The Expected Discounted Penalty at Ruin for a Markov-Modulated Risk Process Perturbed by Diffusion

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Abstract

A Markov-modulated risk process perturbed by diffusion is considered in this paper. In the model the frequencies and distributions of the claims and the variances of the Wiener process are influenced by an external Markovian environment process with a finite number of states. This model is motivated by the flexibility in modeling the claim arrival process, allowing that periods with very frequent arrivals and ones with very few arrivals may alternate. Given the initial surplus and the initial environment state, systems of integro-differential equations for the expected discounted penalty functions at ruin caused by a claim and oscillation are established, respectively; a generalized Lundberg's equation is also obtained. In the two-state model, the expected discounted penalty functions at ruin due to a claim and oscillation are derived when both claim amount distributions are from the rational family. As an illustration, the explicit results are obtained for the ruin probability when claim sizes are exponentially distributed. A numerical example also is given for the case that two classes of claims are Erlang(2) distributed and of a mixture of two exponentials.

1. Introduction

A classical continuous-time surplus process perturbed by diffusion assumes a constant intensity rate of claim arrivals, a certain claim amount distribution, and a constant variance parameter of the Wiener process over the time period. Here we consider an external environment process $\{I(t);\, t \geq 0\}$ that influences the frequencies and the distributions of the claims, and the variance parameters of a continuous-time surplus process perturbed by diffusion. The states of $\{I(t);\, t \geq 0\}$ could describe, for example, the El Niño/La Niña phenomena in property insurance, or economic conditions in unemployment insurance. Asmussen (1989) also pointed out that sojourns of $\{I(t);\, t \geq 0\}$ could be certain types of epidemics in health insurance, or these could be weather types in automobile insurance. This particular type of generalization is motivated partly by the flexibility on the modeling of the arrival process, allowing one to model arrival streams that are more irregular than any renewal process, and partly by the interpretation in the sense that an underlying external environment may involve the insurance business (see Asmussen et al. 1995).

Suppose that $\{I(t);\, t \geq 0\}$ is a homogeneous, irreducible, and recurrent Markov process with finite state space $J = \{1, \ldots, m\}$. Let $N(t)$ denote the number of claims that have occurred before time t. When the Markov process $\{I(t);\, t \geq 0\}$ remains in state $i \in J$ in a small time interval $(s, s + h]$, the number of claims occurring at this interval $(s, s + h]$, $N(s + h) - N(s)$, is assumed to follow a Poisson distribution with a parameter $\lambda_i \in \mathbb{R}^+$, and the corresponding claim sizes have a distribution function

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$F_i(x)$ with density function $f_i(x)$ and finite mean μ_i. In this case $\{N(t);\ t \geq 0\}$ is called a Markov-modulated Poisson process, which is a special case of Cox processes. We further assume that premiums are received continuously at a positive constant rate c. The corresponding continuous-time surplus process $\{U(t);\ t \geq 0\}$ perturbed by diffusion is defined as

$$U(t) = u + ct - S(t) + B(t), \qquad t \geq 0. \tag{1.1}$$

In this equation $u(\geq 0)$ is the initial reserve, $S(t) = \sum_{n=1}^{N(t)} X_n$ (with $S(t) = 0$ if $N(t) = 0$) is the aggregate claim amount under the Markov-modulated Poisson process $\{N(t);\ t \geq 0\}$ where X_n is the amount of the nth claim that is independent of $N(t)$ given the state of the Markov chain at the nth claim time, and $B(t)$ is a normally distributed random variable with mean 0 and variance $D(t)$ for any $t > 0$ where

$$D(t) = \sum_{k=1}^{N_e(t)} \sigma^2_{I_{k-1}}(Z_k - Z_{k-1}) + \sigma^2_{I_{N_e(t)}}(t - Z_{N_e(t)}), \qquad t \geq 0,$$

in which Z_k is the occurring time of the kth transition of $\{I(t);\ t \geq 0\}$ with $Z_0 = 0$, $N_e(t)$ is the number of transitions that have occurred by time t, $I_{k-1} = I(Z_{k-1})$ is the state of the process $\{I(t);\ t \geq 0\}$ during the interval $[Z_{k-1}, Z_k)$, and σ_i^2 is the variance parameter for $\{B(t);\ t \geq 0\}$ when process $\{I(t);\ t \geq 0\}$ remains in state i during the time interval. Further, we assume that $S(t)$ and $B(t)$ are independent given $I(t)$.

Note that if there is only one state for the process $\{I(t);\ t \geq 0\}$, the risk model (1.1) reduces to the classical continuous-time surplus process perturbed by diffusion

$$U(t) = u + ct - S(t) + \sigma W(t),$$

where $\{W(t);\ t \geq 0\}$ is the standard Wiener process. In the case that when $\sigma_i = \sigma$ for all $i \in J$, we have $D(t) = \sigma^2 t$, implying that $\{B(t);\ t \geq 0\}$ is a Wiener process with infinitesimal mean 0 and infinitesimal variance σ^2. If we further assume $\sigma = 0$ (i.e., there is no diffusion component in [1.1]), the risk model (1.1) with $B(t)$ removed is the so-called continuous-time Markov-modulated surplus process; models of this type have been investigated by some authors. Reinhard (1984) and Lu and Li (2005) considered the probability of ruin in a class of Markov-modulated risk models, Bäuerle (1996) considered the expected ruin time, Snoussi (2002) and Lu (2006) studied the severity of ruin, while Ng and Yang (2006) studied the joint distribution of surplus before and after ruin, and Li and Lu (2007) investigated moments of the dividend payments and related problems. See also Rolski et al. (1999) and references therein.

For the surplus process given by (1.1), let $T = \inf\{t : U(t) \leq 0\}$ be the time of ruin. Two important nonnegative random variables in connection with the time of ruin T are $|U(T)|$, the deficit at the time of ruin, and $U(T-)$, the surplus immediately before the time of ruin. Consider a penalty scheme that is defined by a constant w_0 and a nonnegative function $w(x, y)$ for $x, y > 0$; the penalty due at ruin is w_0 if ruin occurs by oscillation and $w(U(T-), |U(T)|)$ if ruin is caused by a claim. Then given the initial surplus $u \geq 0$ and the initial state $i \in J$, the expected discounted penalty function at ruin $\phi(u)$ is

$$\phi(u; i) = w_0\phi_d(u; i) + \phi_w(u; i), \tag{1.2}$$

where for $\delta \geq 0$,

$$\phi_d(u; i) = \mathbb{E}[e^{-\delta T} I(T < \infty, U(T) = 0) | U(0) = u, I(0) = i] \tag{1.3}$$

(with $\phi_d(0; i) = 1$) is the Laplace transform or the expectation of the present value of the time of ruin T due to oscillation, and

$$\phi_w(u; i) = \mathbb{E}[e^{-\delta T}\, w(U(T-), |U(T)|) I(T < \infty, U(T) < 0) | U(0) = u, I(0) = i] \tag{1.4}$$

(with $\phi_w(0; i) = 0$) is the expected discounted penalty function at ruin caused by a claim. Many ruin-related quantities can be analyzed by appropriately choosing special penalty function w, for example, the discounted joint and marginal density functions of $U(T-)$ and $|U(T)|$, the discounted joint moments of $U(T-)$ and $|U(T)|$, the Laplace transform of T with respect to δ, as well as the probability of ruin and the probability of the severity of ruin.

Let $\mathbf{\Sigma} = (\alpha_{ij})_{i,j=1}^{m}$ denote the intensity matrix of $\{I(t); t \geq 0\}$ with $\alpha_{ii} := -\alpha_i = -\Sigma_{j\neq i}\, \alpha_{ij}$ for $i \in J$. The expected discounted function at ruin in the stationary case is $\phi(u) = \Sigma_{i=1}^{m}\, \zeta_i \phi(u; i)$, $u \geq 0$, where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_m)$ is the unique stationary probability distribution of the process $\{I(t); t \geq 0\}$, satisfying $\boldsymbol{\zeta\Sigma} = \mathbf{0}$.

The evaluation of the expected discounted penalty function, first introduced in Gerber and Shiu (1998), was recently discussed for different kinds of risk models; see, for example, Gerber and Landry (1998), Tsai (2001), and Tsai and Willmot (2002a,b) for the classical surplus process perturbed by diffusion, Willmot and Dickson (2003) for the stationary renewal risk model, Lin, Willmot, and Drekic (2003) for the classical risk model with a constant dividend barrier, Li and Garrido (2005) for the Sparre Andersen risk model perturbed by diffusion, Li and Lu (2005) for a two-class risk process, and Garrido and Morales (2006) for the Lévy risk process.

