An Actuarial Premium Pricing Model for Nonnormal Insurance and Financial Risks in Incomplete Markets

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Abstract

A model for pricing insurance and financial risks, based on recent developments in actuarial premium principles with elliptical distributions, is developed for application to incomplete markets and heavy-tailed distributions. The pricing model involves an application of a generalized variance premium principle from insurance pricing to the pricing of a portfolio of nontraded risks relative to a portfolio of traded risks. This pricing model for a portfolio of insurance or financial risks reflects preferences for features of the distributions other than mean and variance, including kurtosis. The model reduces to the Capital Asset Pricing Model for multinormal portfolios and to a form of the CAPM in the case where the traded and nontraded risks have the same elliptical distribution.

1. Introduction

Various pricing or premium principles have been proposed for pricing insurance contracts. Wang (2002) has proposed a framework for enterprise risk management based on the Esscher premium approach to insurance pricing and illustrated how it is consistent with the Capital Asset Pricing Model (CAPM) under specified distributional assumptions. Wang (2003) shows how this approach relates to economic premium principles developed in Bühlmann (1980, 1984). In incomplete markets, the Esscher premium principle in insurance has been applied to pricing financial contracts as in Gerber and Shiu (1994). More recently the family of elliptical distributions has been the topic of actuarial research for models of a portfolio of risks $X_1, \ldots, X_n$ considered as n random variables.

Premium principles for pricing insurance risks have been developed for the elliptical family. Landsman (2004) proposed a generalization of Wang's exponential tilting to elliptical tilting, which is natural for elliptical distributions. The elliptical family is richer than the traditional multivariate normal model, because it contains many important distributions such as the generalized Student, the exponential power, and others as well as the normal distribution. The elliptical family also includes heavy-tailed distributions, which have become popular in the modeling of stock daily returns (see McDonald 1996). An important advantage of the elliptical family is that they have a linear dependence structure similar to the Normal family and they preserve the property of symmetric marginals. However, the elliptical family is more general than the normal distribution and is not specified by only the expectation and variance-covariance structure.

Theoretical models of equilibrium asset pricing were originally developed assuming mean variance preferences for investors or multinormal distributions of returns. Practical applications of portfolio models often assume multinormal returns. The original CAPM (Sharpe 1964; Lintner 1965; Mossin 1966) was developed in the mean variance preference model framework. Asset-pricing models, such as

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the CAPM, have been shown to hold for more general asset return distributions. In general the elliptical class of distributions has been shown to remain consistent with a form of CAPM (Owen and Rabinovitch 1983; Ingersoll 1987). Berk (1997) shows when mean variance maximization is consistent with expected utility maximization. However, CAPM models allow only for the linear dependence structure of the return distribution and do not explicitly reflect other moments or other characteristics of the distribution in the pricing. In insurance risk and option pricing, models based on the normal distribution are not satisfactory.

Insurance markets are incomplete in the sense that individuals cannot trade insurance contracts on assets or property nor can they hold arbitrary amounts of insurance coverage. For this reason standard asset-pricing models such as CAPM cannot be readily used in insurance markets.

This paper develops an insurance premium principle derived in Landsman (2004) for pricing risks in incomplete markets. The pricing principle is novel and interesting since it allows for a preference to be placed on distributions that have the same mean and variance but with, for example, different kurtosis, whenever these moments exist. Even if these moments do not exist, the pricing principle can still be applied. The approach to taking into account incomplete markets is not new and uses a traded portfolio of risks, which is assumed to have a joint elliptical distribution, to price a portfolio of nontraded risks, also assumed to have a joint elliptical distribution, which may be different from that of the traded asset portfolio. An important contribution of this paper is that the pricing principle allows for kurtosis and for the application to the pricing of risks where moments such as variance do not exist.

Calibrating a pricing model on a set of actively traded securities in order to price less actively traded securities is an approach used in practice by many banks in marking their portfolios to model or market. This allows a market price of risk to be determined from the traded asset portfolio. This is then used in the pricing of the nontraded risks taking into account the relative differences in the underlying distributions of the two portfolios. We use this approach to implement the pricing principle.

A main benefit of the pricing principle is that it captures the effect of differences in traded and nontraded distributions allowing for elliptical distributions. The pricing principle reduces to a form of CAPM if the traded and nontraded portfolios have the same underlying distribution. It is derived as an extension of the variance premium principle, which is used in practice by actuaries for determining risk loads for insurance and reinsurance contracts.

2. ESSCHER PREMIUM, ASSET PRICING, AND CAPM

The Esscher premium principle is one of many important insurance premium principles developed for pricing insurance risks. This premium principle determines the net insurance premium for a loss X as

$$\Pi(X) = \frac{E(Xe^{\lambda X})}{E(e^{\lambda X})}, \qquad \lambda > 0.$$

Gerber and Shiu (1994) have developed the Esscher transform approach to option pricing. This is equivalent to using an exponential form for the state price density and is a pricing model that has been considered for asset pricing in incomplete markets.

The Esscher premium principle has been modified to allow for an economic interpretation of the state price density implied in the pricing model. Following the work of Bühlmann (1980, 1984) on economic premium principles, Wang (2002) introduced the exponential tilting of a risk X_1 induced by a reference portfolio X_2, as

$$H_\lambda(X_1, X_2) = \frac{E(X_1 \exp(\lambda X_2))}{E(\exp(\lambda X_2))}, \qquad \lambda > 0.$$

Landsman (2004) showed, provided that the risks have a finite variance-covariance structure, that the Esscher premium and exponential tilting are equivalent to a (co)variance premium principle. This can be written as

$$H_\lambda(X_1, X_2) = \mu_{X_1} + \lambda\rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2} + O(\lambda^2), \qquad \lambda \to 0, \tag{2.1}$$

where μ_{X_1} is the expected value of X_1, ρ_{X_1,X_2} is the correlation between X_1, X_2, σ_{X_1} is the standard deviation of X_1, σ_{X_2} is the standard deviation of X_2, and λ is a parameter reflecting the price trade-off between the mean and covariance. For the case of a multinormal distribution for the risks, the result is exact, referred to as the variance premium, and given by

$$VP(X_1, X_2) = \mu_{X_1} + \lambda\rho_{X_1,X_2}\sigma_{X_1}\sigma_{X_2}, \tag{2.2}$$

which also allows a derivation of the CAPM.

Consider the modeling of asset returns where X_2 is the portfolio of returns on the market portfolio denoted by M and X_1 is the return on a security included in the market portfolio M. P_M is the expected return on the market portfolio M. Define λ as

$$\lambda = \frac{P_M - \mu_M}{\sigma_M^2},$$

where $\mu_M = E(M)$ is the expected return on the market portfolio M and σ_M^2 is the variance of the return on the market portfolio. The variance premium principle then gives the expected return for X_1 as

$$P_{X_1} = VP(X_1, M) = \mu_{X_1} + \frac{\operatorname{cov}(X_1, M)}{\sigma_M^2}[P_M - \mu_M], \tag{2.3}$$

which is the CAPM.

Valdez and Chernih (2003) were the first to develop an extension of the Esscher premium to elliptical portfolios. However, using the Esscher premium for the pricing of elliptical portfolios as in Valdez and Chernih is limited because the Esscher premium does not exist for the most important elliptical distributions such as the Student's t family. Their result also only reflects deviations from the normal distribution in the term of order $O(\lambda^2)$ as is evident from (2.1). For these reasons a more general pricing approach is required.

We develop a generalized version of the variance premium.

