On the Gerber-Shiu Discounted Penalty Function for the Ordinary Renewal Risk Model with Constant Interest

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Abstract

In this paper we study the Gerber-Shiu discounted penalty function for the ordinary renewal risk model modified by the constant interest on the surplus. Explicit answers are expressed by an infinite series, and a relational formula for some important joint density functions is derived. Applications of the results to the compound Poisson model are given. Finally, a lower bound and an upper bound for the ultimate ruin probability are derived.

1. Introduction

We consider an ordinary renewal risk model. The number of claims process $\{N(t); t \geq 0\}$ is assumed to be an ordinary renewal process, where the interclaim times are represented by the independent and identically distributed sequence $\{T_n, n \geq 1\}$ of positive random variables. Here $T_n = S_n - S_{n-1}$, $n \geq 1$ ($S_0 = 0$, which means that there is a claim at time 0), that is, S_n is the $(n + 1)$–th claim occurring time. Let T_1 have a distribution function $K(t)$, a density function $k(t)$, and a finite mean $\mathrm{E}(T_1)$. Let $\bar{K}(t) = 1 - K(t)$.

The individual claim amounts $\{Z_j\}$ are assumed to be i.i.d. positive random variables with common distribution $P(z)$, $P(0) = 0$. For simplicity we assume that $P(z)$ is differentiable, with $P'(z) = p(z)$ being the individual claim amount probability density function, and the mean $\mathrm{E}(Z_1) = \mu < \infty$. The aggregate claim amount up to time t is $X(t) = \sum_{k=1}^{N(t)} Z_k$, $t \geq 0$.

Assume that the insurer receives interest on its surplus at a constant force δ per unit time. For $\delta \geq 0$, let $U_\delta(t)$ denote the surplus at time $t \geq 0$,

$$U_\delta(t) = ue^{\delta t} + c\bar{s}_{\overline{t}|}^{(\delta)} - \int_0^t e^{\delta(t-v)}\, dX(v), \tag{1.1}$$

where u is the surplus after paying the claim at time $S_0 = 0$, c is the positive constant premium income rate, and $\bar{s}_{\overline{t}|}^{(\delta)} = \int_0^t e^{\delta s}\, ds$. See Willmot and Lin (2001, Chapter 2) and Cai and Dickson (2002, 2003) and references therein for details on this well-known model.

The time of ruin is $T_\delta = \inf\{t \geq 0 | U_\delta(t) < 0\}$, where $T_\delta = \infty$ if $U_\delta(t) \geq 0$ for all $t \geq 0$. If ruin occurs, the deficit at ruin is $-U_\delta(T_\delta)$, and the surplus immediately prior to ruin is $U_\delta(T_\delta-)$. The probability of ultimate ruin is

$$\Psi_\delta(u) = \Pr\{T_\delta < \infty \mid U_\delta(0) = u\}. \tag{1.2}$$

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The following notation applies throughout this paper:

$$F_\delta(\alpha, u, x, y) = \mathrm{E}[e^{-\alpha T_\delta} I(U_\delta(T_\delta-) \le x,\ U_\delta(T_\delta) \le y) I(T_\delta < \infty) | U_\delta(0) = u],$$

$$f_\delta(\alpha, u, x, y) = \frac{\partial^2 F_\delta(\alpha, u, x, y)}{\partial x \partial y},$$

where $x > 0$, $y < 0$, and I is the indicator function, that is, $I(A) = 1$ if A is true and $I(A) = 0$ if A is false.

We consider the Gerber-Shiu discounted penalty function of the surplus immediately prior to ruin and the deficit at ruin when ruin occurs as a function of initial surplus u, namely,

$$\Phi_{\delta,\alpha}(u) = \mathrm{E}[e^{-\alpha T_\delta} w(U_\delta(T_\delta-), |U_\delta(T_\delta)|) I(T_\delta < \infty) | U_\delta(0) = u], \tag{1.3}$$

where $w(x, y)$, $0 \le x, y < \infty$, is a nonnegative function and α is a nonnegative parameter. We can interpret $\exp\{-\alpha T_\delta\}$ as the "discount factor."

The function $\Phi_{\delta,\alpha}(u)$ provides a unified means of studying the joint distribution of the surplus immediately prior to ruin and the amount of the surplus at ruin. Note that choosing different forms of the penalty function $w(x, y)$ in equation (1.3) gives rise to different information relating to the amount of the surplus at ruin and the surplus immediately prior to ruin. For example, $\Phi_{\delta,0}(u)$ will represent the k-th order moment of the surplus immediately prior to ruin (or the amount of the surplus at ruin) if we specially choose $w(x, y) = x^k$ (or y^k), represent their joint distribution function if $w(x, y) = I(x \le x_1, y \le y_1)$, represent the distribution function of the amount of the surplus at ruin if $w(x, y) = I(y \le y_1)$, and so on. When $w(x, y) \equiv 1$, the function $\Phi_{\delta,0}(u)$ coincides with the well-known ruin probability $\Psi_\delta(u)$ independently defined by (1.2). The financial explanations of $w(x, y)$ can be found in Gerber and Shiu (1998). The distributions of the quantities in the classical surplus process with constant interest have been studied by many authors including Sundt and Teugels (1997), Yang and Zhang (2001), Cai (2003, 2004), Cardoso and Waters (2003), Konstantinides, Tang, and Tsitsiashvili (2003), Tang and Tsitsiashvili (2004), Wu, Wang, and Zhang (2005), and others.

This paper is organized as follows. In Section 2 we derive an infinite series expression and some recursion formulas for the Gerber-Shiu discounted penalty function and some examples. In Section 3 we derive a lower bound and an exponential-type upper bound for the ultimate ruin probability.

2. THE GERBER-SHIU DISCOUNTED PENALTY FUNCTION

Let

$$g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_n, y_n) = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{(\delta y_{i-1} + c)^{\alpha/\delta}}{(\delta x_i + c)^{1+(\alpha/\delta)}}\, k\left(\frac{1}{\delta} \ln \frac{\delta x_i + c}{\delta y_{i-1} + c}\right) p(x_i - y_i), & \text{if } \delta > 0, \\ \displaystyle\frac{1}{c^n} \prod_{i=1}^{n} e^{-(\alpha/c)(x_i - y_{i-1})}\, k\left(\frac{x_i - y_{i-1}}{c}\right) p(x_i - y_i), & \text{if } \delta = 0, \end{cases} \tag{2.1}$$

with $y_0 = u$.

Lemma 2.1

For $n \ge 1$, let $x_1, \ldots, x_n, y_1, \ldots, y_{n-1} > 0$, $y_n < 0$, such that $x_1 > u \ge 0$ and $x_i > y_i$, $i = 1, 2, \ldots, n$. Assume that $\alpha, \delta \ge 0$, then we have

$$\mathrm{E}[e^{-\alpha S_n} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n, U_\delta(S_n) \in dy_n) | U_\delta(0) = u]$$
$$= g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_n, y_n) dx_1 dy_1 \ldots dx_n dy_n,$$

where $U_\delta(S_n-)$ is the surplus immediately prior to the $(n + 1)$–th claim time S_n.

The proof of this lemma is somewhat lengthy. We therefore omit it and refer the reader to the Appendix.