This paper is organized as follows. In Section 2 we derive systems of integro-differential equations for $\phi_w(u; i)$ and $\phi_d(u; i)$, respectively. Laplace transforms of these functions for $i \in J$ are obtained in Section 3. Section 4 discusses a generalized Lundberg's equation and its roots for the two-state model and gives the explicit expressions for $\phi_w(u; i)$ and $\phi_d(u; i)$ when both claim amount distributions belong to the rational family. Finally, as an illustration, explicit results for the ruin probabilities caused by a claim and oscillation, respectively, are obtained when claim amounts are exponentially distributed. A numerical example also is given.

2. SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS FOR $\phi_w(u; i)$ AND $\phi_d(u; i)$

We first derive a system of integro-differential equations for $\phi_w(u; i)$, $i \in J$. Consider an infinitesimal time interval $[0, t]$, and let $V_i(t) = u + ct + \sigma_i\, W(t)$, $i \in J$. By conditioning on the occurrence of claims and the change of environment in $[0, t]$, we have

$$\begin{aligned}\phi_w(u; i) = {} & (1 - \alpha_i t - \lambda_i t)e^{-\delta t}\mathbb{E}[\phi_w(V_i(t); i)] \\ & + \lambda_i t e^{-\delta t}\mathbb{E}\left[\int_0^{V_i(t)} \phi_w(V_i(t) - x; i)\, dF_i(x) + \int_{V_i(t)}^{\infty} w(V_i(t), x - V_i(t))\, dF_i(x)\right] \\ & + te^{-\delta t} \sum_{k=1,k\neq i}^{m} \alpha_{ik}\mathbb{E}[\phi_w(V_i(t); k)] + o(t), \qquad (2.1)\end{aligned}$$

where $o(t)/t \to 0$ as $t \to 0$. The first term on the right-hand side of (2.1) corresponds to the case where no claim and no change of environment occur in $[0, t]$, and the second term corresponds to the case where a claim x smaller or larger than $V_i(t)$ occurs without change of environment, while the last two terms correspond to the case where the environment changes but no claims occur, and two or more events occur in $[0, t]$, respectively.

Multiplying by $e^{\delta t}$ on both sides of equation (2.1) and rearranging gives

$$\begin{aligned}(\alpha_i + \lambda_i)\, t\mathbb{E}[\phi_w(V_i(t); i)] = {} & \mathbb{E}[\phi_w(V_i(t); i)] - e^{\delta t}\phi_w(u; i) \\ & + \lambda_i t\mathbb{E}\left[\int_0^{V_i(t)} \phi_w(V_i(t) - x; i)\, dF_i(x) + \int_{V_i(t)}^{\infty} w(V_i(t), x - V_i(t))\, dF_i(x)\right] \\ & + t \sum_{k=1,k\neq i}^{m} \alpha_{ik}\mathbb{E}[\phi_w(V_i(t); k)] + o(t). \qquad (2.2)\end{aligned}$$

By Cai and Xu (2006, eq. [2.7] with $\alpha = r = 0$), we have

$$\lim_{t\to 0} \frac{\mathbb{E}[\phi_w(V_i(t); i)] - e^{\delta t}\phi_w(u; i)}{t} = \lim_{t\to 0} \frac{\mathbb{E}[\phi_w(V_i(t); i)] - \phi_w(u; i) - (e^{\delta t} - 1)\phi_w(u; i)}{t}$$
$$= D_i\phi''_w(u; i) + c\phi'_w(u; i) - \delta\phi_w(u; i),$$

where $D_i = \frac{1}{2}\sigma_i^2$ for $i \in J$. Dividing by t on both sides of (2.2), letting $t \to 0$, and noting that $\alpha_{ii} = -\alpha_i$ for $i \in J$ yields a system of integro-differential equations for $\phi_w(u; i)$ given the initial surplus u and the initial environment $i \in J$:

$$(\lambda_i + \delta)\phi_w(u; i) = D_i\phi''_w(u; i) + c\phi'_w(u; i) + \sum_{k=1}^{m} \alpha_{ik}\phi_w(u; k)$$
$$+ \lambda_i \left[\int_0^u \phi_w(u - x; i)\, dF_i(x) + \int_u^\infty w(u, x - u)\, dF_i(x)\right], \qquad u > 0. \qquad (2.3)$$

Next we derive a system of differential equations for $\phi_d(u; i)$, $i \in J$. Similarly, by conditioning on the event that occurs in the infinitesimal time interval $[0, t]$, we have

$$\phi_d(u; i) = (1 - \alpha_i t - \lambda_i t)\, e^{-\delta t}\, \mathbb{E}[\phi_d(V_i(t); i)]$$
$$+ \lambda_i t e^{-\delta t}\mathbb{E}\left[\int_0^{V_i(t)} \phi_d(V_i(t) - x; i)\, dF_i(x)\right]$$
$$+ t e^{-\delta t} \sum_{k=1, k\neq i}^{m} \alpha_{ik}\, \mathbb{E}[\phi_d(V_i(t); k)] + o(t). \qquad (2.4)$$

Using arguments similar to those used in deriving (2.3), we get from (2.4) the following system of integro-differential equations for $\phi_d(u; i)$ given the initial surplus u and the initial environment $i \in J$:

$$(\lambda_i + \delta)\phi_d(u; i) = D_i\, \phi''_d(u; i) + c\, \phi'_d(u; i) + \sum_{k=1}^{m} \alpha_{ik}\phi_d(u; k)$$
$$+ \lambda_i \int_0^u \phi_d(u - x; i)\, dF_i(x), \qquad u > 0. \qquad (2.5)$$

REMARKS

1. When $m = 1$ (i.e., there is no Markovian environment process), implying $\alpha_1 = 0$, equations (2.3) and (2.5) reduce to the one given in Tsai and Willmot (2002a) and (6) in Gerber and Landry (1998) for the classical risk model perturbed by diffusion, respectively. In the case that $\delta = 0$, equation (2.5) becomes equation (4.1) of Dufresne and Gerber (1991) for function $\psi_d(u)$, the probability of ruin caused by oscillation for the surplus process perturbed by diffusion.
2. It is observed from the derivation that mathematically the model can be generalized to the case where the premium rate depends on the external environment process $\{J(t); t \geq 0\}$, simply replacing c by c_i in equations (2.3) and (2.5).

3. LAPLACE TRANSFORMS

We now apply Laplace transforms to solve the system of equations in (2.3). For $i \in J$ and $s \in \mathbb{C}$, let $\hat{\phi}_w(s; i) = \int_0^\infty e^{-su}\, \phi_w(u; i)\, du$, $\hat{f}_i(s) = \int_0^\infty e^{-su}\, f_i(u)\, du$, and $\hat{\omega}_i(s) = \int_0^\infty e^{-su}\, \omega_i(u)\, du$ be the Laplace transforms of $\phi_w(\cdot; i)$, f_i, and ω_i, respectively, where

$$\omega_i(u) = \int_u^\infty w(u, x - u)\, dF_i(x) = \int_0^\infty w(u, x)\, dF_i(x + u), \qquad u \geq 0,$$

with $\omega_i(u) = 1 - F_i(u)$ when $w(u, x) = 1$.

Assume that $\lim_{u\to\infty} e^{-su}\phi_w(u; i) = 0$ and $\lim_{u\to\infty} e^{-su}\phi'_w(u; i) = 0$ hold for $\mathcal{R}(s) > 0$. Taking Laplace transforms on both sides of equation (2.3) and noting that $\phi_w(0; i) = 0$ yields

$$[D_i s^2 + cs - \delta - \lambda_i(1 - \hat{f}_i(s))]\hat{\phi}_w(s; i) + \sum_{k=1}^{m} \alpha_{ik}\hat{\phi}_w(s; k) = D_i\phi'_w(0; i) - \lambda_i\hat{\omega}_i(s), \qquad i \in J, s \in \mathbb{C}. \tag{3.1}$$

The standard properties of the Laplace transform can be found in the literature, for example, Oberhettinger and Badii (1973). For simplicity, define $a_i(s) = D_i s^2 + cs - \delta - \lambda_i(1 - \hat{f}_i(s))$ for $i \in J$. Then the matrix form of (3.1) is given by

$$\mathbf{A}(s)\hat{\boldsymbol{\phi}}_w(s) = \mathbf{B}_w(s), \qquad s \in \mathbb{C}, \tag{3.2}$$

where $\hat{\boldsymbol{\phi}}_w(s) = (\hat{\phi}_w(s; 1), \ldots, \hat{\phi}_w(s; m))^t$, $\mathbf{A}(s) = \mathbf{diag}(a_1(s), \ldots, a_m(s)) + \boldsymbol{\Sigma}$, $\boldsymbol{\Sigma} = (\alpha_{i,j})_{i,j=1}^m$, and $\mathbf{B}_w(s) = (D_1\phi'_w(0; 1) - \lambda_1\hat{\omega}_1(s), \ldots, D_m\phi'_w(0; m) - \lambda_m\hat{\omega}_m(s))^t$. Then the vector of Laplace transforms $\hat{\boldsymbol{\phi}}_w(s)$ can be solved as

$$\hat{\boldsymbol{\phi}}_w(s) = [\mathbf{A}(s)]^{-1}\, \mathbf{B}_w(s), \qquad s \in \mathbb{C}; \tag{3.3}$$

the equation

$$|\mathbf{A}(s)| = 0, \qquad s \in \mathbb{C}, \tag{3.4}$$

is called the characteristic equation of (3.2), or a generalized Lundberg's fundamental equation for the Markov-modulated risk process perturbed by diffusion, where $|\mathbf{A}(s)|$ is the determinant of the matrix $\mathbf{A}(s)$.