3. PRICING WITH THE GENERALIZED VARIANCE PREMIUM

A generalization of the Esscher insurance premium pricing approach is the basis for our derivation of a pricing model that can be applied to insurance and financial risks allowing for higher moments and nontraded risks. The generalization used in this paper is a natural extension of the covariance premium principle in insurance and recovers the CAPM in asset pricing under the assumption of multinormal distributions. The standard CAPM result does not reflect deviations of a risk from the normal distribution. Even though the CAPM has been shown to hold for elliptical distributions, the form of the model does not capture preferences for moments other than mean and variance.

Landsman (2004) introduced a generalized variance premium given by

$$GVP(X_1, X_2; E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)) = \mu_1 + \lambda\beta_{g_2}(h_2)\rho_{12}\sigma_1\sigma_2, \tag{3.1}$$

which will be used to derive an insurance and asset-pricing model that can be regarded as an extension of the classical CAPM to allow for more general preferences other than for mean and variance. In (3.1), (X_1, X_2) have a joint elliptical distribution with density generator h_2, μ_1 is the expected value of X_1, $\rho_{12}\sigma_1\sigma_2$ is the (generalized) covariance between X_1, X_2, λ is a price-of-risk parameter, and $\beta_{g_2}(h_2)$ captures the difference between the joint distribution of X_1, X_2 and a reference distribution that is assumed to have an elliptical distribution with density generator g_2. If the density generator for the joint distribution of X_1, X_2 and the reference distribution are the same, then $\beta_{g_2}(h_2) = 1$.

The classical (co)variance premium principle (2.2) can be interpreted in the form of modern financial pricing by writing it as

$$VP(X_1, X_2) = E(Z_2X_1),$$

where the stochastic discount factor is

$$Z_2 = 1 + \lambda(X_2 - \mu_2),$$

which is an approximation to exponential tilting. The generalized (co)variance premium principle (3.1) has an analogous interpretation with the stochastic discount factor given by

$$Z_2 = 1 + \lambda(X_2 - \mu_2)\beta_{g_2}(h_2)$$

and is an approximation to elliptical tilting.

We use the generalized variance premium and calibrate it to a portfolio with an elliptical distribution, using a reference portfolio for elliptical tilting. Using the calibrated parameters from the traded portfolio prices, the generalized variance premium is then applied to another portfolio of risks, also assumed to have a known elliptical distribution, using the same reference portfolio for elliptical tilting. The pricing model uses a market price of risk that is calibrated to the traded portfolio prices, and allowance is made for preferences over differences in the traded and nontraded portfolios through the $\beta_{g_2}(h_2)$ functional. This includes allowance for heavy tails of the underlying distribution and concentration of probability mass around the center of the distribution.

For completeness, we will briefly review some results for elliptical distributions and for the modification of the variance premium that is the basis of the pricing model. Let $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^{\mathrm{T}}$ be a vector of expectations, $\boldsymbol{\Sigma}$ be an $n \times n$ positive definite matrix, and $g_n(x)$ a nonnegative function. The random vector $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ is said to have an n-variate elliptical distribution with vector of expectations $\boldsymbol{\mu}$, covariance matrix $\boldsymbol{\Sigma}$, and density generator g_n if its density is represented as

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{(2\pi)^{n/2}\sqrt{|\boldsymbol{\Sigma}|}}\, g_n\left[\frac{1}{2}\,(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu})\right],$$

and we write $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. The normalizing constant c_n is explicitly determined by

$$c_n = \Gamma(n/2)\left[\int_0^\infty x^{n/2-1} g_n(x)\, dx\right]^{-1}, \tag{3.2}$$

which is assumed to be finite.

More details on the elliptical family and the conditions for the generator g_n are given in Kelker (1970), Fang, Kotz, and Ng (1987), Embrechts, McNeil, and Straumann (1999), and Landsman and Valdez (2003). In the last paper, results for the Tail-VaR risk measure are derived for the elliptical family.

Let $E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)$ be a bivariate elliptical distribution. In Landsman (2004) the elliptical tilting of risk X_1 induced by X_2 with $E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)$ was defined as

$$H_\lambda(X_1, X_2; E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)) = \frac{1}{C}\int\int_{R^2} x_1 \frac{g_2[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu}) - \lambda x_2]}{g_2[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu})]} f_{\mathbf{x}}(x_1, x_2)\, dx_1\, dx_2, \qquad \lambda > 0, \tag{3.3}$$

where

$$C = \int_{R^2} \frac{g_2[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu}) - \lambda x_2]}{g_2[\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu})]} f_{\mathbf{x}}(x_1, x_2)\, dx_1\, dx_2.$$

It is shown that $H_\lambda(X_1, X_2; E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2))$ coincides with Wang's exponential tilting if $E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)$ is bivariate normal. This immediately follows from (3.3), because for the normal family the generator has the form $g_2(x) = \exp(-x)$. It is also shown that in general the vector $\mathbf{X} = (X_1, X_2)^{\mathrm{T}}$ does not need to have the same distribution as used for elliptical tilting, $E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)$, but if $\mathbf{X}$ is also distributed $E_2(\mu, \boldsymbol{\Sigma}, g_2)$, then

$$H_\lambda(X_1, X_2; E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)) = \mu_1 + \lambda\rho_{12}\sigma_1\sigma_2,$$

where $\sigma_i = \sqrt{\sigma_{ii}}$, $i = 1, 2$,

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2},$$

and σ_{ij}, $i, j = 1, 2$, are the elements of the matrix Σ. Naturally, ρ_{12} coincides with the correlation coefficient between X_1, X_2 if the covariance of $\mathbf{X}$ exists.

If the vector $\mathbf{X}$ is elliptically distributed with the same variance-covariance structure but with a different generator h_2, so that $\mathbf{X} \sim E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, h_2)$, then the following asymptotic representation is derived:

$$H_\lambda(X_1, X_2; E_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_2)) = \mu_1 + \lambda\beta_{g_2}(h_2)\rho_{12}\sigma_1\sigma_2 + o(\lambda), \qquad \lambda \to 0,$$

where

$$\beta_{g_2}(h_2) = \frac{1}{4} c_2 \int_0^\infty x J_2\left(\frac{1}{2} x\right) h_2(x)\, dx \tag{3.4}$$

and

$$J_2(u) = -\frac{d}{du} \log g_2(u), \tag{3.5}$$

where c_2 is given in (3.2).

The new pricing principle developed here uses a generator g_2 as a reference generator for the pricing of risks with different elliptical distributions. The variance premium is the generalized variance premium with respect to a normal distribution reference generator so that this pricing principle includes the CAPM. The difference between the reference generator and the generator of the underlying distribution of the vector $\mathbf{X} = (X_1, X_2)^T$ determines the difference between the generalized variance premium and the standard variance premium.

4. PRICING WITH ELLIPTICAL DISTRIBUTIONS

The new pricing principle for incomplete markets will now be developed in more detail. Later we will discuss the special case of the generalized Student's t model, characterized by the shape (power parameter) p. The generalized Student's t distribution is of interest because of the empirical support in the asset-modeling literature for this distribution for many different countries, as found in studies including Aparicio and Estrada (2001), Kim and Kon (1994), and Broca (2002).