Let

$$F_{\delta,n}(\alpha, u, x, y) = \mathrm{E}[e^{-\alpha T_\delta} I(U_\delta(T_\delta-) \leq x, U_\delta(T_\delta) \leq y, T_\delta = S_n)|U_\delta(0) = u], \tag{2.2}$$

$$f_{\delta,n}(\alpha, u, x, y) = \frac{\partial^2 F_{\delta,n}(\alpha, u, x, y)}{\partial x \partial y}. \tag{2.3}$$

Note that $f_{\delta,n}(\alpha, u, x, y)$ is the density function of $F_{\delta,n}(\alpha, u, x, y)$.

Lemma 2.2

For each $\alpha, u, \delta \geq 0, x > 0, y < 0$, we have

$$F_{\delta,1}(\alpha, u, x, y) = I(x \geq u) \int_u^x dx_1 \int_{-\infty}^y g_{\delta,1}(\alpha, u, x_1, y_1)\, dy_1, \tag{2.4}$$

and for $n > 1$,

$$F_{\delta,n}(\alpha, u, x, y)$$
$$= \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-2}}^\infty dx_{n-1} \int_0^{x_{n-1}} dy_{n-1} \int_{y_{n-1}}^x dx_n \int_{-\infty}^y g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_n, y_n)\, dy_n. \tag{2.5}$$

PROOF

First, by (1.1), we have

$$(U_\delta(0) = u) \cap (T_\delta = T_1) = (U_\delta(T_1-) \geq u, U_\delta(T_1) < 0, U_\delta(0) = u).$$

Then by Lemma 2.1, we get

$$\begin{aligned} F_{\delta,1}(\alpha, u, x, y) &= \mathrm{E}[e^{-\alpha T_1} I(u \leq U_\delta(T_1 -) \leq x, U_\delta(T_1) \leq y)\, I(x \geq u)|U_\delta(0) = u] \\ &= I(x \geq u) \int_u^x dx_1 \int_{-\infty}^y g_{\delta,1}(\alpha, u, x_1, y_1)\, dy_1. \end{aligned} \tag{2.6}$$

When $n \geq 2$,

$$(U_\delta(0) = u) \cap (T_\delta = S_n)$$
$$= (0 < U_\delta(S_1-), 0 < U_\delta(S_1), \ldots, 0 < U_\delta(S_{n-1}), 0 < U_\delta(S_n-), U_\delta(S_n) < 0) \cap (U_\delta(0) = u).$$

In addition, when $T_\delta = S_n$,

$$U_\delta(S_k-) > U_\delta(S_{k-1}), U_\delta(S_k-) > U_\delta(S_k), k = 1, 2, \ldots, n.$$

Hence, by Lemma 2.1, we can easily obtain the expression for $F_{\delta,n}(\alpha, u, x, y)$. This ends the proof. □

Lemma 2.3

Let $n \geq 2$. Then,

(1) For $\delta > 0$,

$$F_{\delta,n}(\alpha, u, x, y) = \int_u^\infty dx_1 \int_0^{x_1} \left(\frac{\delta u + c}{\delta x_1 + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta} \ln \frac{\delta x_1 + c}{\delta u + c}\right) \frac{p(x_1 - y_1)}{\delta x_1 + c} F_{\delta,n-1}(\alpha, y_1, x, y)\, dy_1.$$

(2) For $\delta = 0$,

$$F_{0,n}(\alpha, u, x, y) = \int_u^\infty dx_1 \int_0^{x_1} \frac{1}{c} e^{-(\alpha/c)(x_1 - u)} k\left(\frac{x_1 - u}{c}\right) p(x_1 - y_1) F_{0,n-1}(\alpha, y_1, x, y)\, dy_1.$$

PROOF

Applying Lemma 2.1 and Lemma 2.2 to equations (2.2) and (2.5), we can obtain the results.

REMARK 2.1

In fact, by Lemma 2.3 we derive recursion formulas for $F_{\delta,n}(\alpha, u, x, y)$ and the density function $f_{\delta,n}(\alpha, u, x, y)$.

Let $n \geq 2$. Then,

(1) For $\delta > 0$,

$$f_{\delta,n}(\alpha, u, x, y) = \int_u^\infty dx_1 \int_0^{x_1} \left(\frac{\delta u + c}{\delta x_1 + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta} \ln \frac{\delta x_1 + c}{\delta u + c}\right) \frac{p(x_1 - y_1)}{\delta x_1 + c} f_{\delta,n-1}(\alpha, y_1, x, y)\, dy_1. \tag{2.7}$$

(2) For $\delta = 0$,

$$f_{0,n}(\alpha, u, x, y) = \int_u^\infty dx_1 \int_0^{x_1} \frac{1}{c} e^{-(\alpha/c)(x_1-u)} k\left(\frac{x_1 - u}{c}\right) p(x_1 - y_1) f_{0,n-1}(\alpha, y_1, x, y)\, dy_1. \tag{2.8}$$

REMARK 2.2

For $\alpha, \delta, u \geq 0, x > 0, y < 0, f_{\delta,1}(\alpha, u, x, y) = I(x \geq u) \cdot g_{\delta,1}(\alpha, u, x, y)$.

Proposition 2.1

For each $\alpha, \delta, u \geq 0, x > 0, y < 0$, we have

$$\begin{aligned} F_\delta(\alpha, u, x, y) &= I(x \geq u) \int_u^x dx_1 \int_{-\infty}^y g_{\delta,1}(\alpha, u, x_1, y_1)\, dy_1 \\ &\quad + \sum_{n=2}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-1}}^x dx_n \int_{-\infty}^y g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_n, y_n)\, dy_n, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} f_\delta(\alpha, u, x, y) &= I(x \geq u) g_{\delta,1}(\alpha, u, x, y) \\ &\quad + \sum_{n=2}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \int_{y_{n-2}}^\infty dx_{n-1} \int_0^{x_{n-1}} g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y)\, dy_{n-1}. \end{aligned} \tag{2.10}$$

PROOF

Note that

$$F_\delta(\alpha, u, x, y) = \sum_{n=1}^\infty F_{\delta,n}(\alpha, u, x, y). \tag{2.11}$$

By Lemma 2.2 and (2.11), we can get immediately that (2.9) holds true. From (2.3) and (2.11), we obtain that

$$F_\delta(\alpha, u, x, y) = \int_0^x \int_{-\infty}^y \left(\sum_{n=1}^\infty f_{\delta,n}(\alpha, u, x_1, y_1)\right) dx_1\, dy_1. \tag{2.12}$$

Hence,

$$f_\delta(\alpha, u, x, y) = I(x \geq u)\, g_{\delta,1}(\alpha, u, x, y) + \sum_{n=2}^\infty f_{\delta,n}(\alpha, u, x, y). \tag{2.13}$$

Furthermore, by (2.5) we have for $n \geq 2$,

$$f_{\delta,n}(\alpha, u, x, y) = \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-2}}^\infty dx_{n-1} \int_0^{x_{n-1}} g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y)\, dy_{n-1}. \tag{2.14}$$

Formula (2.10) follows from (2.14). This ends the proof. □

Corollary 2.1

For each $\alpha, \delta, u \geq 0, x > 0, y < 0$, $F_\delta(\alpha, u, x, y)$ satisfies the integral equation

$$F_\delta(\alpha, u, x, y) = I(x \geq u) \int_u^x dx_1 \int_{-\infty}^y g_{\delta,1}(\alpha, u, x_1, y_1)\, dy_1 + \int_u^\infty dx_1 \int_0^{x_1} g_{\delta,1}(\alpha, u, x_1, y_1) F_\delta(\alpha, y_1, x, y)\, dy_1. \tag{2.15}$$

PROOF

Apply Lemma 2.2 to (2.9). □

Furthermore, using Lemma 2.2, we can derive some important mathematical relations about many density functions.