Similarly, let $\hat{\phi}_d(s; i) = \int_0^\infty e^{-su}\, \phi_d(u; i)\, du$ be the Laplace transform of $\phi_d(u; i)$ and assume that $\lim_{u\to\infty} e^{-su}\phi_d(u; i) = 0$ and $\lim_{u\to\infty} e^{-su}\phi'_d(u; i) = 0$ hold for $\mathcal{R}(s) > 0$. Taking Laplace transforms on both sides of equation (2.5) and noting that $\phi_d(0; i) = 1$ gives

$$[D_i s^2 + cs - \delta - \lambda_i(1 - \hat{f}_i(s))]\, \hat{\phi}_d(s; i) + \sum_{k=1}^{m} \alpha_{ik}\hat{\phi}_d(s; k) = D_i\phi'_d(0; i) + D_i s + c, \qquad i \in J, s \in \mathbb{C}, \tag{3.5}$$

or in a matrix form

$$\mathbf{A}(s)\hat{\boldsymbol{\phi}}_d(s) = \mathbf{B}_d(s), \qquad s \in \mathbb{C},$$

where $\mathbf{B}_d(s) = (D_1\phi'_d(0; 1) + D_1 s + c, \ldots, D_m\phi'_d(0; m) + D_m s + c)^t$ and $\hat{\boldsymbol{\phi}}_d(s) = (\hat{\phi}_d(s; 1), \ldots, \hat{\phi}_d(s; m))^t$. The vector $\hat{\boldsymbol{\phi}}_d(s)$ also can be solved as

$$\hat{\boldsymbol{\phi}}_d(s) = [\mathbf{A}(s)]^{-1}\, \mathbf{B}_d(s), \qquad s \in \mathbb{C}.$$

REMARK

It is observed that the system of equations for $\hat{\phi}_d(s; i)$, (3.5), is structurally analogous to the one for $\hat{\phi}_w(s; i)$ given by (3.1). By setting $\hat{\omega}_i(s) = -(D_i s + c)/\lambda_i$, one assumes that the corresponding expressions for $\hat{\phi}_d(s; i)$ can be obtained from the results for $\hat{\phi}_w(s; i)$.

4. THE TWO-STATE MODEL

We now derive explicit expressions for the conditional expected discounted penalty function at ruin caused by a claim and oscillation, respectively, for a two-state model given the initial surplus and the initial state of the environment process. When the Laplace transforms of the claim size distributions can be expressed as certain rational functions, the analysis of the roots of the characteristic equation allows for the inversion of the Laplace transforms of $\phi_w(u; i)$ and $\phi_d(u; i)$.

4.1 The Characteristic Equation

Next we consider the case that the external environmental process $\{I(t); t \geq 0\}$ is a two-state Markov process with intensity rates α_1 and α_2. In this case the matrix $\mathbf{A}(s)$ has the form

$$\mathbf{A}(s) = \begin{pmatrix} a_1(s) - \alpha_1 & \alpha_1 \\ \alpha_2 & a_2(s) - \alpha_2 \end{pmatrix},$$

and the characteristic equation (3.4) is given by

$$[a_1(s) - \alpha_1][a_2(s) - \alpha_2] = \alpha_1\alpha_2. \tag{4.1}$$

The following theorem shows that the equation above has exactly four real roots; two are on the right half complex plane, and the other two are on the left half complex plane. The two positive real roots play a key role in the following sections.

Theorem 1

For $\delta > 0$, the characteristic equation (4.1) has exactly two distinct positive real roots, denoted by $\rho_1(\delta)$ and $\rho_2(\delta)$, and two distinct negative real roots. Moreover, $\rho_1(\delta)$ and $\rho_2(\delta)$ are the only roots on the right half complex plane.

PROOF

Let $\xi(s) = [a_1(s) - \alpha_1][a_2(s) - \alpha_2]$. For $\delta > 0$, consider equation (4.1):

$$\xi(s) = [D_1 s^2 + cs - (\lambda_1 + \alpha_1 + \delta) + \lambda_1 \hat{f}_1(s)]$$
$$\times [D_2 s^2 + cs - (\lambda_2 + \alpha_2 + \delta) + \lambda_2 \hat{f}_2(s)] = \alpha_1\alpha_2.$$

Note that $\xi(0) = (\alpha_1 + \delta)(\alpha_2 + \delta) > \alpha_1\alpha_2$ and $\lim_{s\to\infty} \xi(s) = +\infty$. By the fact that $\xi(s) = 0$ has two positive roots, we conclude that equation (4.1) has at least two positive roots.

If $s \in \{z : |z| = r(r > 0), \Re(z) \geq 0\}$ (the half circle on the complex plane), then $|\xi(s)| > \alpha_1\alpha_2$ for sufficiently large r; if $s \in \{z : \Re(z) = 0\}$ (the imaginary axis), then $|\xi(s)| \geq (\alpha_1 + \delta)(\alpha_2 + \delta) > \alpha_1\alpha_2$, which is the right-hand side of equation (4.1). This implies that, on the boundary of the contour enclosed by the half circle and the imaginary axis, $|\xi(s)| > \alpha_1\alpha_2$. We conclude, by Rouché's Theorem, that on the right half plane, the number of roots to equation (4.1) equals the number of roots of the equation $\xi(s) = 0$. Moreover, again by Rouché's Theorem, the latter only has two roots with positive real parts on the right half complex plane.

It follows that equation (4.1) also has exactly two roots with positive real parts: that is, $\rho_1(\delta)$ and $\rho_2(\delta)$ are the only roots on the right half complex plane. By the same arguments, it can be proved that equation (4.1) also has exactly two negative real roots on the left half complex plane. □

Note that if we assume $\rho_1(\delta) < \rho_2(\delta)$, then $\rho_1(\delta) \to 0^+$ as $\delta \to 0^+$. In the rest of the paper, $\rho_j(\delta)$ are simply denoted by ρ_j for $j = 1, 2$, and $\delta > 0$.

4.2 Laplace Transforms for $\phi_w(u; i)$ and $\phi_d(u; i)$

Equation (3.3) has the form

$$\hat{\boldsymbol{\phi}}_w(s) = \frac{1}{|\mathbf{A}(s)|} \mathbf{A}^*(s)\mathbf{B}_w(s), \tag{4.2}$$

where $\hat{\boldsymbol{\phi}}_w(s) = (\hat{\phi}_w(s; 1), \hat{\phi}_w(s; 2))^t$, $|\mathbf{A}(s)| = [a_1(s) - \alpha_1][a_2(s) - \alpha_2] - \alpha_1\alpha_2$, $\mathbf{A}^*(s)$ is the adjoint of matrix $\mathbf{A}(s)$ given by

$$\mathbf{A}^*(s) = \begin{pmatrix} a_2(s) - \alpha_2 & -\alpha_1 \\ -\alpha_2 & a_1(s) - \alpha_1 \end{pmatrix},$$

and $\mathbf{B}_w(s) = (D_1\phi'_w(0; 1) - \lambda_1\hat{\omega}_1(s), D_2\phi'_w(0; 2) - \lambda_2\hat{\omega}_2(s))^t$.