For the elliptical assumption the multivariate distribution of a portfolio of risks is determined not only by the expectations and variance-covariance structure, but also by the density generator of the elliptical distribution. Different assets or portfolios in general will have different density generators reflecting their different price characteristics. For example, the generalized Student family that we use here has the density generator

$$g_n(u) = \left(1 + \frac{u}{k_p}\right)^{-p},$$

where

$$c_n = \frac{\Gamma(p)}{\Gamma(p - n/2)} (2\pi k_p)^{-n/2},$$

n is a dimension of the portfolio, and k_p is a normalizing constant, with different p for the shape parameter. Varying p allows the fitting of the tail behavior to a wide range of shapes of distributions, some of which are so heavy tailed that the variance is infinite.

The density generators can differ by the functional form of the generator as well as by a parameter such as in the generalized Student case. Since the generator reflects an important characteristic of the joint distribution, it is assumed that investor preferences reflect the density generator as well as the variance-covariance structure of the distribution. This is an enhancement of the standard mean-variance preference assumption, allowing for higher moments, that is the basis of the CAPM.

Let $\mathbf{X} = (X_1, \ldots, X_n)^{\mathrm{T}}$ be a multivariate portfolio of random returns (risks) with a distribution from the elliptical family $E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$. There is, in fact, along with g_n, a sequence $(g_1, \ldots, g_{n-1})$ of density generators of the marginal distributions of this family. We use one of the important properties of the elliptical family of distributions, which motivates its use in asset pricing and portfolio theory: for an $m \times n$ matrix of rank $m \leq n$, $\mathbf{A}$, and an m-dimensional column-vector $\mathbf{b}$, we have

$$\mathbf{AX} + \mathbf{b} \sim E_m(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T, g_m). \tag{4.1}$$

This means that the return on a portfolio of assets, $R = \sum_{j=1}^{n} \gamma_j X_i$, where X_i is the return on asset i, also has a distribution from the same elliptical family. If we set $\boldsymbol{\mu}_{i,R} = (\mu_i, \mu_R)^T$, $\mu_R = \sum_{j=1}^{n} \gamma_j \mu_j$,

$$\boldsymbol{\Sigma}_{i,R} = \begin{pmatrix} \sigma_i^2 & \sigma_{iR} \\ \sigma_{iR} & \sigma_R^2 \end{pmatrix},$$

with $\sigma_i^2 = \sigma_{ii}$, $\sigma_{iR} = \sum_{j=1}^{n} \gamma_j \sigma_{ij}$, $\sigma_R^2 = \sum_{i,j=1}^{n} \gamma_i \gamma_j \sigma_{ij}$, then we obtain the result that the bivariate vector $(X_i, R)^{\mathrm{T}}$ is distributed $E_2(\boldsymbol{\mu}_{i,R}, \boldsymbol{\Sigma}_{i,R}, g_2)$. One of the reasons that the elliptical family can generalize standard multinormal portfolio and asset-pricing results is that it has aggregation properties similar to those of the multinormal distribution.

Now assume that a particular density generator $\mathring{g}_n$ is chosen as a reference generator for an n-dimensional portfolio. As noted previously, together with $\mathring{g}_n$ we have also a sequence of generators $(\mathring{g}_1, \ldots, \mathring{g}_{n-1})$ for all the marginal distributions of dimensions from $n - 1$ to 1.

The generalized (co)variance premium principle (3.1) provides a pricing model for individual returns of the $\mathbf{X}$-portfolio given by

$$GVP(X_i, R; E_2(\boldsymbol{\mu}_{i,R}, \boldsymbol{\Sigma}, \mathring{g}_2)) = \mu_i + \lambda \beta_{\mathring{g}_2}(g_2) \rho_{iR} \sigma_i \sigma_R, \qquad i = 1, \ldots, n, \tag{4.2}$$

where $\beta_{\mathring{g}_2}(g_2)$ reflects preferences over other features of the distribution apart from the covariance structure.

Now consider traded portfolio returns with a vector of portfolio proportions $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^{\mathrm{T}}$ of the individual assets and denote by P_M the expected market return (price) of the portfolio $M = \boldsymbol{\alpha}^T \mathbf{X}$. The model then gives the expected return (price) of the portfolio as

$$P_M = \mu_M + \lambda \beta_{\mathring{g}_2}(g_2) \sigma_M^2,$$

where $\mu_M = \boldsymbol{\alpha}^T \boldsymbol{\mu}$, $\sigma_M^2 = \boldsymbol{\alpha}^T \boldsymbol{\Sigma} \boldsymbol{\alpha}$. We can then determine λ in (4.2) from the traded prices as

$$\lambda_M = \frac{P_M - \mu_M}{\beta_{\mathring{g}_2}(g_2) \sigma_M^2}. \tag{4.3}$$

This is a portfolio-based price of risk consistent with the market price of risk in standard portfolio theory, but taking into account other features of the elliptical distribution, including the concentration of the probability mass around the center of the distribution and heavy tails, through the coefficient $\beta_{\mathring{g}_2}(g_2)$.

This traded market price of risk λ_M is then applied to price other nontraded asset portfolios. Assume that $\mathbf{Y} = (Y_1, \ldots, Y_m)^{\mathrm{T}}$ are the returns on a portfolio of risks from the elliptical family with another sequence of density generators $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_m)$, so that $\mathbf{Y} \sim E_m(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}, \tilde{g}_m)$. Let $\tilde{M} = \tilde{\boldsymbol{\alpha}}^T \mathbf{Y}$ be an arbitrary portfolio from $\mathbf{Y}$ that we will refer to as a nontraded basis portfolio. Then for any risk Y_i, $i = 1, 2, \ldots, m$ we can substitute (4.3) into (3.1) to obtain the new pricing model for an incomplete market given by

$$P_{Y_i} = \mu_{Y_i} + \frac{P_M - \mu_M}{\sigma_M^2} \frac{\beta_{\mathring{g}_2}(\tilde{g}_2)}{\beta_{\mathring{g}_2}(g_2)} \rho_{i\tilde{M}} \sigma_{Y_i} \sigma_{\tilde{M}}, \qquad i = 1, \ldots, n,$$

where variance-covariance elements are obtained from the matrix $\tilde{\tilde{\boldsymbol{\Sigma}}}$.

If the generator of the reference portfolio and the generator of the traded portfolio are the same, so that $\mathring{g}_2 = g_2$, then it has been shown by Landsman (2004) that

$$\beta_{\mathring{g}_2}(g_2) = \beta_{g_2}(g_2) = 1, \tag{4.4}$$

and formulas for λ and for expected returns (prices) simplify to an extension of the CAPM model with

$$\lambda_M = \frac{P_M - \mu_M}{\sigma_M^2}$$

and

$$P_{Y_i} = \mu_{Y_i} + \frac{P_M - \mu_M}{\sigma_M^2} \beta_{g_2}(\tilde{g}_2) \rho_{i\tilde{M}} \sigma_{Y_i} \sigma_{\tilde{M}}, \qquad i = 1, \ldots, n. \tag{4.5}$$

For the case of a complete market where pricing is of the securities from the traded portfolio, we take M as the basis portfolio, and, because in this case $g_2 = \tilde{g}_2$, we have $\beta_{g_2}(\tilde{g}_2) = \beta_{g_2}(g_2) = 1$. Formula (4.5) then reduces to the classical CAPM

$$P_{Y_i} = \mu_{Y_i} + \frac{P_M - \mu_M}{\sigma_M^2} \operatorname{cov}(Y_i, M). \tag{4.6}$$