Let $f_\delta(\alpha, u, x)$ denote the density function of $F_\delta(\alpha, u, x, 0)$, defined by

$$f_\delta(\alpha, u, x)\, dx = \mathrm{E}[e^{-\alpha T_\delta} I(U_\delta(T_\delta -) \in dx, T_\delta < \infty) | U_\delta(0) = u]. \tag{2.16}$$

We define

$$h_\delta(a, u, x, y)\, dx\, dy = \mathrm{E}[e^{-\alpha T_\delta} I(U_\delta(T_\delta -) \in dx, |U_\delta(T_\delta)| \in dy, T_\delta < \infty) | U_\delta(0) = u], \tag{2.17}$$

where $\alpha, \delta, u \geq 0, x > 0, y > 0$.

Corollary 2.2

For $\alpha, \delta, u \geq 0$, and $x, y > 0$, we have

$$h_\delta(\alpha, u, x, y) = f_\delta(\alpha, u, x) \frac{p(x + y)}{\bar{P}(x)}. \tag{2.18}$$

PROOF

Similar to (2.10), we can obtain easily the expression of $f_\delta(\alpha\ u, x)$, and for $\alpha, \delta, u \geq 0, x > 0, z < 0$, we get that

$$\frac{f_\delta(\alpha, u, x, z)}{f_\delta(\alpha, u, x)} = \frac{p(x - z)}{\bar{P}(x)}. \tag{2.19}$$

Let $y = -z$ in (2.19) and note that

$$h_\delta(\alpha, u, x, y) = f_\delta(\alpha, u, x, -z). \tag{2.20}$$

Then we get the result. □

REMARK 2.4

When $\alpha, \delta, u = 0$, this result coincides with formula (2.3) in Willmot (2006) and generalizes Theorem 4.3 in Cai and Dickson (2002).

Lemma 2.4

Let $\Phi_{\delta,\alpha,n}(u) = \mathrm{E}[e^{-\alpha T_\delta} w(U(T_\delta -), |U(T_\delta)|)\, I(T_\delta = S_n)|U_\delta(0) = u]$. Then we get for $\alpha, \delta, u \geq 0, n \geq 2$,

$$\Phi_{\delta,\alpha,n}(u) = \begin{cases} \int_u^\infty dx \int_0^x \left(\frac{\delta u + c}{\delta x + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta}\ln\frac{\delta x + c}{\delta u + c}\right)\frac{p(x-y)}{\delta x + c}\,\Phi_{\delta,\alpha,(n-1)}(y)\,dy, & \text{if } \delta > 0, \\ \int_u^\infty dx \int_0^x \frac{1}{c}\, e^{(-\alpha/c)(x-u)}\, k\left(\frac{x-u}{c}\right) p(x-y)\Phi_{\delta,\alpha,(n-1)}(y)\,dy, & \text{if } \delta = 0, \end{cases} \tag{2.21}$$

and

$$\Phi_{\delta,\alpha,1}(u) = \begin{cases} \int_u^\infty dx \int_{-\infty}^0 w(x, |y|)\left(\frac{\delta u + c}{\delta x + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta}\ln\frac{\delta x + c}{\delta u + c}\right)\frac{p(x-y)}{\delta x + c}\,dy, & \text{if } \delta > 0, \\ \int_u^\infty dx \int_{-\infty}^0 w(x, |y|)\,\frac{1}{c}\, e^{(-\alpha/c)(x-u)}\, k\left(\frac{x-u}{c}\right) p(x-y)\,dy, & \text{if } \delta = 0. \end{cases} \tag{2.22}$$

PROOF

Obviously, we have

$$\Phi_{\delta,\alpha,n}(u) = \int_0^\infty dx \int_{-\infty}^0 w(x, |y|) f_{\delta,n}(\alpha, u, x, y)\,dy, \qquad n \geq 1. \tag{2.23}$$

By (2.7) and (2.8), we get the following recursive calculation formula:

$$\Phi_{\delta,\alpha,n}(u) = \int_u^\infty dx \int_0^x g_{\delta,1}(\alpha, u, x, y)\Phi_{\delta,\alpha,(n-1)}(y)\,dy, \qquad n \geq 2. \tag{2.24}$$

By Remark 2.2, the expression for $\Phi_{\delta,\alpha,1}(u)$ can be obtained. □

Theorem 2.1

For $\alpha, u \geq 0$, we have

$$\Phi_{\delta,\alpha}(u) = \int_u^\infty dx \int_{-\infty}^0 w(x, |y|) g_{\delta,1}(\alpha, u, x, y)\,dy + \sum_{n=2}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-2}}^\infty dx_{n-1} \int_0^{x_{n-1}} dy_{n-1} \int_{y_{n-1}}^\infty dx \int_{-\infty}^0 w(x, |y|)\, g_{\delta,n}(\alpha, u, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y)\,dy. \tag{2.25}$$

PROOF

Note that

$$\Phi_{\delta,\alpha}(u) = \sum_{n=1}^\infty \Phi_{\delta,\alpha,n}(u).$$

By Lemma 2.1 and Lemma 2.4, using exactly similar arguments to Proposition 2.1, we get the exact expression for the Gerber-Shiu discounted penalty function. □

An important special case of the ordinary renewal process is the compound Poisson process. In this case the positive net profit condition is $cE(T_1) > \mu$.

Corollary 2.3

For each $\delta > 0$, $\alpha, u \geq 0$, and $K(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$,

$$\Phi_{\delta,\alpha}(u) = \frac{c\Phi_{\delta,\alpha}(0)}{c + \delta u} - \frac{\lambda}{c + \delta u}\int_0^u A(t)\, dt + \int_0^u k_{\delta,\alpha}(u, t)\Phi_{\delta,\alpha}(t)\, dt, \tag{2.26}$$

where $A(y) = \int_y^\infty w(y, x - y)p(x)\, dx$, $k_{\delta,\alpha}(u, t) = (\delta + \alpha + \lambda\bar{P}(u - t))/(c + \delta u)$.