Since $\hat{\phi}_w(s; 1)$ and $\hat{\phi}_w(s; 2)$ are finite for all s with $\mathcal{R}(s) \geq 0$ and the fact that ρ_1 and ρ_2 are zeros of the denominator of (4.2), which has been shown in Theorem 1, we have that the numerator in (4.2) is also zero when $s = \rho_1$ or ρ_2, that is,

$$\mathbf{A}^*(\rho_1)\mathbf{B}_w(\rho_1) = \mathbf{A}^*(\rho_2)\mathbf{B}_w(\rho_2) = \mathbf{0},$$

which is equivalent to

$$[\mathbf{A}^*(\rho_2) - \mathbf{A}^*(\rho_1)]\mathbf{B}_w(\rho_2) = \mathbf{A}^*(\rho_1)[\mathbf{B}_w(\rho_1) - \mathbf{B}_w(\rho_2)]. \tag{4.3}$$

Here we recall the definition of the divided difference (see Gerber and Shiu 2005). For a function $h(s)$, its first divided difference with respect to distinct numbers ρ_1 and ρ_2 is defined as $h[\rho_1, \rho_2] = [h(\rho_2) - h(\rho_1)]/(\rho_2 - \rho_1)$. The higher-order divided differences can be obtained recursively; for example, the second divided difference of h with respect to distinct numbers ρ_1, ρ_2, and ρ_3 is given by $h[\rho_1, \rho_2, \rho_3] = (h[\rho_1, \rho_3] - h[\rho_1, \rho_2])/(\rho_3 - \rho_2)$. The definition of the divided difference obviously can be extended to any vector or matrix that is a function of a single variable. For example, for matrix $\mathbf{A}^*(s)$,

$$\mathbf{A}^*[\rho_1, \rho_2] = \frac{\mathbf{A}^*(\rho_2) - \mathbf{A}^*(\rho_1)}{\rho_2 - \rho_1} = \begin{pmatrix} a_2[\rho_1, \rho_2] & 0 \\ 0 & a_1[\rho_1, \rho_2] \end{pmatrix}, \tag{4.4}$$

$$\mathbf{A}^*[\rho_1, \rho_2, \rho_3] = \frac{\mathbf{A}^*[\rho_1, \rho_3] - \mathbf{A}^*[\rho_1, \rho_2]}{\rho_3 - \rho_2} = \begin{pmatrix} a_2[\rho_1, \rho_2, \rho_3] & 0 \\ 0 & a_1[\rho_1, \rho_2, \rho_3] \end{pmatrix}. \tag{4.5}$$

By using the notation of the divided difference for functions and matrices, equation (4.3) can be rewritten as

$$\mathbf{A}^*[\rho_1, \rho_2]\mathbf{B}_w(\rho_2) = \mathbf{A}^*(\rho_1)\begin{pmatrix} \lambda_1\hat{\omega}_1[\rho_1, \rho_2] \\ \lambda_2\hat{\omega}_2[\rho_1, \rho_2] \end{pmatrix} = \mathbf{A}^*(\rho_1)\mathbf{\Lambda}\hat{\boldsymbol{\omega}}[\rho_1, \rho_2], \tag{4.6}$$

where $\mathbf{\Lambda} = \mathbf{diag}(\lambda_1, \lambda_2)$, and $\hat{\boldsymbol{\omega}}[\rho_1, \rho_2] = (\hat{\omega}_1[\rho_1, \rho_2], \hat{\omega}_2[\rho_1, \rho_2])^t$. Then $\mathbf{B}_w(\rho_2)$ can be easily solved by equation (4.6) as

$$\mathbf{B}_w(\rho_2) = \{\mathbf{A}^*[\rho_1, \rho_2]\}^{-1}\,\mathbf{A}^*(\rho_1)\mathbf{\Lambda}\hat{\boldsymbol{\omega}}[\rho_1, \rho_2], \tag{4.7}$$

where ρ_1 and ρ_2 are the only positive real roots of equation (4.1).

Note that $\hat{\omega}_i[\rho_1, \rho_2] = -D_i/\lambda_i$ when $\hat{\omega}_i(s) = -(D_is + c)/\lambda_i$ for $i = 1, 2$. By the remark in Section 3, we obtain that

$$\mathbf{B}_d(\rho_2) = -\{\mathbf{A}^*[\rho_1, \rho_2]\}^{-1}\mathbf{A}^*(\rho_1)\mathbf{D}, \tag{4.8}$$

where $\mathbf{D} = (D_1, D_2)^t$.

By the fact that $s = \rho_2$ is a root of the numerator in (4.2), we get

$$\begin{aligned} \mathbf{A}^*(s)\mathbf{B}_w(s) &= \mathbf{A}^*(s)\mathbf{B}_w(s) - \mathbf{A}^*(\rho_2)\mathbf{B}_w(\rho_2) \\ &= \mathbf{A}^*(s)[\mathbf{B}_w(s) - \mathbf{B}_w(\rho_2)] - [\mathbf{A}^*(\rho_2) - \mathbf{A}^*(s)]\mathbf{B}_w(\rho_2) \\ &= (s - \rho_2)\{-\mathbf{A}^*(s)\mathbf{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, s] + \mathbf{A}^*[\rho_2, s]\mathbf{B}_w(\rho_2)\}. \end{aligned} \tag{4.9}$$

Note that $s = \rho_1$ is also a root of the numerator in (4.2), implying that $s = \rho_1$ is a zero of the expression within the braces in (4.9). Then by the same technique the expression of (4.9) can be further derived as

$$\mathbf{A}^*(s)\mathbf{B}_w(s) = (s - \rho_1)(s - \rho_2)\{-\mathbf{A}^*[\rho_1, s]\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, s] \\ -\mathbf{A}^*(\rho_1)\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, \rho_1, s] + \mathbf{A}^*[\rho_2, \rho_1, s]\mathbf{B}_w(\rho_2)\}. \tag{4.10}$$

From (4.10), Laplace transform expressions (4.2) can be written as

$$\hat{\boldsymbol{\phi}}_w(s) = \frac{(s - \rho_1)(s - \rho_2)}{|\mathbf{A}(s)|}\{-\mathbf{A}^*[\rho_1, s]\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, s] \\ -\mathbf{A}^*(\rho_1)\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, \rho_1, s] + \mathbf{A}^*[\rho_2, \rho_1, s]\mathbf{B}_w(\rho_2)\}, \tag{4.11}$$

where $\mathbf{B}_w(\rho_2)$ is given by (4.7). By the same argument in deriving $\mathbf{B}_d(\rho_2)$, and the fact that $\hat{\omega}_i[\rho_2, \rho_1, s] = 0$ in this case for $i = 1, 2$, we get the following Laplace transform expressions for $\phi_d(s; i)$:

$$\hat{\boldsymbol{\phi}}_d(s) = \frac{(s - \rho_1)(s - \rho_2)}{|\mathbf{A}(s)|}\{\mathbf{A}^*[\rho_1, s]\mathbf{D} + \mathbf{A}^*[\rho_2, \rho_1, s]\mathbf{B}_d(\rho_2)\}, \tag{4.12}$$

where $\mathbf{B}_d(\rho_2)$ is given by (4.8).

4.3 Explicit Results for $\phi_w(u; i)$ and $\phi_d(u; i)$

In some cases the functions $\phi_w(u; i)$ and $\phi_d(u; i)$ can be explicitly and analytically determined by inversion of (4.11) and (4.12). Consider the case where the claim size distributions f_1 and f_2 are from the rational family, namely, their Laplace transformations can be expressed as a ratio of polynomials:

$$\hat{f}_1(s) = \frac{p^{\dagger}_{k-1}(s)}{p_k(s)}, \qquad \hat{f}_2(s) = \frac{q^{\dagger}_{l-1}(s)}{q_l(s)}, \qquad k, l \in \mathbb{N}^+, \tag{4.13}$$

where p_k and q_l are polynomials of degrees k and l, respectively, while $p^{\dagger}_{k-1}$ and $q^{\dagger}_{l-1}$ are polynomials of degrees $k - 1$ and $l - 1$, or less, respectively; all have leading coefficient 1 and satisfy $p^{\dagger}_{k-1}(0) = p_k(0)$ and $q^{\dagger}_{l-1}(0) = q_l(0)$. Further, equations $p_k(s) = 0$ and $q_l(s) = 0$ have roots with only negative real parts.

The rational family is a wide class of distributions, which includes, as special cases, Erlang, Coxian, and some phase-type distributions, as well as mixtures of these (see Cohen 1982; Tijms 1994). Note that distributions with damped sine and cosine functions as part of their densities also belong to the rational family.

To obtain expressions that can be inverted easily, we multiply both numerator and denominator of equation (4.2) by $p_k(s)q_l(s)$, yielding

$$\hat{\boldsymbol{\phi}}_w(s) = \frac{1}{|\mathbf{A}(s)|p_k(s)q_l(s)}\mathbf{A}^*(s)\mathbf{B}_w(s)p_k(s)q_l(s). \tag{4.14}$$

First, we look at the denominator in (4.14), denoted by $D_{k+l+4}(s)$:

$$D_{k+l+4}(s) = |\mathbf{A}(s)|p_k(s)q_l(s) = ([a_1(s) - \alpha_1][a_2(s) - \alpha_2] - \alpha_1\alpha_2)p_k(s)q_l(s),$$

which is clearly a polynomial of degree $k + l + 4$ with the leading coefficient D_1D_2, and therefore equation $D_{k+l+4}(s) = 0$ has $k + l + 4$ roots on the complex plane. By Theorem 1, $|\mathbf{A}(s)| = 0$ has two positive real roots ρ_1 and ρ_2, and two negative real roots. Thus, we can rewrite $D_{k+l+4}(s)$ as

$$D_{k+l+4}(s) = D_1D_2(s - \rho_1)(s - \rho_2)\prod_{i=1}^{k+l+2}(s + R_i),$$

where all R_i's have positive real parts by Theorem 1 and the definition of the rational distribution. For simplicity, we assume that these R_i's are distinct. It turns out by (4.10) that the term $(s - \rho_1)(s - \rho_2)$ can be canceled from both numerator and denominator of (4.14).