Although the main idea underlying formula (4.5) is the generalization of the (co)variance premium principle, developed from the actuarial approach to premium pricing, the linear regression interpretation of the CAPM (as a best linear approximation) considered in Magill and Quinzii (1996, Section 17.6) assists in understanding the new pricing principle. In fact, the right-hand side of (4.6), which we refer to as the classical CAPM, can be rewritten as

$$P_{Y_i} = \hat{E}[Y_i | (1, (M - \mu_M))]_{|M=P_M},$$

where $\hat{E}[X|(Y, Z)]$ is the orthogonal projection of the random variable X onto the linear space of random vectors (Y, Z). Thus if

$$(\hat{c}_o, \hat{c}_1) = \arg \inf_{c_o, c_1} (Y_i - (c_o + c_1(M - \mu_M)))^2, \tag{4.7}$$

then we have

$$P_{Y_i} = [\hat{c}_o + \hat{c}_1(M - \mu_M)]_{|M=P_M}, \tag{4.8}$$

with

$$\hat{c}_o = \mu_{Y_i},$$

$$\hat{c}_1 = \frac{\operatorname{cov}(Y_i, M)}{\sigma_M^2}.$$

The extended version of the CAPM presented in (4.5), where traded and nontraded portfolios are treated differently for pricing, can be interpreted using the ideas of projection and the regression interpretation starting with

$$P_{Y_i} = \hat{E}[Y_i | (1, (\tilde{M} - \mu_{\tilde{M}}))]_{|\tilde{M}=P_{\tilde{M}}}.$$

Similar to (4.7) and (4.8) we have, denoting

$$(\hat{c}_o,\ \hat{c}_1) = \arg \inf_{c_o, c_1} (Y_i - (c_o + c_1(\tilde{M} - \mu_{\tilde{M}})))^2,$$

the result

$$P_{Y_i} = [\hat{c}_o + \hat{c}_1(\tilde{M} - \mu_{\tilde{M}})]_{|\tilde{M}=P_{\tilde{M}}},$$

where now

$$\hat{c}_o = \mu_{Y_i},$$

$$\hat{c}_1 = \frac{\text{cov}(Y_i, \tilde{M})}{\sigma^2_{\tilde{M}}}.$$

Now if we take

$$P_{\tilde{M}} = \mu_{\tilde{M}} + \beta_{g_2}(\tilde{g}_2)(P_M - \mu_M)\,\frac{\sigma^2_{\tilde{M}}}{\sigma^2_M},$$

we obtain our generalized pricing principle. Thus the coefficient $\beta_{g_2}(\tilde{g}_2)$ is used to modify the price of risk of the traded portfolio to derive the price of the nontraded basis portfolio. In the case of a complete market, or pricing securities from the traded portfolio, we take $\tilde{M} = M$, so that there is no difference in the variances of the market and basis portfolios, resulting in $(\sigma^2_{\tilde{M}}/\sigma^2_M) = 1$ and $\beta_{g_2}(g_2) = 1$, and we then have $P_{\tilde{M}} = P_M$, which shows that our generalized pricing principle reduces to the classical CAPM (4.6).

To apply the model in practice it is necessary to consider the determination of the appropriate reference portfolio. We would also like to consider specific distributions such as the special case of multivariate Student distributions that have empirical support in daily stock return modeling. In the next section we consider the selection of the reference portfolio and derive the form of the pricing principle for the multivariate Student distribution.

5. MULTIVARIATE STUDENT PRICING MODEL

We will derive an explicit form of the model presented in the previous section by assuming that the asset portfolio distribution belongs to the Student family. We consider the multivariate Student distribution in the form given by

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{c_{n,p}}{(2\pi)^{n/2}\sqrt{|\boldsymbol{\Sigma}|}}\left[1 + \frac{(\mathbf{x} - \boldsymbol{\mu})^T\,\boldsymbol{\Sigma}^{-1}\,(\mathbf{x} - \boldsymbol{\mu})}{2k_{n,p}}\right]^{-p}, \tag{5.1}$$

where

$$c_{n,p} = \frac{\Gamma(p)}{\Gamma(p - n/2)k^{n/2}_{n,p}}, \tag{5.2}$$

and $k_{n,p}$ is the normalized constant (see details in Landsman and Valdez (2003)).

For any given expectation vector $\boldsymbol{\mu}$ and "covariance" matrix $\boldsymbol{\Sigma}$, the Student distribution has a shape parameter p, which when varied will change the tail behavior and probability mass concentration in the center of the distribution. The multivariate normal distribution is simply the limiting case of the Student when $p = \infty$. The density generator of this family is

$$g_n(u) = \frac{1}{\left(1 + \dfrac{u}{k_{n,p}}\right)^p}. \tag{5.3}$$

The following theorem shows that the density generator of the Student family depends on the dimension of this family.

Theorem 1

Suppose $g_n(u)$ is a density generator of an n-variate Student family with shape (power) parameter p, having form (5.3). The density generator of its $n-1$–variate has power parameter equal to $p-\frac{1}{2}$, given by

$$g_{n-1}(u) = \frac{1}{\left(1+\dfrac{u}{k_{n-1,p}}\right)^{p-1/2}}, \tag{5.4}$$

where $k_{n-1,p} = k_{n,p}$.

PROOF

In the integral form of the marginal $n-1$–variate Student density with zero expectation and unit matrix Σ we obtain, by changing the variable of integration to $z = x_n/(1+(x_1^2+\cdots+x_{n-1}^2)/(2k_{n,p}))^{1/2}$,

$$\begin{aligned} f(x_1,\ldots,x_n) &= \frac{c_{n,p}}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\left(1+\frac{x_1^2+\cdots+x_{n-1}^2}{2k_{n,p}}+\frac{x_n^2}{2k_{n,p}}\right)^{-p} dx_n \\ &= \frac{1}{(2\pi)^{(n-1)/2}}\frac{c_{n,p}}{c_{1,p}}\left(1+\frac{x_1^2+\cdots+x_{n-1}^2}{2k_{n,p}}\right)^{-p}\left(1+\frac{x_1^2+\cdots+x_{n-1}^2}{2k_{n,p}}\right)^{1/2} \\ &\times \int_{-\infty}^{\infty}\frac{c_{1,p}}{(2\pi)^{1/2}}\left(1+\frac{z^2}{2k_{n,p}}\right)^{-p} dz. \end{aligned} \tag{5.5}$$

As the last integral in (5.5) is equal to $\sqrt{(k_{n,p}/k_{1,p}}$ and automatically from (5.2)

$$\frac{c_{n,p}}{c_{1,p}}\sqrt{\frac{k_{n,p}}{k_{1,p}}} = c_{n-1,p},$$

with $k_{n-1,p} = k_{n,p}$, we have

$$f(x_1,\ldots,x_n) = \frac{c_{n-1,p}}{(2\pi)^{(n-1)/2}}\left(1+\frac{x_1^2+\cdots+x_{n-1}^2}{2k_{n,p}}\right)^{-(p-1/2)}.$$

Thus the density generator of the Student $n-1$–variate marginal distribution has the form given by (5.4). ■

5.1 Univariate Generalized Student's *t* Distribution

From (5.1) the univariate Generalized Student's *t* distribution has the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2k_{1,p}}B\left(\dfrac{1}{2}, p-\dfrac{1}{2}\right)}\left[1+\frac{(x-\mu)^2}{2k_{1,p}\sigma^2}\right]^{-p}, \qquad p > \frac{1}{2}. \tag{5.6}$$

If $p \le \frac{1}{2}$, the density does not exist. For $\frac{1}{2} < p \le 1$, the expectation does not exist. If $1 < p \le \frac{3}{2}$, the expectation exists, but the variance is still infinite. For $p > \frac{3}{2}$, the generalized Student's *t* distribution has finite variance. For this case if we put $k_{1,p} = p - \frac{3}{2}$, the variances corresponding to the densities (5.6) are all equal to

$$V(X) = \sigma^2, \qquad p > \frac{3}{2}.$$

Therefore we use

$$k_{1,p} = \begin{cases} p - \dfrac{3}{2}, & p > \dfrac{3}{2} \\ \dfrac{1}{2}, & \dfrac{1}{2} < p \le \dfrac{3}{2} \end{cases}. \tag{5.7}$$

We will refer to the distribution with density (5.6) as the univariate generalized Student's *t* distribution (UGST). This distribution is discussed by many authors, for example, McDonald (1996).