Proof

By Theorem 2.1 and equation (2.7), we get

$$\begin{aligned}\Phi_{\delta,\alpha}(u) &= \int_u^\infty dx \int_{-\infty}^0 w(x, -y)g_{\delta,1}(\alpha, u, x, y)\, dy + \sum_{n=2}^\infty \int_0^\infty dx \int_{-\infty}^0 w(x, -y)f_{\delta,n}(\alpha, u, x, y)\, dy \\ &= J_1 + J_2,\end{aligned}$$

where

$$\begin{aligned}J_1 &= \int_u^\infty dx \int_{-\infty}^0 w(x, -y)\left(\frac{\delta u + c}{\delta x + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta}\ln\frac{\delta x + c}{\delta u + c}\right)\frac{1}{\delta x + c}\, p(x - y)\, dy \\ &= \int_u^\infty dx \int_{-\infty}^0 w(x, -y)\lambda(\delta u + c)^{(\lambda+\alpha)/\delta}(\delta x + c)^{-(\lambda+\alpha)/\delta - 1}p(x - y)\, dy \\ &= \lambda(\delta u + c^{(\lambda+\alpha)/\delta})\int_u^\infty (\delta x + c)^{-(\lambda+\alpha)/\delta-1}\left(\int_x^\infty w(x, z - x)p(z)\, dz\right) dx, \\ J_2 &= \int_0^{+\infty} dx \int_{-\infty}^0 w(x, -y)\sum_{n=2}^\infty \left[\int_u^\infty dx_1 \int_0^{x_1}\left(\frac{\delta u + c}{\delta x_1 + c}\right)^{\alpha/\delta}\right. \\ &\qquad \left.\lambda\left(\frac{\delta x_1 + c}{\delta u + c}\right)^{-(\lambda/\delta)}\frac{p(x_1 - y_1)}{\delta x_1 + c}f_{\delta,n-1}(\alpha, y_1, x, y)\, dy_1\right] \\ &= \int_u^\infty dx_1 \int_0^{x_1}\left(\int_0^{+\infty} dx \int_{-\infty}^0 w(x, -y)\sum_{n=1}^\infty f_{\delta,n}(\alpha, y_1, x, y)\, dy\right) \\ &\qquad \lambda(\delta u + c)^{(\lambda+\alpha)/\delta}(\delta x_1 + c)^{-(\lambda+\alpha)/\delta-1}p(x_1 - y_1)\, dy_1 \\ &= \lambda(\delta u + c)^{(\lambda+\alpha)/\delta}\int_u^\infty dx \int_0^x (\delta x + c)^{-(\lambda+\alpha)/\delta-1}\Phi_{\delta,\alpha}(y)p(x - y)\, dy.\end{aligned}$$

Hence,

$$\Phi_{\delta,\alpha}(u) = \lambda(\delta u + c)^{(\lambda+\alpha)/\delta}\int_u^\infty (\delta x + c)^{-(\lambda+\alpha)/\delta-1}\left(\int_0^x \Phi_{\delta,\alpha}(x - y)p(y)\, dy + A(x)\right) dx. \tag{2.27}$$

Using arguments similar to those of Cai and Dickson (2002) and Gerber and Shiu (1998), by differentiating (2.27) with respect to u and then integrating (see, e.g., Cai and Dickson 2002), we get

$$\Phi_{\delta,\alpha}(u) = \frac{c\Phi_{\delta,\alpha}(0)}{c + \delta u} - \frac{\lambda}{c + \delta u}\int_0^u A(t)\, dt + \int_0^u k_{\delta,\alpha}(u, t)\Phi_{\delta,\alpha}(t)\, dt.$$

□

Remark 2.5

Cai and Dickson (2002) have derived this integral equation and find the initial value $\Phi_{\delta,0}(0)$ by using Laplace transforms. But they did not determine the initial value $\Phi_{\delta,\alpha}(0)$ when $\delta > 0$ and $\alpha > 0$. They leave this as an open question. In this paper, using Theorem 2.1, when δ, α, $u > 0$, both $\Phi_{\delta,\alpha}(u)$ and $\Phi_{\delta,\alpha}(0)$ can be given in an infinite series.

REMARK 2.6

Equation (2.26) is a type of the Volterra integral equation

$$\varphi(x) = l(x) + \int_0^x k(x, s)\varphi(s)\, ds. \tag{2.28}$$

See, for example, Cai and Dickson (2003, p. 392 [2.7]) and Mikhlin (1957). If $l(x)$ is absolutely integrable and the kernel $k(x, s)$ is continuous, then for each $x > 0$, the unique solution for $\varphi(x)$ has the following representation:

$$\varphi(x) = l(x) + \int_0^x K(x, s)l(s)\, ds, \tag{2.29}$$

where $K(x, s) = \sum_{m=1}^{\infty} k_m(x, s)$, with $k_1(x, s) = k(x, s)$, and $k_m(x, s) = \int_s^x k(x, t)k_{m-1}(t, s)\, dt$, $m = 2, 3, \ldots$, $x > s \geq 0$.

COROLLARY 2.4

For δ, $u \geq 0$, and $K(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$,

$$\Psi_\delta(u) = \frac{c\Psi_\delta(0)}{c + \delta u} - \frac{\lambda}{c + \delta u}\int_0^u \bar{P}(t)\, dt + \int_0^u k_\delta(u, t)\Psi_\delta(t)\, dt, \tag{2.30}$$

where

$$k_\delta(u, t) = \frac{\delta + \lambda\bar{P}(u - t)}{c + \delta u}.$$

PROOF

It follows Corollary 2.3 with $\alpha = 0$ and $w(x, y) \equiv 1$. □

Example 2.1

We consider an important special case of the Sparre Andersen model, which is the compound Poisson risk model, namely, in which $K(t) = 1 - e^{-\lambda t}$, $t \geq 0$, $\lambda > 0$. Let Z_1 have an exponential distribution with $P(z) = 1 - e^{-\beta z}$, $z \geq 0$, $\beta > 0$. In this case an explicit formula for the ruin probability $\psi(u)$ and the distribution function of the deficit at ruin are available.

(1) The ruin probability $\psi(u)$, namely, with δ, $\alpha = 0$, $w(x, |y|) \equiv 1$, we have $\psi(u) = \Phi_{0,0}(u) = \sum_{n=1}^{\infty} \psi_n(u)$, where $\psi_n(u) = \Phi_{0,0,n}(u)$.

From Lemma 2.4,

$$\begin{aligned}
\psi_1(u) &= \int_u^\infty dx \int_{-\infty}^0 \frac{1}{c}\lambda e^{-\lambda((x-u)/c)}\beta e^{-\beta(x-y)}\, dy \\
&= \frac{\lambda}{\lambda + \beta c} e^{-\beta u}, \\
\psi_2(u) &= \int_u^\infty dx \int_0^x \frac{1}{c}\lambda e^{-\lambda((x-u)/c)}\beta e^{-\beta(x-y)}\psi_1(y)\, dy \\
&= \left(\frac{\lambda}{\lambda + \beta c}\right)^2 \beta \left(u + \frac{c}{\lambda + \beta c}\right) e^{-\beta u},
\end{aligned}$$

$$\psi_3(u) = \int_u^\infty dx \int_0^x \frac{1}{c} \lambda e^{-\lambda((x-u)/c)} \beta e^{-\beta(x-y)} \psi_2(y) \quad dy$$
$$= \left(\frac{\lambda}{\lambda + \beta c}\right)^3 \beta^2 \left(\frac{1}{2} u^2 + 2 \frac{c}{\lambda + \beta c} u + 2 \left(\frac{c}{\lambda + \beta c}\right)^2\right) e^{-\beta u}.$$

We assume that

$$\psi_n(u) = \left(\frac{\lambda}{\lambda + \beta c}\right)^n \beta^{n-1} P_{n-1}(u) e^{-\beta u}, \qquad n \geq 1,$$

where

$$P_{n-1}(u) = \sum_{k=1}^{n} a_{n-k}^{(n-1)} \left(\frac{c}{\lambda + \beta c}\right)^{k-1} u^{n-k}$$
$$= a_{n-1}^{(n-1)} u^{n-1} + a_{n-2}^{(n-1)} \frac{c}{\lambda + \beta c} u^{n-2} + a_{n-3}^{(n-1)} \left(\frac{c}{\lambda + \beta c}\right)^2 u^{n-3}$$
$$+ \cdots + a_1^{(n-1)} \left(\frac{c}{\lambda + \beta c}\right)^{n-2} u + a_0^{(n-1)} \left(\frac{c}{\lambda + \beta c}\right)^{n-1}, \qquad n \geq 1.$$