Next, for analyzing the numerator in (4.14), noting the expressions of (4.4) and (4.5), and $\mathbf{A}^*(s)\mathbf{B}_w(s)$ given by (4.10), we further denote that

$$h_i(s) = a_i[\rho_1, s]p_k(s)q_l(s), \qquad i = 1, 2, \tag{4.15}$$

$$g_i(s) = a_i[\rho_1, \rho_2, s]p_k(s)q_l(s), \qquad i = 1, 2, \tag{4.16}$$

where $a_i[\rho_1, s] = D_i(s + \rho_1) + c + \lambda_i\hat{f}_i[\rho_1, s]$ and $a_i[\rho_1, \rho_2, s] = D_i + \lambda_i\hat{f}_i[\rho_1, \rho_2, s]$. It is not difficult to see that $h_i(s)$and $g_i(s)$ are polynomials of degrees $k + l + 1$ and $k + l$, respectively. Then it follows from (4.11) that (4.14) can be expressed as

$$\hat{\boldsymbol{\phi}}_w(s) = \frac{1}{D_2D_2\prod_{j=1}^{k+l+2}(s + R_j)}\{-\mathbf{diag}(h_2(s), h_1(s))\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, s]$$
$$-\mathbf{A}^*(\rho_1)\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, \rho_1, s]p_k(s)q_l(s) + \mathbf{diag}(g_2(s), g_1(s))\mathbf{B}_w(\rho_2)\}. \tag{4.17}$$

Furthermore, by decomposing the rational expressions in (4.17) into partial fractions, we get

$$\hat{\boldsymbol{\phi}}_w(s) = \frac{1}{D_1D_2}\sum_{j=1}^{k+l+2}\frac{1}{s + R_j}\{-\mathbf{diag}(h_{2j}, h_{1j})\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, s]$$
$$-g_j\,\mathbf{A}^*(\rho_1)\boldsymbol{\Lambda}\hat{\boldsymbol{\omega}}[\rho_2, \rho_1, s] + \mathbf{diag}(g_{2j}, g_{1j})\mathbf{B}_w(\rho_2)\}, \tag{4.18}$$

where coefficients h_{ij}, g_{ij}, and g_i, for $i = 1, 2$ and $j = 1, \ldots, k + l + 2$, are given by

$$h_{ij} = \frac{h_i(-R_j)}{\prod_{v=1,v\neq j}^{k+l+2}(R_v - R_j)}, \qquad g_{ij} = \frac{g_i(-R_j)}{\prod_{v=1,v\neq j}^{k+l+2}(R_v - R_j)}, \qquad g_j = \frac{p_k(-R_j)q_l(-R_j)}{\prod_{v=1,v\neq j}^{k+l+2}(R_v - R_j)}. \tag{4.19}$$

To determine function $\boldsymbol{\phi}_w$ in (4.18), we introduce the operator T_r, for an integrable real-valued function h with respect to a complex number r ($\Re(r) \geq 0$):

$$T_r h(y) = \int_y^\infty e^{-r(x-y)}h(x)\,dx, \qquad y \geq 0. \tag{4.20}$$

It can be seen that $T_sh(0) = \hat{h}(s)$, the Laplace transform of h. The further composition operators of T_r can be derived recursively, for $r_1 \neq r_2 \in \mathbb{C}$, as

$$T_{r_1}T_{r_2}h(y) = T_{r_2}T_{r_1}h(y) = \frac{T_{r_1}h(y) - T_{r_2}h(y)}{r_2 - r_1}, \qquad y \geq 0. \tag{4.21}$$

Properties of this operator can be found in Li and Garrido (2004). Gerber and Shiu (2005) also presented the following result on the relationship between the operator T_r and the corresponding divided difference:

$$\left[\left(\prod_{i=1}^m T_{r_i}\right)h\right](0) = (-1)^{m-1}\hat{h}[r_1, r_2, \ldots, r_m]. \tag{4.22}$$

By (4.22), we have $\hat{\boldsymbol{\omega}}[\rho_2, s] = -T_sT_{\rho 2}\boldsymbol{\omega}(0)$ and $\hat{\boldsymbol{\omega}}[\rho_2, \rho_1, s] = T_sT_{\rho 1}T_{\rho 2}\boldsymbol{\omega}(0)$. Thus inverting the Laplace transform (4.18), and noting that $T_sh(0) = \hat{h}(s)$, we get the following expressions for $\boldsymbol{\phi}_w(u) = (\phi_w(u; 1), \phi_w(u; 2))^t$:

$$\boldsymbol{\phi}_w(u) = \frac{1}{D_1D_2}\sum_{j=1}^{k+l+2}\{e^{-R_ju}\bigstar(\mathbf{diag}(h_{2j}, h_{1j})\boldsymbol{\Lambda}[T_{\rho 2}\boldsymbol{\omega}(u)]$$
$$-\, g_j\,\mathbf{A}^*(\rho_1)\boldsymbol{\Lambda}[T_{\rho 1}T_{\rho 2}\boldsymbol{\omega}(u)]) + e^{-R_ju}\mathbf{diag}(g_{2j}, g_{1j})\mathbf{B}_w(\rho_2)\}, \tag{4.23}$$

where $\mathbf{B}_w(\rho_2)$ is given by (4.7), ρ_1 and ρ_2 are the only positive real roots of the characteristic equation (4.1), and coefficients g_{ij}, h_{ij}, and g_j are given by (4.19).

Note that $e^{-R_ju} \bigstar T_{\rho 2}\omega_i(u)$ and $e^{-R_ju} \bigstar T_{\rho 1}T_{\rho 2}\omega_i(u)$ in formula (4.23) are convolutions that can be calculated by the following property of T_r (see eq. (10.2) of Gerber and Shiu 2005):

$$T_r[f \star g(x)] = f \star [T_r g(x)] + [T_r g(0)][T_r f(x)], \qquad x \geq 0.$$

We also remark that if there are pairs of complex roots to equation $D_{k+l+4}(s) = 0$, the expected discounted penalty functions involve trigonometric functions, which can be seen in the last section.

Similarly, by inverting the Laplace transform (4.12), we can get the following formulas for $\boldsymbol{\phi}_d(u) = (\phi_d(u;1), \phi_d(u;2))^t$:

$$\boldsymbol{\phi}_d(u) = \frac{1}{D_1 D_2} \sum_{j=1}^{k+l+2} e^{-R_j u} \{\mathbf{diag}(h_{2j}, h_{1j})\mathbf{D} + \mathbf{diag}(g_{2j}, g_{1j})\mathbf{B}_d(\rho_2)\}, \tag{4.24}$$

where $\mathbf{B}_d(\rho_2)$ is given by (4.8).

4.4 Ruin Probability with Exponential Claim Sizes

As an illustration, consider a special case that $\delta = 0$, $w_0 = 1$, and the penalty function $w(x, y) = 1$. Then (1.3) and (1.4) become the conditional ruin probabilities caused by a claim and oscillation given the initial surplus u and initial state i, denoted by $\psi_s(u; i)$ and $\psi_d(u; i)$, respectively. For the exponential claim size distributions case, their Laplace transforms are of the form $\hat{f}_1(s) = p_0^\dagger(s)/p_1(s) = \beta_1/(s + \beta_1)$ and $\hat{f}_2(s) = q_0^\dagger(s)/q_1(s) = \beta_2/(s + \beta_2)$, and accordingly we have $\omega_i(u) = 1 - F_i(u) = e^{-\beta_i u}$ for $u \geq 0$ and $i = 1, 2$. Also, $\rho_1 = 0$ and $\rho_2 = \rho$ is the only positive real root of equation (4.1), and equation $D_6(s) = 0$ has four other roots $-R_i$, $i = 1, 2, 3, 4$, where all R_i's have positive real parts.