We will show how the pricing model takes into account preferences over distributions allowing for other characteristics of the distributions other than the variance.

In Figure 1 the univariate generalized Student's *t* distribution densities are given with the same mean $\mu = 0.5$ and $\sigma^2 = 1$ and with differing values for $p = 1, 1.5, 1.6, 1.8, 2.2$, and ∞. For $p = 1.6, 1.8, 2.2$ and ∞ the variances exist and are all equal with $\sigma^2 = 1$. Differing values of p can be seen to capture characteristics of the distribution on which a ranking of risk preferences can be determined. Distributions with smaller values of p, provided $p > 1.5$, are preferred for any distribution with equal mean and variance. This is so since the density contains more mass close to the center of the distribution for smaller values of p. However, for values of $p < 1.5$ the variance does not exist although the densities have the same μ and σ^2 parameters. Preferences can still be ranked over distributions in this case since smaller values of p result in heavier tails so that higher values of p will now be preferred.

This preference ordering is given by the coefficient $\beta_{p_0}(p)$. Figure 2 shows the coefficient $\beta_{p_0}(p)$ for varying values of p with $p_0 = 1.5$. By comparing the densities in Figure 1 with the behavior of the coefficient $\beta_{p_0}(p)$ in Figure 2 we confirm that the preference ranking of densities is given by the coefficient $\beta_{p_0}(p)$. This preference ordering does not depend on the reference density parameter p_0 and coincides with an ordering according to the kurtosis of the densities for values of the shape parameter where the kurtosis exists, since the coefficient of kurtosis is given by

Figure 1

Student's *t* Densities with $\mu = 0.5$ and $\sigma^2 = 1$ for Varying Values of *p*

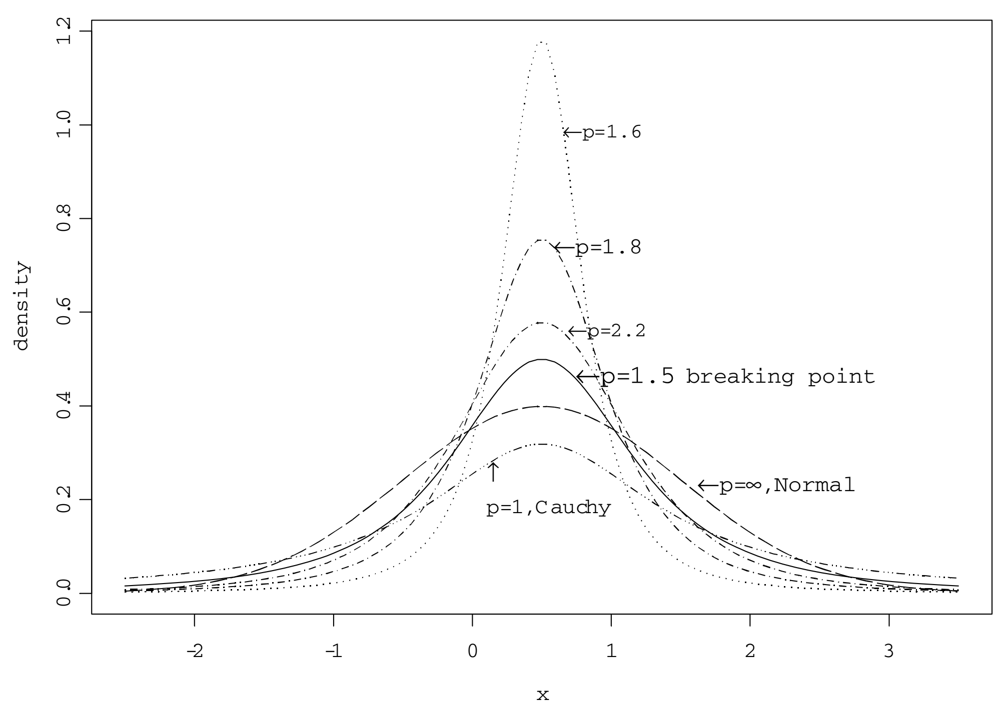
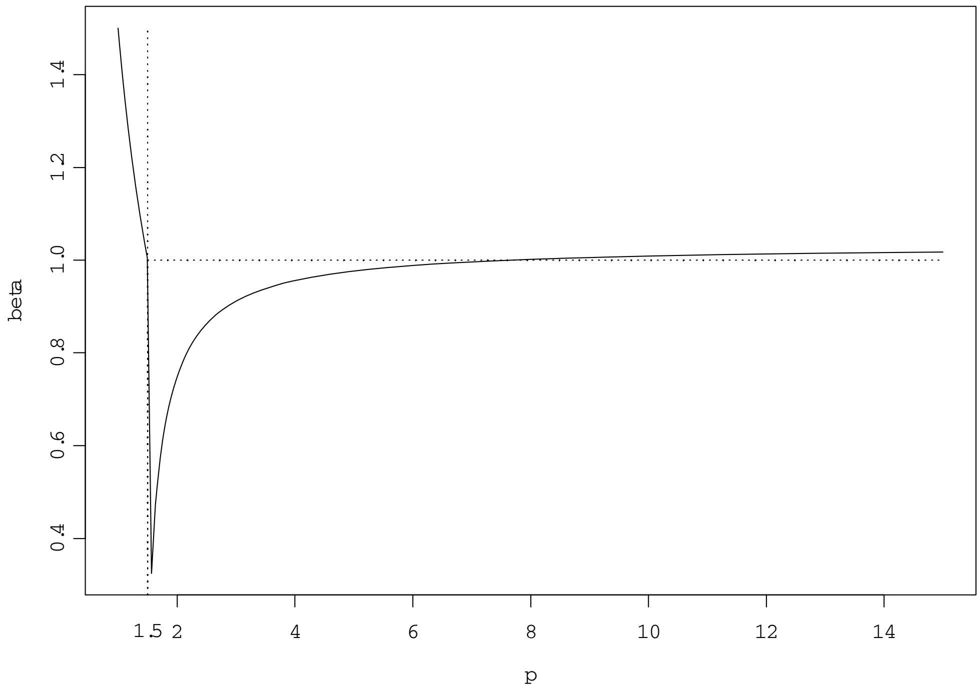


Figure 2
Coefficient $\beta_{p_0}(p)$ for Varying p and with Reference Point $p_0 = 1.5$



$$\gamma = \frac{6}{2p - 5}, \qquad p > \frac{5}{2}.$$

For values of p where the kurtosis does not exist, the coefficient $\beta_{p_0}(p)$ does exist and provides a preference ordering for the densities. The coefficient $\beta_{p_0}(p)$ preserves the natural ordering of the densities even allowing for the case $p = 1.5$, which we refer to as a breaking point, and for $p = \infty$, which is the normal distribution case. This is the preference ordering that is the basis of the pricing model and exists even when the variance or coefficient of kurtosis does not exist. This is an important aspect of the risk-ordering properties of the pricing model.