Then

$$\psi_{n+1}(u) = \int_u^\infty dx \int_0^x \frac{1}{c} \lambda e^{-\lambda((x-u)/c)} \beta e^{-\beta(x-y)} \psi_n(y)\, dy$$
$$= \left(\frac{\lambda}{\lambda + \beta c}\right)^n \beta^{n-1} \frac{\lambda\beta}{c} e^{\lambda u/c} \left[\frac{a_{n-1}^{(n-1)}}{n} \int_u^\infty e^{-((\lambda/c)+\beta)x} x^n \, dx\right.$$
$$+ \frac{c}{\lambda + \beta c} \frac{a_{n-2}^{(n-1)}}{n-1} \int_u^\infty e^{-((\lambda/c)+\beta)x} x^{n-1} \, dx + \cdots + \left(\frac{c}{\lambda + \beta c}\right)^{n-2} \frac{a_1^{(n-1)}}{2} \int_u^\infty e^{-((\lambda/c)+\beta)x} x^2 \, dx$$
$$\left. + \left(\frac{c}{\lambda + \beta c}\right)^{n-1} \frac{a_0^{(n-1)}}{1} \int_u^\infty e^{-((\lambda/c)+\beta)x} x \, dx\right].$$

By the formula

$$\int_u^\infty x^n e^{-\alpha x} \, dx = \left[\frac{1}{a} u^n + \frac{n}{a^2} u^{n-1} + \frac{n(n-1)}{a^3} u^{n-2} + \cdots + \frac{n(n-1)\cdots 2}{a^n} u + \frac{n!}{a^{n+1}}\right] e^{-au},$$

we obtain

$$\int_u^\infty e^{-(\lambda/c+\beta)x} x^n \, dx = \left[\frac{c}{\lambda + \beta c} u^n + \left(\frac{c}{\lambda + \beta c}\right)^2 n u^{n-1} + \cdots + \left(\frac{c}{\lambda + \beta c}\right)^{n+1} n!\right] e^{-(\lambda/c+\beta)u},$$
$$\int_u^\infty e^{-(\lambda/c+\beta)x} x^{n-1} \, dx = \left[\frac{c}{\lambda + \beta c} u^{n-1} + \left(\frac{c}{\lambda + \beta c}\right)^2 (n-1) u^{n-2}\right.$$
$$\left. + \cdots + \left(\frac{c}{\lambda + \beta c}\right)^n (n-1)!\right] e^{-(\lambda/c+\beta)u} \cdots$$
$$\int_u^\infty e^{-(\lambda/c+\beta)x}\, x \, dx = \left[\frac{c}{\lambda + \beta c} u + \left(\frac{c}{\lambda + \beta c}\right)^2\right] e^{-(\lambda/c+\beta)u}.$$

Therefore

$$\psi_{n+1}(u) = \left(\frac{\lambda}{\lambda + \beta c}\right)^{n+1} \beta^n \left[\frac{a_{n-1}^{(n-1)}}{n} u^n + \left(a_{n-1}^{(n-1)} + \frac{a_{n-2}^{(n-1)}}{n-1}\right) \frac{c}{\lambda + \beta c} u^{n-1} \right.$$
$$+ \left((n-1)a_{n-1}^{(n-1)} + a_{n-2}^{(n-1)} + \frac{a_{n-3}^{(n-1)}}{n-2}\right)\left(\frac{c}{\lambda + \beta c}\right)^2 u^{n-2}$$
$$+ \left((n-1)(n-2)a_{n-1}^{(n-1)} + (n-2)a_{n-2}^{(n-1)} + a_{n-3}^{(n-1)} + \frac{a_{n-4}^{(n-1)}}{n-3}\right)\left(\frac{c}{\lambda + \beta c}\right)^3 u^{n-3}$$
$$+ \cdots + \left((n-1)(n-2)\cdots 2a_{n-1}^{(n-1)} + \cdots + a_1^{(n-1)} + a_0^{(n-1)}\right)\left(\frac{c}{\lambda + \beta c}\right)^{n-1} u$$
$$\left. + ((n-1)!a_{n-1}^{(n-1)} + (n-2)!a_{n-2}^{(n-1)} + \cdots + a_1^{(n-1)} + a_0^{(n-1)}) \left(\frac{c}{\lambda + \beta c}\right)^n\right] e^{-\beta u}.$$

Note that we also have

$$\psi_{n+1}(u) = \left(\frac{\lambda}{\lambda + \beta c}\right)^{n+1} \beta^n P_n(u) e^{-\beta u},$$

where

$$P_n(u) = a_n^{(n)} u^n + a_{n-1}^{(n)} \frac{c}{\lambda + \beta c} u^{n-1} + a_{n-2}^{(n)} \left(\frac{c}{\lambda + \beta c}\right)^2 u^{n-2}$$
$$+ \cdots + a_1^{(n)} \left(\frac{c}{\lambda + \beta c}\right)^{n-1} u + a_0^{(n)} \left(\frac{c}{\lambda + \beta c}\right)^n.$$

Then we get the recursive relation formulas

$$a_n^{(n)} = \frac{a_{n-1}^{(n-1)}}{n}, \qquad a_{n-1}^{(n)} = a_{n-1}^{(n-1)} + \frac{a_{n-2}^{(n-1)}}{n-1},$$
$$a_{n-2}^{(n)} = (n-1)a_{n-1}^{(n-1)} + a_{n-2}^{(n-1)} + \frac{a_{n-3}^{(n-1)}}{n-2},$$
$$a_{n-3}^{(n)} = (n-1)(n-2)\, a_{n-1}^{(n-1)} + (n-2)\, a_{n-2}^{(n-1)} + a_{n-3}^{(n-1)} + \frac{a_{n-4}^{(n-1)}}{n-3},$$
$$a_1^{(n)} = (n-1)(n-2)\cdots 2\, a_{n-1}^{(n-1)} + (n-2)(n-3)\cdots 2\, a_{n-2}^{(n-1)} + \cdots + a_1^{(n-1)} + a_0^{(n-1)},$$
$$a_0^{(n)} = (n-1)!a_{n-1}^{(n-1)} + (n-2)!a_{n-2}^{(n-1)} + \cdots + a_1^{(n-1)} + a_0^{(n-1)}.$$

By solving these recursive formulas, we get the general forms

$$a_k^{(n)} = \frac{(k+1)(n+2)(n+3)\cdots(n+(n-k))}{(n-k)!k!} = \frac{(k+1)\prod_{i=2}^{n-k}(n+i)}{(n-k)!k!}, \qquad k = 0, 1, 2, \ldots, n-1,$$
$$a_n^{(n)} = \frac{1}{n!}.$$

