
Approximate and Exact Values: Mixture of Two Exponentials

	$\psi(1,t)$		$\psi(10,t)$	
t	Approx.	Exact	Approx.	Exact
2	0.3111815	0.3111800	0.0086734	0.0086734
4	0.4338986	0.4338971	0.0224371	0.0224369
6	0.5034887	0.5034873	0.0383388	0.0383385
8	0.5495090	0.5495076	0.0548492	0.0548488
10	0.5827389	0.5827376	0.0711844	0.0711839
20	0.6706340	0.6706329	0.1422087	0.1422078
40	0.7358076	0.7358065	0.2347063	0.2347051

factor of 20 used by the authors. What these tables show is that for the values of u and t considered by the authors, the approximations are correct to five decimal places and hardly differ from the exact values in the sixth decimal place. The computing time involved to produce each table was just under one minute.

Whilst there is a tremendous satisfaction in obtaining an explicit solution to a problem, it seems to us that approximate calculation of finite time ruin probabilities in the classical risk model is every bit as satisfactory as an exact calculation. It is hard to imagine situations in which we would want to compute a ruin probability to a huge number of decimal places. The advantages offered by our algorithm are that it is accurate, it can be applied to any individual claim amount distribution, and computer programming is straightforward. Its main disadvantage is that it is a recursive algorithm, and so computer run time must increase as either u or t increases, although this can be offset by applying a truncation procedure. In contrast, the main advantage offered by an exact calculation is precisely that the calculation is exact. However, a degree of error is involved as solutions are expressed in terms of infinite sums which must be truncated to produce numerical solutions. A further issue with exact solutions is that although the solution is expressed in terms of simple functions (powers, exponentials, factorials) significant care has to be exercised in programming to avoid numerical underflow or overflow for large values of u and/or t.

We view this paper as an important addition to the actuarial literature. It has also had the effect of reinforcing our view that approximate calculation of finite time ruin probabilities by our algorithm is both an efficient and an accurate approach.

Table 2
Approximate and Exact Values: Sum of Two Exponentials

	$\psi(1,t)$		$\psi(10,t)$	
t	Approx.	Exact	Approx.	Exact
2	0.3619133	0.3619122	0.0002544	0.0002544
4	0.4804158	0.4804148	0.0018053	0.0018053
6	0.5437348	0.5437340	0.0050871	0.0050869
8	0.5844615	0.5844607	0.0098601	0.0098598
10	0.6133861	0.6133853	0.0157086	0.0157082
20	0.6880862	0.6880854	0.0505445	0.0505434
40	0.7415016	0.7415004	0.1102928	0.1102909

Table 3
Approximate and Exact Values: Sum of Two Gammas

	$\psi(1,t)$		$\psi(10,t)$	
t	Approx.	Exact	Approx.	Exact
2	0.3490735	0.3490723	0.0020146	0.0020146
4	0.4700509	0.4700497	0.0076405	0.0076404
6	0.5357349	0.5357338	0.0159737	0.0159734
8	0.5783363	0.5783352	0.0259209	0.0259205
10	0.6087601	0.6087590	0.0366778	0.0366772
20	0.6880546	0.6880537	0.0899278	0.0899268
40	0.7456923	0.7456915	0.1677384	0.1677368

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AUTHORS' REPLY

We wish to thank Professors Dickson and Waters for their interesting discussion of our paper. We agree that the numerical algorithm given by them in Dickson and Waters (1991) is quite accurate for the claim size distributions that we considered in the paper. We wish to make a few further remarks concerning the issues raised in their discussion.

It seems to us that the importance of finding exact solutions to problems of this nature should not be underestimated. This logic was also employed in the paper by Dickson and Willmot (2005) on finite time ruin probabilities. In particular, we find it interesting that the discussants concluded that their numerical algorithm is accurate by comparing their numbers to the exact values obtained by us. This clearly highlights the importance of the availability of exact solutions. Moreover, we are unaware of whether or not their numerical approximations are accurate for all choices of parameter values and all choices of claim size distributions. Apart from computational issues associated with exact results, we firmly believe that there is not and will not be a good substitute for the availability of exact computational procedures, which necessarily serve as a benchmark for any approximation. Of course, we fully agree that if the approximation is known to be accurate then it should certainly be used.

Turning next to computational issues, we should mention that in employing their algorithm we chose to use a scaling factor of 20 rather than a larger value such as 100 because that was the recommended value for sufficient accuracy that was given in their paper. Certainly, we could have increased the accuracy by an increase in the scaling factor as was done in the discussion, but our focus was on the exact procedure and not on the approximation. Also, we remark that although the approximation algorithm is not trivial and does involve a reasonable amount of computational resources, it is certainly simpler than the exact procedure.

We wonder if the importance of this latter issue is not over-estimated, however, as personal computers are readily available nowadays at virtually no extra cost for their usage. Moreover, this problem will almost certainly become less important in the future as computational power continues to increase.

The exact procedure that we employ to calculate finite time ruin probabilities does involve numerical issues, as does any method. We do agree that care must be taken when evaluating alternating series as well as infinite sums, but neither of these problems appear to be insurmountable in this situation. In fact, these types of problems are by no means uncommon and there are plenty of resources to deal with them (eg. Press et al, 1986). We do agree, however, that care needs to be taken when evaluating quantities needed for the exact procedure, but the computational error that is introduced may be controlled.

Finally, we remark that our focus in this paper was twofold. First, our aim was to show that many claim size distributions of interest are of mixed Erlang form (including some for which this is not immediately obvious). Second, we wished to show that many quantities of interest in risk theoretic calculations including aggregate claims distributions, stop-loss moments, infinite time ruin probabilities, and finite time ruin probabilities, may all be calculated in a straightforward manner by series expansions. In particular, the numerical root finding techniques often needed for partial fraction expansions based on the Laplace transform or for eigenvalues in phase-type analysis may be avoided. These models are thus useful in many situations, only one of which is the evaluation of finite time ruin probabilities.

Again, we thank the discussants for their insightful comments.

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Discussions on this paper can be submitted until October 1, 2007. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.