5.2 Generalized (Co)variance Premium Principle with the Student Family

Suppose $\mathbf{X} = (X_1, \ldots, X_n)^T$ is a multivariate portfolio of risks having a Student distribution with vector of expectations $\boldsymbol{\mu}$, covariance structure matrix $\boldsymbol{\Sigma}$, and density generator from (5.3). We define

$$k_{n,p} = \begin{cases} p - (n+2)/2, & p > (n+2)/2 \\ \frac{1}{2}, & n/2 < p \leq (n+2)/2 \end{cases}. \tag{5.8}$$

Denote the density generator (5.3) with $k_{n,p}$ from (5.8) by $g_{n,p}$. For the case $n = 2$ we have

$$k_{2,p} = \begin{cases} p - 2, & p > 2 \\ \frac{1}{2}, & 1 < p \leq 2 \end{cases}.$$

Note that the bivariate Cauchy is generalized Student's t (GST) with $p = 1.5$. The following theorem immediately follows from Theorem 1 and from properties of the univariate GST distribution.

Theorem 2

(1) *For* $p > (n/2) + 1$, $k_{n,p} = p - (n/2 + 1)$ *reduces to the property that* $\boldsymbol{\Sigma} = cov(\mathbf{X})$; (2) *For* $(n/2) + 1 \geq p > (n/2) + (1/2)$ *the covariance matrix does not exist, but the vector of expectations exists; and* (3) *For* $(n/2) + (1/2) \geq p > (n/2)$ *the expectations do not exist, but the density of* $\mathbf{X}$ *exists.*

In Figure 3 we show graphs of bivariate GST densities with common vector of expectations $\mu = (0.9, 0.9)^{T}$ and matrix

$$\boldsymbol{\Sigma} = \begin{Vmatrix} 1 & \rho \\ \rho & 1 \end{Vmatrix}, \qquad \rho = 0.4,$$

for different parameters $p = 1.5, 2.1, 4, \infty$.

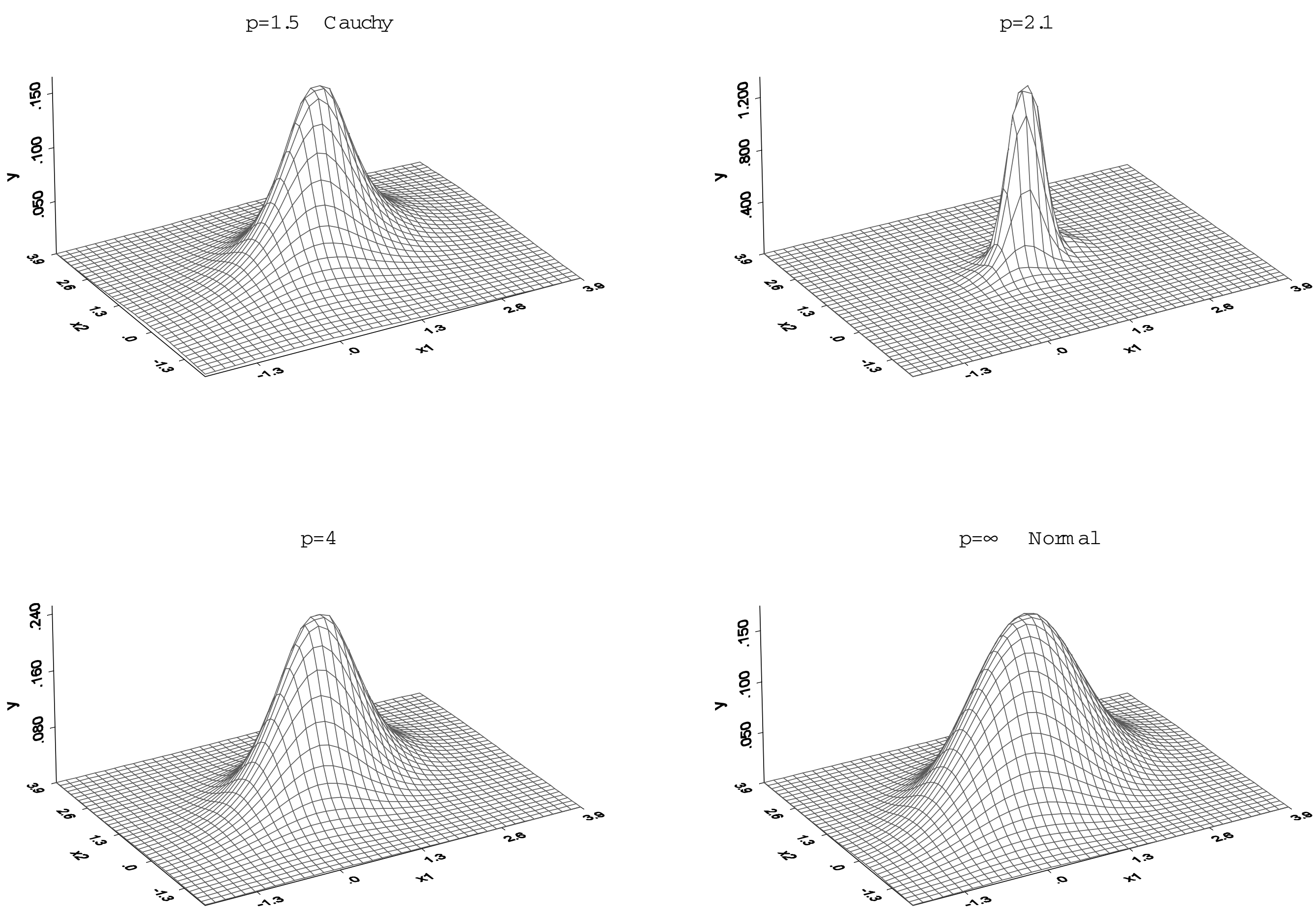
The following theorem shows how to determine the form of the coefficient $\beta_{\mathring{g}_2}(g_2)$ in the generalized (co)variance premium formula (4.2) for the Student family, where

$$\mathring{g}_2 = g_{2,p_0} = \frac{1}{\left(1 + \dfrac{u}{k_{2,p_0}}\right)^{p_0}},$$

$$g_2 = g_{2,p} = \frac{1}{\left(1 + \dfrac{u}{k_{2,p}}\right)^{p}}.$$

Figure 3

Densities of Bivariate GST for *p* = 1.5, 2.1, 4, ∞ and ρ = 0.4



Theorem 3

For $p, p_0 > 1$,

$$\beta_{g_{2,p_0}}(g_{2,p}) = p_0 \left(1 - (p-1) \int_0^\infty \left(1 + \frac{k_{2,p}}{k_{2,p_0}} z\right)^{-1} (1+z)^{-p}\, dz\right).$$

When $p = p_0$,

$$\beta_{g_{2,p_0}}(g_{2,p}) = 1, \tag{5.9}$$

and for the case $1 < p, p_0 \le 2$,

$$\beta_{g_{2,p_0}}(g_{2,p}) = \frac{p_0}{p}.$$

PROOF

From (3.5) and (3.4) we have

$$J_{2,p_0}(u) = \frac{p_0}{k_{2,p_0}} \frac{1}{\left(1 + \frac{u}{k_{2,p_0}}\right)},$$

and, taking into account (5.2), we have, after changing variables,

$$\begin{aligned}
\beta_{g_{2,p_0}}(g_{2,p}) = \beta_{p_0}(p) &= \frac{1}{4} c_{2,p} \int_0^\infty x J_2\left(\frac{1}{2}x\right) g_{2,p}\left(\frac{1}{2}x\right) dx \\
&= \frac{(p-1)}{4k_{2,p}} \frac{p_0}{k_{2,p_0}} \int_0^\infty x \frac{1}{\left(1 + \frac{x}{2k_{2,p_0}}\right)} \frac{1}{\left(1 + \frac{x}{2k_{2,p}}\right)^p} dx \\
&= p_0 \left(1 - (p-1) \int_0^\infty \left(1 + \frac{k_{2,p}}{k_{2,p_0}} z\right)^{-1} (1+z)^{-p}\, dz\right), \qquad p_0, p > 1.
\end{aligned} \tag{5.10}$$