Therefore, we deduce that

$$\psi(u) = \sum_{n=0}^{\infty} \left(\frac{\lambda}{\lambda + \beta c}\right)^{n+1} \beta^n P_n(u) e^{-\beta u},$$

where

$$P_0(u) = 1, \qquad P_1(u) = u + \frac{c}{\lambda + \beta c}, \qquad P_2(u) = \frac{1}{2}u^2 + 2\frac{cu}{\lambda + \beta c} + 2\left(\frac{c}{\lambda + \beta c}\right)^2,$$

and for $n \geq 3$,

$$P_n(u) = \frac{1}{n!}u^n + \frac{n}{(n-1)!}\frac{c}{\lambda + \beta c}u^{n-1} + \frac{(n-1)(n+2)}{2!(n-2)!}\left(\frac{c}{\lambda + \beta c}\right)^2 u^{n-2} + \cdots$$
$$+ \frac{2(n+2)(n+3)\cdots(2n-1)}{(n-1)!}\left(\frac{c}{\lambda + \beta c}\right)^{n-1} u$$
$$+ \frac{(n+2)(n+3)\cdots(2n)}{n!}\left(\frac{c}{\lambda + \beta c}\right)^n.$$

(2) The distribution function of the deficit at ruin, namely, with $\delta, \alpha = 0$, $w(U_0(T_0-), |U_0(T_0)|) = I(|U_0(T_0)| \leq z)$, then $F(z) = \sum_{n=0}^{\infty} F_{n+1}(z|u)$, where $F_n(z|u) \overset{\Delta}{=} \Phi_{0,0,n}(u) = \mathrm{E}[I(|U_0(T_0)| \leq z)I(T_0 = S_n)|U_\delta(0) = u]$, $n \geq 1$.

From Lemma 2.4, we get

$$F_1(z|u) = \int_u^\infty dx \int_{-\infty}^0 \frac{1}{c}\lambda e^{-\lambda(x-u)/c}\beta e^{-\beta(x-y)} I(|y| \leq z)\, dy$$
$$= \int_u^\infty dx \int_{-z}^0 \frac{1}{c}\lambda e^{-\lambda(x-u)/c}\beta e^{-\beta(x-y)}\, dy$$
$$= (1 - e^{-\beta z})\frac{\lambda}{\lambda + \beta c} e^{-\beta u}.$$

Following the steps shown in the previous paragraph, one can obtain that

$$F_{n+1}(z|u) = (1 - e^{-\beta z})\left(\frac{\lambda}{\lambda + \beta c}\right)^{n+1}\beta^n P_n(u) e^{-\beta u}.$$

Note that $F_n(z|u)/\phi_n(u) = 1 - e^{-\beta z} = P(z)$, for $n = 1, 2, \ldots$, so we have

$$F(z) = \sum_{n=0}^{\infty}(1 - e^{-\beta z})\left(\frac{\lambda}{\lambda + \beta c}\right)^{n+1}\beta^n P_n(u) e^{-\beta u},$$

where $P_n(u)$, $n \geq 0$ are the same as the ones in (1) of this example.

3. LOWER AND UPPER BOUNDS FOR ULTIMATE RUIN PROBABILITY $\Psi_\delta(u)$

Although in Section 2 we have the infinite series expression for the Gerber-Shiu discounted penalty function $\Phi_{\delta,\alpha}(u)$, it is rather complicated to calculate $\Psi_\delta(u)$. In this section we will derive an upper bound for the ultimate ruin probability $\Psi_\delta(u)$. The ideas and methods we use are motivated by Cai and Dickson (2002, 2003) and Cai (2004). We assume that the positive net profit condition holds in this section, namely, $c\mathrm{E}(T_1) > \mu$. $\mathrm{E}(e^{tZ_1})$ exists for $0 < t < \xi$, and $\lim_{t\to\xi}\mathrm{E}(e^{tZ_1}) = \infty$.

Note that $\Psi_\delta(u) = F_\delta(0, u, \infty, 0)$. Then using Theorem 2.1 and Corollary 2.1, for $\delta, u \geq 0$ we get

$$\Psi_\delta(u) = \int_u^\infty dx \int_{-\infty}^0 g_{\delta,1}(0, u, x, y)\, dy$$
$$+ \sum_{n=2}^{\infty}\int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-2}}^\infty dx_{n-1}\int_0^{x_{n-1}} dy_{n-1}\int_{y_{n-1}}^\infty dx \int_{-\infty}^0 g_{\delta,n}(0, u, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y)\, dy,$$
(3.1)

or

$$\Psi_\delta(u) = \int_u^\infty dx_1 \int_{-\infty}^0 g_{\delta,1}(0, u, x_1, y_1)\, dy_1 + \int_u^\infty dx_1 \int_0^{x_1} g_{\delta,1}(0, u, x_1, y_1)\Psi_\delta(y_1)\, dy_1. \tag{3.2}$$

Theorem 3.1

For $\delta,\ u \geq 0$

$$\Psi_\delta(u) \geq \frac{\int_u^\infty dx_1 \int_{-\infty}^0 g_{\delta,1}(0, u, x_1, y_1)\, dy_1}{1 - \int_u^\infty dx_1 \int_0^u g_{\delta,1}(0, u, x_1, y_1)\, dy_1}. \tag{3.3}$$

PROOF

Since $\Psi_\delta(u)$ is a decreasing function, we have

$$\int_u^\infty dx_1 \int_0^{x_1} g_{\delta,1}(0, u, x_1, y_1)\Psi_\delta(y_1)\, dy_1 \geq \int_u^\infty dx_1 \int_0^u g_{\delta,1}(0, u, x_1, y_1)\Psi_\delta(u)\, dy_1.$$

Then by (3.2), we get

$$\Psi_\delta(u) \geq \int_u^\infty dx_1 \int_{-\infty}^0 g_{\delta,1}(0, u, x_1, y_1)\, dy_1 + \Psi_\delta(u) \int_u^\infty dx_1 \int_0^u g_{\delta,1}(0, u, x_1, y_1)\, dy_1.$$

It implies that (3.3) holds true. □

Next, using the recursive equation (3.2), we derive an upper bound for $\Psi_\delta(u)$.

Lemma 3.1

There exists a unique positive number R, such that

$$\mathrm{E}[\exp\{-R(c\bar{s}_{\overline{T_1}|}^{(\delta)} - Z_1)\}] = M, \tag{3.4}$$

where

$$M = \min_{0 \leq r \leq \xi} \mathrm{E}[\exp\{-r(c\bar{s}_{\overline{T_1}|}^{(\delta)} - Z_1)\}] < 1.$$

PROOF

Let

$$f(r, \delta) = \mathrm{E}[\exp\{-r(c\bar{s}_{\overline{T_1}|}^{(\delta)} - Z_1)\}], \qquad \delta,\ r \geq 0.$$

Then $f(0, \delta) \equiv 1$ and $\lim_{r\to\xi} f(r, \delta) = \infty$. For each fixed δ, it is easy to prove that $(\partial^2 f(r, \delta)/\partial r^2) > 0$, that is, $f(r, \delta)$ is convex in r. It is also easy to see that for each fixed r, $f(r, \delta)$ is decreasing function in δ. By the positive net profit condition, the function $f(r, 0)$ must have a unique minimum at $r = R(0)$; therefore, for each fixed δ, $\delta > 0$, the function $f(r, \delta)$ has a unique minimum M at $r = R(\delta)$, and $M \leq 1$. For simplicity, let $R = R(\delta)$. Thus the result (3.4) follows. □

Theorem 3.2

For each $u \geq 0$, $\delta > 0$, we have

$$\Psi_\delta(u) \leq \frac{M\beta}{1 - M} e^{-Ru}, \tag{3.5}$$

where

$$\beta^{-1} = \inf_{x \geq 0} \frac{\int_x^\infty e^{Ry} p(y)\, dy}{e^{Rx} \bar{P}(x)}$$

and M and R are the symbols defined in Lemma 3.1.

