"On Optimal Dividend Strategies in the Compound Poisson Model," by Elias S. W. Shiu and Hans U. Gerber, April 2006

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Professors Gerber and Shiu have again produced a very comprehensive paper that provides important insights into solving different problems. I would like to show, using an identical approach as in Appendix A, how the value of a threshold strategy can be calculated when the claim size distribution is a finite scale and shape mixture of Erlangs.

It is well known that the class of finite mixtures of Erlangs with the same scale parameter, which is a subset of this class of distributions, can be used to approximate (arbitrarily accurately) any absolutely continuous distribution on $(0, \infty)$ (see, e.g., Tijms 1994, pp. 162–64; Willmot and Lin 2001, p. 14). Therefore, it becomes theoretically possible to approximate $V(x; b)$ for non-phase-type claims, such as Pareto or lognormal claims, where an explicit form of $V(x; b)$ is unlikely to be available.

The claim amount density for a finite scale and shape mixture of Erlangs is given by

$$p(y) = \sum_{j=1}^{r} \sum_{i=1}^{n_j} A_{i,j} \frac{\beta_j^i y^{i-1} e^{-\beta_j y}}{(i-1)!}, \; y > 0, \tag{D.1}$$

with $\beta_j > 0$ and $A_{i,j} \geq 0$ for $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, r$, and $\sum_{j=1}^{r} \sum_{i=1}^{n_j} A_{i,j} = 1$. Furthermore, without loss of generality, let us assume β_j's are distinct, and n_j is such that $A_{n_j,j} \neq 0$ for $j = 1, 2, \ldots, r$. Let us also define the quantity $n = \sum_{j=1}^{r} n_j$. The Laplace transform of (D.1) is

$$\hat{p}(\xi) = \sum_{j=1}^{r} \sum_{i=1}^{n_j} A_{i,j} \left(\frac{\beta_j}{\beta_j + \xi} \right)^i. \tag{D.2}$$

Let us first assume that $b > 0$. Note from (5.6) that the function $h(\cdot)$ satisfies the same integro-differential equation as the function $m(\cdot)$ in Gerber (1979, pp. 147–48). Thus, using the arguments and results in Cheung, Dickson, and Drekic (2006, Section 3) with slight modifications, it can immediately be deduced that $h(\cdot)$ can be written as (A.4), where $\rho_0, \rho_1, \ldots, \rho_n$ are the solutions of (A.7) with $\hat{p}(\xi)$ given by (D.2), and $C_0, C_1, \ldots, C_n$ satisfy

$$\sum_{k=0}^{n} C_k \left\{ \sum_{i=q+1}^{n_j} \frac{A_{i,j} \beta_j^i}{(\beta_j + \rho_k)^{i-q}} \right\} = 0, \; q = 0, 1, \ldots, n_j - 1; \; j = 1, 2, \ldots, r. \tag{D.3}$$

For the same reason outlined in Appendix A, then let $C_0, C_1, \ldots, C_n$ be a particular solution of (D.3).

Similarly, (A.11) also holds true under the present assumption on the claim size distribution. If one substitutes (D.1), (5.7) with $h(\cdot)$ given by (A.4), and (A.11) into (5.2), and then uses the identity

$$\int_0^u y^{i-1} e^{-vy} \, dy = \frac{(i-1)!}{v^i} \left\{ 1 - \sum_{j=0}^{i-1} e^{-vu} \frac{(vu)^i}{i!} \right\}, \; y > 0; \; i = 1, 2, \ldots \tag{D.4}$$

along with the binomial expansion several times, followed by changing the order of summations and using (3), it is found after some tedious algebra that

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$$\sum_{k=0}^{n} D_k \left\{(c-\alpha)u_k - (\lambda+\delta) + \lambda \sum_{j=1}^{r}\sum_{i=1}^{n_j} A_{i,j}\left(\frac{\beta_j}{\beta_j+u_k}\right)^i\right\} e^{u_k x}$$
$$+ \lambda \sum_{j=1}^{r}\sum_{q=0}^{n_j-1}\frac{1}{q!}\left\{-\frac{\alpha}{\delta}\sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^m(-b)^{m-q}e^{\beta_j b}}{(m-q)!} - \sum_{k=0}^{n} D_k \sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{(\beta_j+u_k)b}}{(\beta_j+u_k)^{i-m}(m-q)!}\right.$$
$$\left. + \gamma\sum_{k=0}^{n} C_k \sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{(\beta_j+\rho_k)b}}{(\beta_j+\rho_k)^{i-m}(m-q)!}\right\} x^q e^{-\beta_j x} = 0. \quad \text{(D.5)}$$