In the case where $p = p_0$ we have

$$\begin{aligned}
\beta_{p_0}(p_0) &= p_0 \left(1 - (p_0 - 1) \int_0^\infty (1+z)^{-(p_0+1)}\, dz\right) \\
&= p_0 \left(1 - \frac{(p_0 - 1)}{p_0}\right) = 1, \qquad p_0 > 1,
\end{aligned}$$

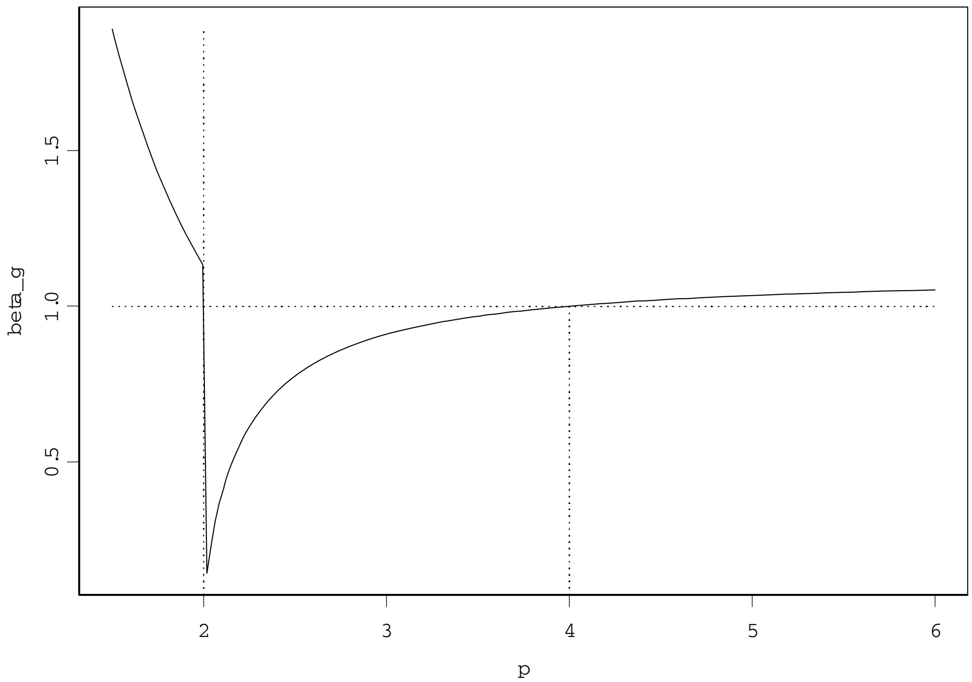
and for $1 < p, p_0 \le 2$

$$\beta_{p_0}(p_0) = p_0 \left(1 - (p-1) \int_0^\infty (1+z)^{-(p+1)}\, dz\right) = \frac{p_0}{p}.$$

■

Note that the result (5.9) conforms with (4.4). In Figure 4 the reference density generator has shape (power) parameter $p_0 = 4$. We can see in the figure that $p = 2$ is a breaking point around which the behavior of $\beta_{2,p_0}(p)$ is different. This is a natural point to use to select the reference density for pricing. This determines $p_0 = 2$ as the shape parameter for the reference density generator. We see that the selection of the density generator for the reference portfolio for the pricing model is intuitive and reflects the breaking point in preferences given by the $\beta_{2,p_0}(p)$ coefficient.

Figure 4
Varying $\beta_{2,p_0}(p)$ in p, $p_0 = 4$



Using this reference density generator with $p_0 = 2$ we determine values of the coefficient $\beta_{2,p_0}(p)$, and a graph of this coefficient is given in Figure 5.

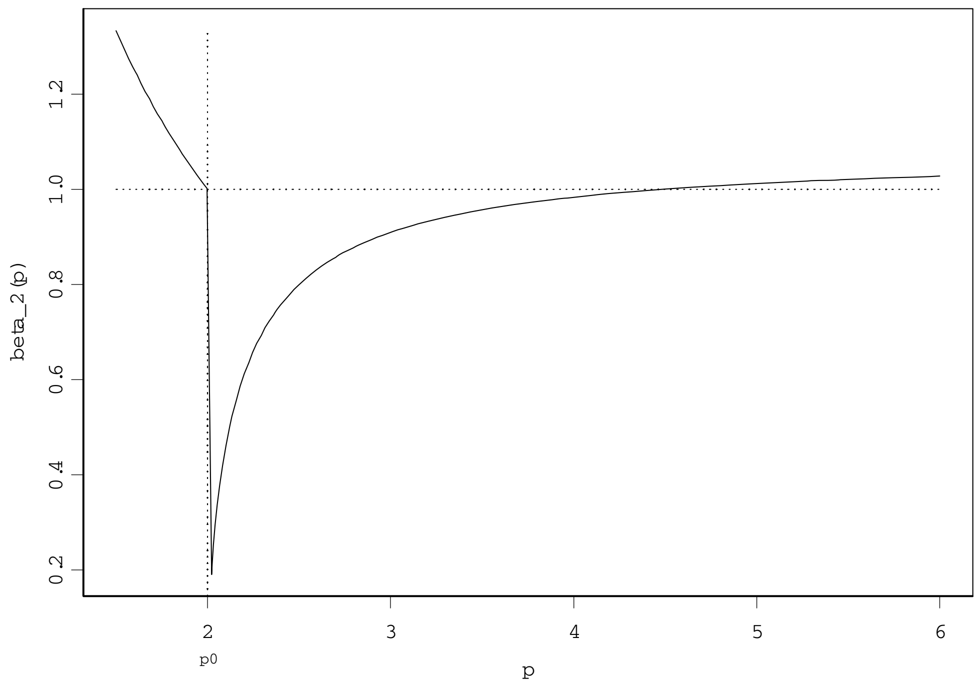
This shows that by selecting the reference portfolio using the shape parameter where the behavior of $\beta_{\mathring{g}_2}(g_2)$ changes, the breaking point, we obtain a well-behaved form for the adjustment for the departures from normality in the pricing model. As the shape parameter for the portfolio to be priced increases relative to the reference portfolio, the value of $\beta_{\mathring{g}_2}(g_2)$ increases monotonically. The reference portfolio does not have a significant bearing on the pricing model results since it is the relative differences between the traded asset portfolio and the nontraded asset portfolio to be priced that is captured in the model. Both of these portfolios are calibrated to the same reference portfolio, and it is the relative differences that are important.

6. APPLICATION TO FINANCIAL ASSET DATA

To illustrate the application of the pricing result we provide an example using stock market data. This example is used to demonstrate the application of the model given in this paper. The aim of this paper is not to provide empirical tests. We aim to show that the proposed generalized covariance pricing model is relatively straightforward to apply and how it reflects preferences for different portfolios. Because the aim of the model is to price nontraded risks relative to traded risks in order to allow for incomplete markets, as in the case of insurance products and infrequently traded asset markets, empirical tests are difficult because of a lack of available data.