In this case we obtain that $a_i(0) = 0$, $\hat{\omega}_i(\rho) = 1/(\beta_i + \rho)$, $T_\rho\omega_i(u) = e^{-\beta_i u}/(\beta_i + \rho)$, $\hat{\omega}_i[0, \rho] = -1/[\beta_i(\beta_i + \rho)]$, and $T_\rho T_0 \omega_i(u) = T_\rho\omega_i(u)/\beta_i$. It follows from (4.7) and (4.8) that

$$\mathbf{B}_s(\rho) = \begin{pmatrix} \dfrac{\alpha_2}{a_2[0,\rho]} & \dfrac{\alpha_1}{a_2[0,\rho]} \\ \dfrac{\alpha_2}{a_1[0,\rho]} & \dfrac{\alpha_1}{a_1[0,\rho]} \end{pmatrix} \begin{pmatrix} \dfrac{\lambda_1}{\beta_1\ (\beta_1 + \rho)} \\ \dfrac{\lambda_2}{\beta_2\ (\beta_2 + \rho)} \end{pmatrix}, \qquad \mathbf{B}_d(\rho) = \begin{pmatrix} \dfrac{\alpha_2}{a_2[0,\rho]} & \dfrac{\alpha_1}{a_2[0,\rho]} \\ \dfrac{\alpha_2}{a_1[0,\rho]} & \dfrac{\alpha_1}{a_1[0,\rho]} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}, \tag{4.25}$$

where $a_i[0, \rho] = D_i\,\rho + c - \lambda_i/(\beta_i + \rho)$ for $i = 1, 2$. Moreover, (4.23) and (4.24) give the following formulas for $\psi_s(u; i)$ and $\psi_d(u; i)$ for $u \geq 0$ and $i = 1, 2$:

$$\psi_s(u) = \frac{1}{D_1 D_2} \sum_{j=1}^{4} \left\{ \left[\begin{pmatrix} h_{2j} & 0 \\ 0 & h_{1j} \end{pmatrix} + g_j \begin{pmatrix} \dfrac{\alpha_2}{\beta_1} & \dfrac{\alpha_1}{\beta_2} \\ \dfrac{\alpha_2}{\beta_1} & \dfrac{\alpha_1}{\beta_2} \end{pmatrix} \right] e^{-R_j u} \star T_\rho \boldsymbol{\omega}(u) \right.$$
$$\left. + e^{-R_j u} \begin{pmatrix} g_{2j} & 0 \\ 0 & g_{1j} \end{pmatrix} \mathbf{B}_s(\rho) \right\},$$
$$\psi_d(u) = \frac{1}{D_1 D_2} \sum_{j=1}^{4} e^{-R_j u} \left[\begin{pmatrix} D_1 h_{2j} \\ D_2 h_{1j} \end{pmatrix} + \begin{pmatrix} g_{2j} & 0 \\ 0 & g_{1j} \end{pmatrix} \mathbf{B}_d(\rho) \right],$$

where $T_\rho\boldsymbol{\omega}(u) = (e^{-\beta_1 u}/(\beta_1 + \rho), e^{-\beta_2 u}/(\beta_2 + \rho))^t$; $\mathbf{B}_s(\rho)$ and $\mathbf{B}_d(\rho)$ are given by (4.25). The coefficients g_{ij}, h_{ij}, g_j can be obtained from (4.19) with $p_1(s)q_1(s) = (s + \beta_1)(s + \beta_2)$, and

$$h_i(s) = \left[D_i s + c - \frac{\lambda_i}{s + \beta_i} \right] (s + \beta_1)(s + \beta_2), \qquad i = 1, 2,$$

$$g_i(s) = \left[D_i + \frac{\lambda_i}{(\rho + \beta_i)(s + \beta_i)} \right] (s + \beta_1)(s + \beta_2), \qquad i = 1, 2.$$

REMARK

In the case where all the parameters are identical for two states, the model reduces to the classical risk process perturbed by diffusion. See the Appendix for more detailed discussions.

4.5 A Numerical Example

In this section we give a numerical example for the ruin probabilities due to a claim and oscillation, $\psi_s(u; i)$ and $\psi_d(u; i)$, respectively, when claim sizes from two classes are Erlang(2) distributed and of a mixture of two exponentials.

Let $f_1(x) = \beta^2\, x\, e^{-\beta x}$ and $f_2(x) = \xi\, \beta_1\, e^{-\beta_1 x} + (1 - \xi)\beta_2\, e^{-\beta_2 x}$, for $x \geq 0$ and $0 \leq \xi \leq 1$, with $\mu_1 = 2/\beta$ and $\mu_2 = \xi/\beta_1 + (1 - \xi)/\beta_2$. The Laplace transforms of f_1 and f_2 are given by

$$\hat{f}_1(s) = \frac{p_1^\dagger(s)}{p_2(s)} = \frac{\beta^2}{(s + \beta)^2}, \qquad \hat{f}_2(s) = \frac{q_1^\dagger(s)}{q_2(s)} = \frac{[\xi\, \beta_1 + (1 - \xi)\beta_2]\, s + \beta_1\beta_2}{(s + \beta_1)(s + \beta_2)}.$$

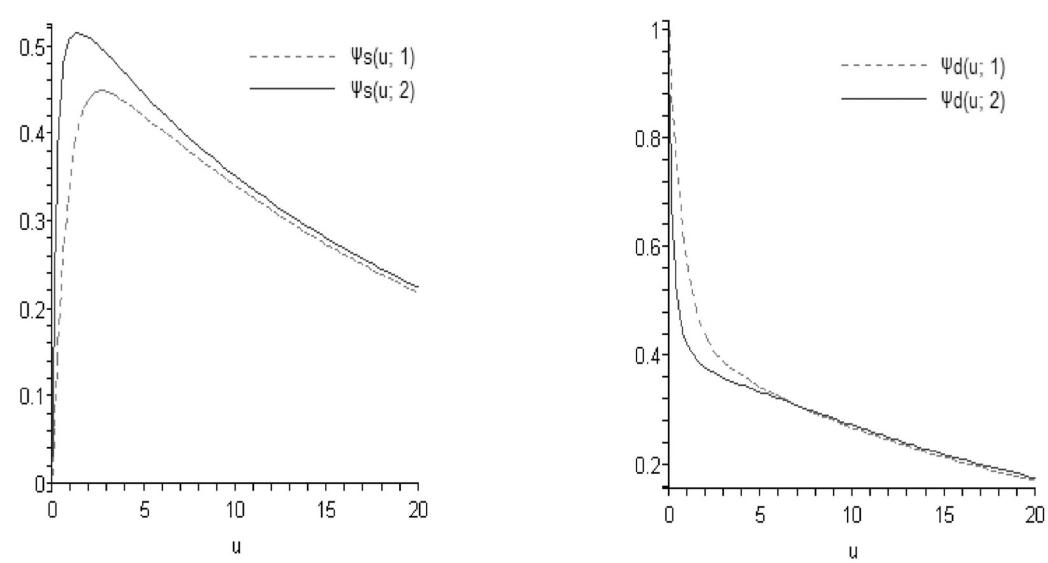
By setting $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$, we get that $\zeta_1 = 2/3$ and $\zeta_2 = 1/3$, the stationary probability distribution for the environment process. Further let $\lambda_1 = 0.5$, $\lambda_2 = 2$, $c = 1.35$, $\beta = 1$, $\beta_1 = 2$, $\beta_2 = 1/2$, $\xi = 0.8$, $D_1 = 2$, and $D_2 = 1/2$, implying that $\mu_1 = 2$ and $\mu_2 = 0.8$.

In this case equation $D_8(s) = |\mathbf{A}(s)|(s + \beta)^2(s + \beta_1)(s + \beta_2) = 0$ has exactly eight roots: 0, $\rho = 0.85182$, $-R_1 = -4.48728$, $-R_2 = -1.17364$, $-R_3 = -0.39284$, $-R_4 = -0.04471$, and $-R_{5,6} = -1.31418 \pm 0.42044\, i$. The ruin probabilities, given by (4.23) and (4.24), are obtained as follows:

$$\begin{pmatrix} \psi_s(u;1) \\ \psi_s(u;2) \\ \psi_d(u;1) \\ \psi_d(u;2) \end{pmatrix} = \begin{pmatrix} 0.00505 & 0.00119 & -0.05633 & 0.53508 \\ -0.50608 & -0.06027 & 0.05091 & 0.54866 \\ -0.00522 & -0.00143 & 0.06987 & 0.41810 \\ 0.52323 & 0.07259 & -0.06316 & 0.42871 \end{pmatrix} \begin{pmatrix} e^{-4.48728u} \\ e^{-1.17364u} \\ e^{-0.39284u} \\ e^{-0.04471u} \end{pmatrix}$$
$$+ \begin{pmatrix} -0.48499 & 0.00798 \\ -0.03323 & -0.14221 \\ 0.51868 & -0.01904 \\ 0.03863 & 0.15142 \end{pmatrix} \begin{pmatrix} \cos(0.42044u) \\ \sin(0.42044u) \end{pmatrix} e^{-1.31418u}.$$

Figure 1 gives the ruin probabilities caused by a claim and due to oscillation, $\psi_s(u; i)$ (left) and $\psi_d(u; i)$ (right), for the initial surplus $u \in [0, 20]$ and $i = 1, 2$. From this graph it can be observed that both $\psi_s(u; 1)$ and $\psi_s(u; 2)$ increase as u increases, reaching their maximums around $u = 3$ and $u = 1$, respectively, and then decrease to 0 with a crossing point around $u = 8$, while both $\psi_d(u; 1)$ and

Figure 1
Ruin Probabilities $\psi_s(u; i)$ and $\psi_d(u; i)$



$\psi_d(u;\,2)$ decrease and become very close as u increases, and $\psi_d(u;\,1)$ is consistently greater than $\psi_d(u;\,2)$ for $u > 0$.