By setting the coefficient of $\exp(u_k x)$ to be 0, it can be seen that $u_0, u_1, \ldots, u_n$ are solutions of (A.7) with $\hat{p}(\xi)$ given by (D.2) and c replaced by $c - \alpha$. It is known that there must be a positive solution when $\delta > 0$ (see Gerber and Shiu 1998). If this root is denoted by u_0, then it follows from (A.11) and the asymptotic formula (5.3) that $D_0 = 0$. Equating the coefficients $x^q e^{-\beta_j x}$ of with 0 yields

$$\gamma\sum_{k=0}^{n} C_k \sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{\rho_k b}}{(\beta_j+\rho_k)^{i-m}(m-q)!} = \frac{\alpha}{\delta}\sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^m(-b)^{m-q}}{(m-q)!}$$
$$+ \sum_{k=1}^{n} D_k \sum_{i=q+1}^{n_j}\sum_{m=q}^{i-1}\frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{u_k b}}{(\beta_j+u_k)^{i-m}(m-q)!},\ q = 0, 1, \ldots, n_j - 1;\ j = 1, 2, \ldots, r. \quad \text{(D.6)}$$

Together with the continuity condition (A.16), one obtains $n + 1$ linear equations to be solved for γ, $D_1, D_2, \ldots, D_n$.

For the case $b = 0$, (A.17) holds true, with $D_1, D_2, \ldots, D_n$ being the solution of the following system of linear equations:

$$\sum_{k=1}^{n} D_k \left\{\sum_{i=q+1}^{n_j}\frac{A_{i,j}\beta_j^i}{(\beta_j+u_k)^{i-q}}\right\} = -\frac{\alpha}{\delta}\sum_{i=q+1}^{n_j} A_{i,j}\beta_j^q,\ q = 0, 1, \ldots, n_j - 1;\ j = 1, 2, \ldots, r. \quad \text{(D.7)}$$

The above result follows by first noting that (D.6) holds true for $b = 0$ if 0^0 is defined to be 1, and then using (D.3). Note that one requires $\rho_0, \rho_1, \ldots, \rho_n$ to be distinct as well as $u_0, u_1, \ldots, u_n$ to be distinct; otherwise, the two systems of linear equations are insolvable.

Tables 1 and 2 show some numerical results when the claim size density is given by

$$p(y) = \frac{1}{3}(3e^{-3y}) + \frac{2}{3}\left(\frac{3^4\, y^3 e^{-3y}}{3!}\right),\ y > 0. \quad \text{(D.8)}$$

For Tables 3 and 4, let us assume the following claim size density:

$$p(y) = \frac{1}{2}(2^2\, y e^{-2y}) + \frac{1}{8}(2.5e^{-2.5y}) + \frac{3}{8}\left(\frac{2.5^3\, y^2 e^{-2.5y}}{2}\right),\ y > 0. \quad \text{(D.9)}$$

Both distributions have mean 1, and let us always assume $\lambda = 1$ and $\theta = 0.1$ so that $c = 1.1$. The value of δ is assumed to be 0.001 in Tables 1 and 3 and 0.002 in Tables 2 and 4. Although the interest rates

Table 1
Claim Size Density (D.8), δ = 0.001, *b and *V*(*x*; *b**) for Various Choices of α**

	0.05	0.1	0.5	1	1.099
b^*	15.19	24.40	36.42	37.49	37.59
$V(0; b^*)$	3.77	6.15	7.31	7.31	7.31
$V(10; b^*)$	30.55	49.77	59.19	59.24	59.24
$V(20; b^*)$	41.03	66.00	79.67	79.74	79.74
$V(30; b^*)$	45.84	76.87	91.57	91.65	91.65
$V(40; b^*)$	48.08	83.77	101.66	101.76	101.76
$V(50; b^*)$	49.11	88.62	111.45	111.67	111.68

Table 2
Claim Size Density (D.8), δ = 0.002, *b and *V*(*x*; *b**) for Various Choices of α**

	0.05	0.1	0.5	1	1.099
b^*	7.99	14.87	24.87	25.93	26.04
$V(0; b^*)$	1.87	2.94	3.66	3.66	3.66
$V(10; b^*)$	15.50	24.37	30.35	30.41	30.41
$V(20; b^*)$	21.06	34.49	43.14	43.21	43.21
$V(30; b^*)$	23.37	40.58	53.17	53.30	53.30
$V(40; b^*)$	24.32	44.28	62.68	63.10	63.12
$V(50; b^*)$	24.72	46.53	71.73	72.68	72.75

chosen seem to be quite low, by a change of time unit it is clear that the choice of $\lambda = 1$ and $\delta = 0.001$ ($\lambda = 1$ and $\delta = 0.002$) is equivalent to the choice of $\lambda = 100$ and $\delta = 0.1$ ($\lambda = 100$ and $\delta = 0.2$). For various values of α, the software package Mathematica was used to numerically maximize $V(x; b)$ with respect to b for different values of x; it was found that the optimal level b^* is independent of x. In fact, when the claim size is distributed as a finite mixture of Erlangs with the same scale parameter, that is, the claim size density is given by (D.1) with $r = 1$, Cheung, Dickson, and Drekic (2006, eq. [4.7]) have derived an equation of the form