We consider a portfolio of returns of 15 stocks from the Dow Jones stock index (3M, Alcoa, Boeing, Caterpillar, Exxon Mobil, FedEx, General Motors, Honeywell Intl, United Technologies, United Parcel Service, CLB, International Paper, Walt Disney, Du Pont de Nemours, Eastman Kodak, Wal-Mart) and denote these by $\mathbf{X} = (X_1, \ldots, X_n)^T$ with $n = 15$. Based on the data of daily stock returns of the above list of companies for 2003, Anderson-Darling test statistics do not reject the hypothesis that $\mathbf{X}$ is generated from a multivariate GST distribution with power parameter $p_n = 12.86$. This is the distribution we assume for this portfolio. We will use this portfolio to price another portfolio of assets.

Figure 5
Varying $\beta_{2,p_0}(p)$ in p, p_0 = 2



It is important to note that the problem of fitting a multivariate distribution is complex and with limited published research. Test statistics that work very well for the univariate case are not well defined, and application in the multivariate case is problematic. However, under the elliptical distribution hypothesis, we reduce the multivariate problem to a univariate one using the following elegant decomposition given in Fang, Kotz, and Ng (1987), Section 2.5:

$$\mathbf{X} \overset{D}{=} \boldsymbol{\mu} + r\mathbf{A}^T\mathrm{U}^{(n)},$$

where $r > 0$ is random variable with density

$$f(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1} g_n(\tfrac{1}{2}r^2), \qquad r > 0,$$

$\mathbf{A}$ is the matrix such that $\mathbf{A}^T\mathbf{A} = \boldsymbol{\Sigma}$, and $\mathbf{U}^{(n)}$ is a random vector uniformly distributed on the unit sphere surface in R^n and independent of r. We use this decomposition to estimate and test the distribution for our empirical data.

Consider, for example, a portfolio of returns $\mathbf{Y} = (Y_1, \ldots, Y_m)^{\mathrm{T}}$ of $m = 10$ of high-technology companies from NASDAQ/Computers (Adobe Systems, Compuware, NVIDIA, Peoplesoft, Veritas Software, SanDisk, Microsoft, Symantec, Citrix Systems, Intuit). Based on the data of daily stock returns for the 2003 year this second portfolio was estimated to be multivariate GST with power parameter $p_m = 7.34$.

We use the formula (4.5) for pricing risks in portfolio $\mathbf{Y}$. Denoting the distribution of these portfolios as $\mathbf{X} \sim E_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g_n)$ and $\mathbf{Y} \sim E_m(\tilde{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Sigma}}, \tilde{g}_m)$, where g_n and $\tilde{g}_m$ are GST density generators (5.3) with p equal to p_n and p_m, respectively, we can evaluate k_{n,p_n} and k_{m,p_m} from (5.3). Theorems 1 and 2 state that g_2 and $\tilde{g}_2$ are Student density generators with $p_0 = p_n - n/2 + 1 = 8.36$ and $p = p_m - m/2 + 1 = 3.34$, respectively. We can then derive the coefficient that reflects the relative differences between the two portfolios, $\beta_{g_2}(\tilde{g}_2)$, in (4.5) to obtain

$$\beta_{\tilde{g}_2}(\tilde{g}_2) = \beta_{g_{2,p_0}}(g_{2,p}) = 0.91.$$

This is illustrated in Figure 6.

The expected return (price) formula we have derived can then be used to price the stocks from the NASDAQ's subset **Y**, using the market price of the portfolio from the Dow Jones's subset **X**, and taking into account the distribution of the portfolio of risks to be priced relative to the Dow Jones reference portfolio. This gives a pricing model for the portfolio

$$P_{Y_i} = \tilde{\mu}_i + 0.91\,\frac{P_M - \mu_M}{\sigma_M^2}\,\rho_{i\tilde{M}}\sigma_{Y_i}\sigma_{\tilde{M}}, \qquad i = 1, \ldots, n.$$

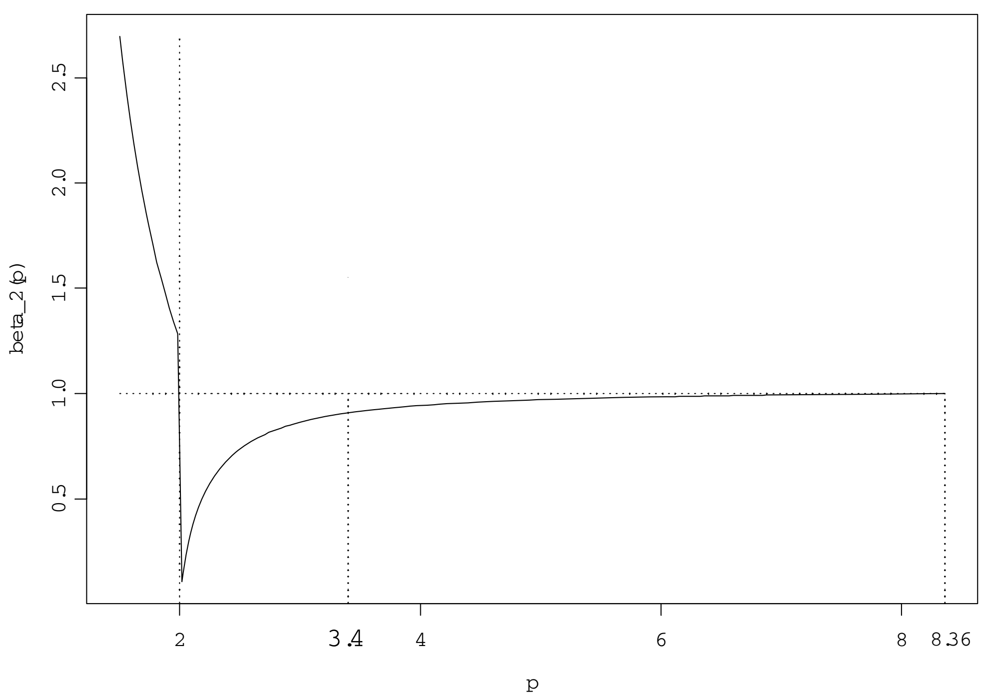
The pricing formula takes into account the relative differences between the portfolios. The 0.91 factor multiplying the price of risk reflects the preference for risks with smaller shape parameters implied by the pricing model. This pricing model is an extension of the CAPM to allow for differences in the traded and nontraded portfolios with preferences reflected in the β coefficient.

7. CONCLUSION

We have developed a new pricing principle that can be applied to incomplete market pricing. We have used a generalization of the variance premium recently proposed in the insurance literature to develop a new model that incorporates features of distributions other than mean and variance for relative pricing. The pricing model assumes elliptical distributions and uses a traded portfolio and a reference portfolio to calibrate a price of risk. This is then used to determine prices of another portfolio taking into account the differences between the distributions of the two portfolios. An important property of the model is that preferences for features of the underlying distributions other than the mean and variance are determined by a coefficient that provides a natural ordering of elliptical distributions. For multinormal distributions the model reduces to the standard CAPM. A CAPM result also holds where

Figure 6

Varying $\beta_{2,p_0}(p)$ in p, p_0 = 8.36



the traded and nontraded portfolios have the same elliptical distributions. An example is given based on traded portfolios to illustrate the intuition behind the pricing model and to demonstrate its application. The pricing model has the properties that it not only reflects preferences over distributions for moments other than the mean and variance, but it can be applied to distributions where these moments do not exist.

8. ACKNOWLEDGMENTS

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