Figure 2 illustrates the total ruin probabilities (left), given by $\psi_t(u;\,i) = \psi_s(u;\,i) + \psi_d(u;\,i)$ (a special case of (1.2) when $w_0 = 1$, $w(x, y) = 1$, and $\delta = 0$) for $0 \le u \le 20$ and $i = 1, 2$, and the ruin probabilities (right) $\psi_s(u)$, $\psi_d(u)$, and $\psi_t(u) = \psi_s(u) + \psi_d(u)$ in the stationary case for $0 \le u \le 20$. It shows that both total ruin probabilities $\psi_t(u;\,1)$ and $\psi_t(u;\,2)$ are decreasing functions of u, and $\psi_t(u;\,2)$ is overall smaller than $\psi_t(u;\,1)$. It also can be seen from the right-hand graph that $\psi_s(u)$ increases, then decreases as u increases, while both $\psi_d(u)$ and $\psi_t(u)$ decrease as u increases, similar to the phenomena observed from Figure 1.

APPENDIX

This appendix verifies that if all the parameters are identical for two states, the results in Section 4.4 are the same as the known ones for the equivalent classical risk model. In this case the intensity rates α_1 and α_2 can be chosen arbitrarily. Consequently, equation $|\mathbf{A}(s)|p_1(s)q_1(s) = 0$ becomes

$$D^2 s\left[s^2 + \left(\frac{c}{D} + \beta\right)s + \frac{1}{D}(c\beta - \lambda)\right] \times \left[s^3 + \left(\frac{c}{D} + \beta\right)s^2 + \frac{1}{D}(c\beta - \lambda - \alpha_1 - \alpha_2)s - \frac{\beta}{D}(\alpha_1 + \alpha_2)\right] = 0.$$

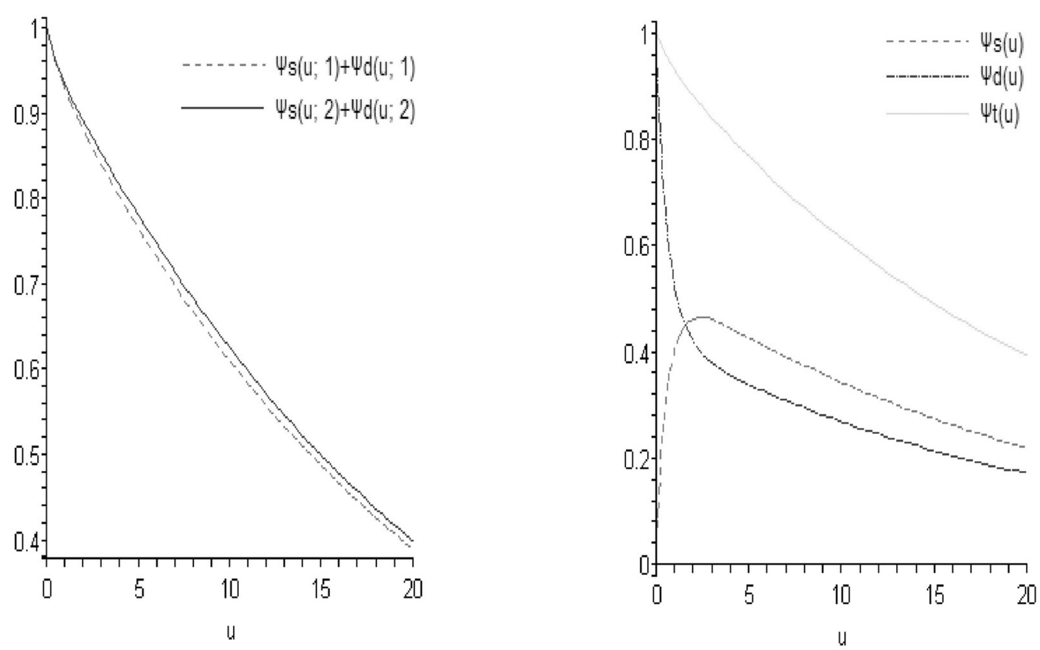
Let $-R_1$ and $-R_2$ be the roots of $s^2 + (c/D + \beta)s + (c\beta - \lambda)/D = 0$, then ρ, $-R_3$ and $-R_4$ are the roots of

$$\left[s^3 + \left(\frac{c}{D} + \beta\right)s^2 + \frac{1}{D}(c\beta - \lambda - \alpha_1 - \alpha_2)s - \frac{\beta}{D}(\alpha_1 + \alpha_2)\right] = 0. \tag{A.1}$$

As ρ is the only positive root of equation (A.1), it is not difficult to get that $-R_3$ and $-R_4$ are the negative roots of the following equation:

Figure 2

Ruin Probabilities with Initial State and Stationary Case



$$l(s) := (s + \beta)^2 + \left(\frac{c}{D} + \rho - \beta\right)(s + \beta) - \frac{\beta\lambda}{D(\rho + \beta)} = 0. \tag{A.2}$$

In this case the Laplace transforms of $\hat{\psi}_s(s)$ in (4.17), and $\hat{\psi}_d(s)$ by similarity, can be simplified to

$$\hat{\psi}_s(s) = \frac{-\lambda h(s)\hat{\omega}[\rho, s] + g(s)B_s(\rho)}{D^2 \prod_{j=1}^4 (s + R_j)} = \frac{\frac{\lambda}{D\beta} l(s)}{\prod_{j=1}^4 (s + R_j)} = \frac{\lambda}{D\beta} \sum_{j=1}^4 \frac{l_j}{s + R_j},$$

$$\hat{\psi}_d(s) = \frac{\lambda D\, h(s) + g(s)B_d(\rho)}{D^2 \prod_{j=1}^4 (s + R_j)} = \frac{(s + \beta)l(s)}{\prod_{j=1}^4 (s + R_j)} = \sum_{j=1}^4 \frac{(-R_j + \beta)\, l_j}{s + R_j},$$

where $l_j = l(-R_j)/\Pi_{v=1,v\neq j}^4 (R_v - R_j)$ for $j = 1, \ldots, 4$. By the fact that $l(-R_3) = l(-R_4) = 0$, implying $l_3 = l_4 = 0$, we get that the following explicit and simple expressions for the ruin probabilities caused by a claim and due to oscillation for the classical risk model perturbed by diffusion with exponentially distributed claim sizes:

$$\begin{pmatrix} \psi_s(u) \\ \psi_d(u) \end{pmatrix} = \begin{pmatrix} \frac{\lambda l_1}{D\beta} & \frac{\lambda l_2}{D\beta} \\ (-R_1 + \beta)l_1 & (-R_2 + \beta)l_2 \end{pmatrix} \begin{pmatrix} e^{-R_1 u} \\ e^{-R_2 u} \end{pmatrix}, \qquad u \geq 0, \tag{A.3}$$

where $l_1 = l(-R_1)/[(R_2 - R_1)(R_3 - R_1)(R_4 - R_1)]$ and $l_2 = l(-R_2)/[(R_1 - R_2)(R_3 - R_2)(R_4 - R_2)]$ with function l given by (A.2). It can be shown that (A.3) is consistent with equations (5.3) and (5.4) of Tsai (2006). Letting $c = 1.2$, and $\lambda = \beta = \sigma = 1$, we obtain that $\rho = 0.7987$, $R_1 = 0.122$, $R_2 = 3.278$, $R_3 = 3.623$, $R_4 = 0.576$; moreover, (A.3) gives

$$\begin{pmatrix} \psi_s(u) \\ \psi_d(u) \end{pmatrix} = \begin{pmatrix} 0.6337 & -0.6337 \\ 0.2782 & 0.7218 \end{pmatrix} \begin{pmatrix} e^{-0.122u} \\ e^{-3.278u} \end{pmatrix}, \qquad u \geq 0,$$

which is the same as the numerical results obtained in Tsai (2006).