Proof

By the definition of β, we get

$$\bar{P}(z) \leq \beta e^{-Rz} \int_z^\infty e^{Ry} p(y)\, dy \leq \beta \mathrm{E}[e^{RZ_1}] e^{-Rz}, \qquad z \geq 0. \tag{3.6}$$

Then equation (2.25) with $w(x, |y|) \equiv 1$ and $\alpha = 0$ becomes

$$\Psi_\delta(u) = \Phi_{\delta,0}(u) = \int_u^\infty dx \int_{-\infty}^0 g_{\delta,1}(0, u, x, y)\, dy$$
$$+ \sum_{n=2}^\infty \int_u^\infty dx_1 \int_0^{x_1} dy_1 \cdots \int_{y_{n-2}}^\infty dx_{n-1} \int_0^{x_{n-1}} dy_{n-1} \int_{y_{n-1}}^\infty dx \int_{-\infty}^0 g_{\delta,n}(0, u, x_1, y_1, \ldots, x_{n-1}, y_{n-1}, x, y)\, dy. \tag{3.7}$$

By Lemma 3.1 and (3.6),

$$\begin{aligned}
\int_u^\infty dx \int_{-\infty}^0 g_{\delta,1}(0, u, x, y)\, dy &= \int_u^\infty dx \int_{-\infty}^0 k\left(\frac{1}{\delta} \ln \frac{\delta x + c}{\delta u + c}\right) \frac{1}{\delta x + c}\, p(x - y)\, dy \\
&= \int_u^\infty k\left(\frac{1}{\delta} \ln \frac{\delta x + c}{\delta u + c}\right) \frac{1}{\delta x + c}\, \bar{P}(x)\, dx \\
&= \int_0^\infty k(z) \bar{P}\left(\frac{e^{\delta z}(\delta u + c) - c}{\delta}\right) dz \\
&\leq \beta \mathrm{E}[e^{RZ_1}] \int_0^\infty k(z) e^{-R\,(e^{\delta z}(\delta u + c) - c)/\delta}\, dz \\
&\leq \beta \mathrm{E}[e^{RZ_1}] e^{-Ru} \mathrm{E}[e^{-Rc/\delta}(e^{\delta T_1} - 1)] \\
&= M\beta e^{-Ru}.
\end{aligned}$$

Now we assume that for some integer $k = n \geq 1$,

$$\int_0^\infty \int_{-\infty}^0 f_{\delta,k}(0, u, x, y)\, dx\, dy \leq M^k \beta e^{-Ru}. \tag{3.8}$$

Then we will prove that the inequality (3.8) is true for $k = n + 1$.

In fact, by equations (2.7) and (3.7), we have

$$\int_0^\infty \int_{-\infty}^0 f_{\delta,n+1}(0, u, x, y)\, dx\, dy$$
$$= \int_u^\infty dx_1 \int_0^{x_1} k\left(\frac{1}{\delta}\ln\frac{\delta x_1 + c}{\delta u + c}\right)\frac{p(x_1 - y_1)}{\delta x_1 + c}\left[\int_0^\infty \int_{-\infty}^0 f_{\delta,n}(0, y_1, x, y)\, dx\, dy\right] dy_1$$
$$\leq M^n\beta \int_u^\infty dx_1 \int_0^{x_1} k\left(\frac{1}{\delta}\ln\frac{\delta x_1 + c}{\delta u + c}\right)\frac{p(x_1 - y_1)}{\delta x_1 + c}\, e^{-Ry_1}\, dy_1$$
$$= M^n\beta \int_0^\infty \left[k(z)e^{-(R/\delta)(e^{\delta z}(\delta u + c) - c)} \int_0^{(e^{\delta z}(u+(c/\delta)) - (c/\delta))} p(y)e^{Ry}\, dy\right] dz$$
$$\leq M^n\beta e^{-Ru} \int_0^\infty \left[k(z)e^{-R(c/\delta)(e^{\delta z} - 1)} \int_0^{(e^{\delta z}(u+(c/\delta)) - (c/\delta))} p(y)e^{Ry}\, dy\right] dz$$
$$\leq M^{n+1}\beta e^{-Ru}.$$

Hence equation (3.8) holds for each $n \geq 1$. Thus, inequality (3.5) follows. □

Corollary 3.1

If $M \geq 1/2$ (where M is the symbol defined in Lemma 3.1), then we have

$$\Psi_\delta(u) \leq e^{-Ru}, \qquad u \geq 0. \tag{3.9}$$

PROOF

By the definition of β, we have $\beta < 1$. When $M \geq 1/2$, we get $M/(1 - M) < 1$.

Hence, $M\beta/(1 - M) < 1$, and the result (3.9) follows from (3.5). □

APPENDIX

The goal of this appendix is to prove Lemma 2.1.

We first consider the case with $\delta > 0$. Let

$$J = \mathrm{E}[e^{-\alpha S_n} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n, U_\delta(S_n) \in dy_n)|U_\delta(0) = u].$$

Since $U_\delta(S_n) = U_\delta(S_n-) - Z_n$, and the independence of Z_n and $U_\delta(S_n-)$, we have

$$J = \mathrm{E}[e^{-\alpha S_n} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n, U_\delta(S_n) \in dy_n)|U_\delta(0) = u]$$
$$= \mathrm{E}[\mathrm{E}\{e^{-\alpha S_n} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n, U_\delta(S_n) \in dy_n)$$
$$|S_1, \ldots, S_n, Z_1, \ldots, Z_{n-1}\}|U_\delta(0) = u]$$
$$= J_1 \cdot p(x_n - y_n)\, dy_n,$$

where

$$J_1 = \mathrm{E}[e^{-\alpha S_n} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n)|U_\delta(0) = u],$$

and

$$U_\delta(S_n-) = e^{\delta T_n}U_\delta(S_{n-1}) + \frac{c}{\delta}(e^{\delta T_n} - 1).$$

Therefore,

$$\begin{aligned} J_1 &= \mathrm{E}[\mathrm{E}\{e^{-\alpha(S_{n-1}+T_n)} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_n-) \in dx_n) \\ &\quad |S_1, \ldots, S_{n-1}, Z_1, \ldots, Z_{n-1}\}|U_\delta(0) = u] \\ &= \mathrm{E}[e^{-\alpha S_{n-1}} I(U_\delta(S_1-) \in dx_1, U_\delta(S_1) \in dy_1, \ldots, U_\delta(S_{n-1}) \in dy_{n-1})|U_\delta(0) = u] \\ &\quad \mathrm{E}\left[e^{-\alpha T_n} I\left(\left(y_{n-1}e^{\delta T_n} + \frac{c}{\delta}(e^{\delta T_n} - 1)\right) \in dx_n\right) |U_\delta(0) = u\right], \end{aligned}$$

where

$$\begin{aligned} &\mathrm{E}\left[e^{-\alpha T_n} I\left(\left(y_{n-1}e^{\delta T_n} + \frac{c}{\delta}(e^{\delta T_n} - 1)\right) \in dx_n\right) |U_\delta(0) = u\right] \\ &= \mathrm{E}\left[(e^{\delta T_n})^{-\alpha/\delta} I\left(e^{\delta T_n} \in \frac{\frac{c}{\delta} + dx_n}{y_{n-1} + \frac{c}{\delta}}\right) |U_\delta(0) = u\right] \\ &= \mathrm{E}\left[\left(\frac{\delta x_n + c}{\delta y_{n-1} + c}\right)^{-\alpha/\delta} I\left(T_1 \in \frac{1}{\delta}\ln \frac{\frac{c}{\delta} + dx_n}{y_{n-1} + \frac{c}{\delta}}\right) |U_\delta(0) = u\right] \\ &= \left(\frac{\delta y_{n-1} + c}{\delta x_n + c}\right)^{\alpha/\delta} k\left(\frac{1}{\delta}\ln \frac{\delta x_n + c}{\delta y_{n-1} + c}\right) \frac{dx_n}{\delta x_n + c}. \end{aligned} \tag{A.1}$$