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DISCUSSION

BANGWON KO*

Professors Lu and Tsai have modeled the surplus process as a Markov-modulated process perturbed by diffusion and obtained many interesting results. In this discussion, the systems of integro-differential equations (2.3) and (2.5) in the paper are derived without resorting to equation (2.7) in Cai and Xu (2006). The derivation is motivated by formulas in McDonald (2006), which is a textbook for Society of Actuaries Examinations FM, MFE, and C.

Let $\varphi(u; i)$ denote the expected discounted value of a contingent payoff, given current surplus u and environment $i \in J$. Then

$$\varphi(u; i)(\delta dt) = \mathrm{E}[d\varphi(U(t); i) | U(t) = u, I(t) = i]. \tag{D.1}$$

Introducing $\varphi(u; i)$ gives a unified way to obtain equations (2.3) and (2.5), because the quantities $\phi_d(u; i)$ and $\phi_w(u; i)$ in equations (1.3) and (1.4) correspond to the cases where the contingent payoffs are $I(T < \infty, U(T) = 0)$ and $w(U(T-), |U(T)|)I(T < \infty, U(T) < 0)$, respectively. In pricing tradable assets, δ is the risk-free force of interest, and the expectation in equation (D.1) is taken with respect to a risk-neutral probability measure or an equivalent martingale measure; a consequence of equation (D.1) is the celebrated Black-Scholes differential equation (McDonald 2006, eq. 21.31).

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We can calculate the right-hand side of equation (D.1) by conditioning on whether or not there is an environment change,

$$E[d\varphi(U(t; i)|U(t) = u, I(t) = i] = (1 - \alpha_i dt)E[d\varphi(U(t); i)|U(t) = u, I(t) = i, I(t + dt) = i]$$
$$+ \sum_{k=1,k\neq i}^{m} (\alpha_{ik}dt)E[d\varphi(U(t); i)|U(t) = u, I(t) = i, I(t + dt) = k]. \quad (D.2)$$

Now,

$$E[d\varphi(U(t); i)|U(t) = u, I(t) = i, I(t + dt) = i]$$
$$= \varphi'(u; i)(cdt) + \varphi''(u; i)\left(\frac{1}{2}\sigma_i^2 dt\right) + E[\varphi(u - X^i; i) - \varphi(u; i)](\lambda_i dt), \quad (D.3)$$

and

$$E[d\varphi(U(t); i)|U(t) = u, I(t) = i, I(t + dt) = k] = \varphi(u; k) - \varphi(u, i), \qquad k \neq i. \quad (D.4)$$

Equation (D.3) follows from Itô's lemma for jump-diffusion processes (McDonald 2006, Section 20.8). Here the random variable X^i represents the claim size in environment i, and its distribution is denoted by $F_i(\cdot)$ in the paper. As mentioned in the Remark in the last paragraph of Section 2, if the premium rate depends on the external environment, then c in equation (D.3) is to be replaced by c_i. But it does not seem feasible in practice that the premium rate can change according to the external environment.

It follows from equations (D.1), (D.2), (D.3), and (D.4) and ignoring terms of order $(dt)^2$ that

$$\delta\varphi(u; i) = c\varphi'(u; i) + \frac{1}{2}\sigma_i^2\varphi''(u; i) + \lambda_i E[\varphi(u - X^i; i) - \varphi(u; i)] + \sum_{k=1,k\neq i}^{m} \alpha_{ik}[\varphi(u; k) - \varphi(u; i)],$$

or

$$(\lambda_i + \delta)\varphi(u; i) = c\varphi'(u; i) + \frac{1}{2}\sigma_i^2\varphi''(u; i) + \lambda_i E[\varphi(u - X^i; i)] + \sum_{k=1}^{m} \alpha_{ik}\varphi(u; k) \quad (D.5)$$

because $\Sigma_{i=1}^{m} \alpha_{ik} = 0$. If $\varphi(u; i)$ has the contingent payoff of $w(U(T-),|U(T)|)I(T < \infty, U(T) < 0)$, then the expectation in equation (D.5) is calculated as

$$\int_0^u \phi_w(u - x; i)\, dF_i(x) + \int_u^\infty w(u, x - u)\, dF_i(x),$$

yielding equation (2.3). Similarly, if the contingent payoff is $I(T < \infty, U(T) = 0)$, the expectation in equation (D.5) is expressed as

$$\int_0^u \phi_d(u - x; i)\, dF_i(x);$$

hence we obtain equation (2.5).

The derivation above can be applied to dividend strategy problems; see Ng (2006).

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AUTHORS' REPLY

We are grateful to have such an insightful discussion. Mr. Ko provides a unified and efficient method to derive the systems of integro-differential equations satisfied by the expected discounted penalty functions at ruin caused by a claim and oscillation by introducing a concept of the contingent payoff for the underlying model. We believe that this method can be used for further research on the risk processes perturbed by diffusion.

Mr. Ko also pointed out that equations (2.3) and (2.5) can be derived (without using equation (2.7) of Cai and Xu 2006) by Itô's lemma for jump-diffusion processes. Actually, equation (2.7) of Cai and Xu (2006) is a special case of equation (2.6) of Paulsen and Gjessing (1997), which was also derived by Itô's lemma for a diffusion-perturbed risk process with stochastic interest rate. Alternatively, a traditional approach can be used to obtain equations (2.3) and (2.5) (see Tsai and Willmot 2002). We illustrate the derivation below.

First, we write equation (2.2) again as follows:

$$\begin{aligned}\phi_w(u; i) &= (1 - \alpha_i t - \lambda_i t)e^{-\delta t}\mathbb{E}[\phi_w(V_i(t); i)] \\ &\quad + \lambda_i t e^{-\delta t}\mathbb{E}\left[\int_0^{V_i(t)} \phi_w(V_i(t) - x; i)\, dF_i(x) + \int_{V_i(t)}^{\infty} w(V_i(t), x - V_i(t))\, dF_i(x)\right] \\ &\quad + t e^{-\delta t} \sum_{k=1,k\neq i}^{m} \alpha_{ik}\mathbb{E}[\phi_w(V_i(t); k)] + o(t), \qquad (1)\end{aligned}$$

where $V_i(t) = u + ct + \sigma_i W(t)$, $i \in J$, and $o(t)/t \to 0$ as $t \to 0$. By expanding $\phi_w(V_i(t); i)$ as a Taylor series with respect to u, and by the fact that $\mathbb{E}[W(t)] = \mathbb{E}[W^3(t)] = 0$ and $\mathbb{E}[W^2(t)] = \mathrm{Var}[W(t)] = t$, we have

$$\mathbb{E}[\phi_w(V_i(t); i)] = \phi_w(u; i) + [c_i\phi'_w(u; i) + D_i\phi''_w(u; i)]t + o(t),$$

where $D_i = \frac{1}{2}\sigma_i^2$ for $i \in J$. By substituting the preceding expression into (1), and noting that $e^{-\delta t} = 1 - \delta t + o(t)$, we then get

$$\begin{aligned}\phi_w(u; i) &= [1 - (\alpha_i + \lambda_i + \delta)t]\{\phi_w(u; i) + [c_i\phi'_w(u; i) + D_i\phi''_w(u; i)]t\} \\ &\quad + \lambda_i t\mathbb{E}\left[\int_0^{V_i(t)} \phi_w(V_i(t) - x; i)\, dF_i(x) + \int_{V_i(t)}^{\infty} w(V_i(t), x - V_i(t))\, dF_i(x)\right] \\ &\quad + t \sum_{k=1,k\neq i}^{m} \alpha_{ik}\{\phi_w(u; k) + [c_i\phi'_w(u; k) + D_i\phi''_w(u; k)]t\} + o(t). \qquad (2)\end{aligned}$$

Now, dividing by t on both sides of (2) and letting $t \to 0$, and noting that $\alpha_{ii} = -\alpha_i$ for $i \in J$, gives a system of integro-differential equations satisfied by $\phi_w(u; i)$ given the initial surplus u and the initial environment $i \in J$:

$$\begin{aligned}(\lambda_i + \delta)\phi_w(u; i) &= D_i\phi''_w(u; i) + c_i\phi'_w(u; i) + \sum_{k=1}^{m} \alpha_{ik}\phi_w(u; k) \\ &\quad + \lambda_i\left[\int_0^u \phi_w(u - x; i)\, dF_i(x) + \int_u^\infty w(u, x - u)\, dF_i(x)\right], \qquad u > 0. \qquad (3)\end{aligned}$$

which is equation (2.3). A similar approach applied to ϕ_d gives equation (2.5).

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Discussions on this paper can be submitted until October 1, 2007. The authors reserve the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.