Substituting (A.1) into J_1, then substituting J_1 into J, and using the inductive method, we get (2.1) in the case with $\delta > 0$.

Note that for $\delta = 0$,

$$U_0(S_n) = U_0(S_n-) - Z_n, \qquad U_0(S_n-) = U_0(S_{n-1}) + cT_n.$$

Following the steps of the proof shown in the previous paragraph, one can verify that the equality (2.1) holds true with $\delta = 0$. This completes the proof of Lemma 2.1.

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REFERENCES

CAI, JUN. 2003. Ruin Probability under Stochastic Rates of Interest. *Insurance: Mathematics and Economics* 32(3): 482–82.

———. 2004. Ruin Probabilities and Penalty Functions with Stochastic Rates of Interest. *Stochastic Processes and Their Application* 112(1): 53–78.

CAI, JUN, AND DAVID C. M. DICKSON. 2002. On the Expected Discounted Penalty Function at Ruin of a Surplus Process with Interest. *Insurance: Mathematics and Economics* 30(3): 389–404.

———. 2003. Upper Bounds for Ultimate Ruin Probabilities in the Sparre Andersen Model with Interest. *Insurance: Mathematics and Economics* 32(1): 61–71.

CARDOSO, RUI M. R., AND HOWARD R. WATERS. 2003. Recursive Calculation of Finite Time Ruin Probabilities under Interest Force. *Insurance: Mathematics and Economics* 33(3): 659–76.

GERBER, HANS U., AND ELIAS S. W. SHIU. 1998. On the Time Value of Ruin. *North American Actuarial Journal* 2(1): 48–78.

KONSTANTINIDES, DIMITRIOS G., QIHE TANG, AND GURAMI TSITSIASHVILI. 2003. Estimates for Ruin Probability in the Classical Risk Model with Constant Interest Force in the Presence of Heavy Tails. *Insurance: Mathematics and Economics* 32(1): 158–158.

MIKHLIN, SOLOMON G. 1957. *Integral Equations*. London: Pergamon Press.

SUNDT, BJØRN, AND JOZEF L. TEUGELS. 1997. The Adjustment Function in Ruin Estimates under Interest Force. *Insurance: Mathematics and Economics* 19(2): 85–94.

TANG, QIHE, AND GURAMI TSITSIASHVILI. 2004. Finite and Infinite-Time Ruin in the Presence of Stochastic Returns on Investments. *Advances in Applied Probability* 36(4): 1278–99.

WILLMOT, GORDON E. 2006. On the Discounted Penalty Function in the Renewal Risk Model with General Interclaim Times. *Insurance: Mathematics and Economics*, in press.

WILLMOT, GORDON E., AND X. SHELDON LIN. 2001. *Lundberg Approximations for Compound Distributions with Insurance Applications*. New York: Springer.

WU, RONG, GUOJING WANG, AND CHUNSHENG ZHANG. 2005. On a Joint Distribution for the Risk Process with Constant Interest Force. *Insurance: Mathematics and Economics* 36(3): 365–74.

YANG, HAILIANG, AND LIHONG ZHANG. 2001. On the Distribution of Surplus Immediately after Ruin under Interest Force. *Insurance: Mathematics and Economics* 29(1): 247–55.

DISCUSSIONS

BANGWON KO*

Professors Wu and Lu and Mr. Fang have obtained many interesting results, as they extend the Gerber and Shiu model by allowing the surplus to earn interest. In this discussion I would like to point out that Corollary 2.2 can be obtained directly, as in Dufresne and Gerber (1988). This is based on the conditional probability formula

$$\Pr(A \cap B) = \Pr(A)\Pr(B|A). \tag{D.1}$$

Here the event A is considered as the occurrence of ruin between time t and time $t + dt$, with the surplus immediately before ruin, $U_\delta(T_\delta-)$, being between x and $x + dx$, given that the initial surplus is u (u is the surplus after the claim at time 0 is paid). The event B is considered as the deficit at ruin, $|U_\delta(T_\delta)|$, being between y and $y + dy$. Now, conditioning on A, the event B means that the claim causing ruin is of size $(x + y)$. However, the claim size random variable does not depend on ruin time or the surplus immediately before ruin. Thus,

$$\Pr(B|A) = \frac{p(x + y)}{1 - P(x)}\, dy.$$

To be more precise, let us define by $f(x, y, t|u)$ the joint probability density function of $U_\delta(T_\delta-)$, $|U_\delta(T_\delta)|$ and T_δ, given that the initial surplus is u. Then the argument above is expressed as

$$f(x, y, t|u) = \left[\int_0^\infty f(x, z, t|u)\, dz\right] \frac{p(x + y)}{1 - P(x)}. \tag{D.2}$$

Now, Corollary 2.2 of the paper follows from multiplying both sides by $e^{-\alpha t}$, integrating with respect to t, and observing that

$$h_\delta(\alpha, u, x, y) = \int_0^\infty e^{-\alpha t} f(x, y, t|u)\, dt$$

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and

$$f_\delta(\alpha, u, x) = \int_0^\infty \int_0^\infty e^{-\alpha t} f(x, y, t|u)\, dy\, dt.$$

Formula (D.2) has been cited by Dufresne and Gerber (1988) for the classical ruin model with $\delta = 0$. Note from the proof that formula (D.2) actually holds for models more general than Gerber and Shiu's. For example, the interest rate need not be constant.

REFERENCE

DUFRESNE, FRANÇOIS, AND HANS U. GERBER. 1988. The Surpluses Immediately before and at Ruin, and the Amount of the Claim Causing Ruin. *Insurance: Mathematics and Economics* 7: 193–99.

AUTHORS' REPLY

Professor Ko obtained the following formula directly, by the conditional probability formula,

$$h_\delta(\alpha, u, x, y) = f_\delta(\alpha, u, x)\, \frac{p(x + y)}{\bar{P}(x)}.$$

We are very grateful for the discussion, which enhances the value of our paper. It illustrates that formula (3) in Dufresne and Gerber (1988) holds for models more general than the classical compound Poisson model.

REFERENCE

DUFRESNE, FRANÇOIS, AND HANS U. GERBER. 1988. The Surpluses Immediately before and at Ruin, and the Amount of the Claim Causing Ruin. *Insurance: Mathematics and Economics* 7: 193–99.

Additional discussions on this paper can be submitted until October 1, 2007. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.