$$\sum_{k=0}^{n} W_k e^{\rho_k b} = 0 \tag{D.10}$$

to be solved for b^* (if positive), where $W_0, W_1, \ldots, W_n$ are constant coefficients independent of b and x. Although Cheung, Dickson, and Drekic assume the condition

$$c - \alpha > \lambda \int_0^\infty y p(y)\, dy \tag{D.11}$$

throughout the entire paper, numerically the approach produces identical results for $V(x; b)$ and b^* as the method described here for $\alpha \in (0, c)$. Detailed comments concerning other numerical findings can be found in Cheung, Dickson, and Drekic's paper.

Furthermore, when the claim size density is given by (D.1), the function $L(x; b)$ for $b > 0$ is given by (B.1) and (B.5), where $\rho_0, \rho_1, \ldots, \rho_n$ and $u_0, u_1, \ldots, u_n$ have the same definition as before, $D_0 = 0$, and $C_0, C_1, \ldots, C_n$ and $D_1, D_2, \ldots, D_n$ can be solved from a system of $2n + 1$ linear equations that consists of (B.8) and the following:

$$\sum_{k=0}^{n} C_k \left\{ \sum_{i=q+1}^{n_j} \frac{A_{i,j}\beta_j^i}{(\beta_j + \rho_k)^{i-q}} \right\} = \sum_{i=q+1}^{n_j} A_{i,j}\beta_j^q,\ q = 0, 1, \ldots, n_j - 1;\ j = 1, 2, \ldots, r \tag{D.12}$$

and

Table 3
Claim Size Density (D.9), δ = 0.001, *b and *V*(*x*; *b**) for Various Choices of α**

	0.05	0.1	0.5	1	1.099
b^*	15.05	24.12	35.92	36.94	37.04
$V(0; b^*)$	3.78	6.16	7.31	7.32	7.32
$V(10; b^*)$	31.03	50.63	60.08	60.12	60.12
$V(20; b^*)$	41.41	67.72	80.36	80.42	80.43
$V(30; b^*)$	46.09	77.49	92.14	92.21	92.21
$V(40; b^*)$	48.22	84.29	102.20	102.30	102.30
$V(50; b^*)$	49.19	89.04	111.98	112.21	112.22

Table 4
Claim Size Density (D.9), $\delta = 0.002$, b^* and $V(x; b^*)$ for Various Choices of α

	0.05	0.1	0.5	1	1.099
b^*	8.01	14.79	24.61	25.63	25.73
$V(0; b^*)$	1.87	2.94	3.65	3.65	3.65
$V(10; b^*)$	15.71	24.74	30.73	30.78	30.78
$V(20; b^*)$	21.22	34.82	43.51	43.52	43.52
$V(30; b^*)$	23.47	40.85	53.46	53.59	53.59
$V(40; b^*)$	24.38	44.48	62.96	63.38	63.41
$V(50; b^*)$	24.75	46.67	72.00	72.96	73.03

$$\sum_{k=0}^{n} C_k \sum_{i=q+1}^{n_j} \sum_{m=q}^{i-1} \frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{\rho_k b}}{(\beta_j + \rho_k)^{i-m}(m - q)!} = \sum_{k=1}^{n} D_k \sum_{i=q+1}^{n_j} \sum_{m=q}^{i-1} \frac{A_{i,j}\beta_j^i(-b)^{m-q}e^{u_k b}}{(\beta_j + u_k)^{i-m}(m - q)!},$$
$$q = 0, 1, \ldots, n_j - 1; j = 1, 2, \ldots, r. \quad \text{(D.13)}$$

For the case $b = 0$, one has that

$$L(x; 0) = \sum_{k=1}^{n} D_k e^{u_k x}, \; x \geq 0, \quad \text{(D.14)}$$

where $D_1, D_2, \ldots, D_n$ can be solved from

$$\sum_{k=1}^{n} D_k \left\{ \sum_{i=q+1}^{n_j} \frac{A_{i,j}\beta_j^i}{(\beta_j + u_k)^{i-q}} \right\} = \sum_{i=q+1}^{n_j} A_{i,j}\beta_j^q, \; q = 0, 1, \ldots, n_j - 1; j = 1, 2, \ldots, r. \quad \text{(D.15)}$$

Note that (D.3), (D.6), (D.7), (D.12), and (D.13) can be reduced to (A.9), (A.15), (A.18), (B.3), and (B.7), respectively, by restricting $n_j = 1$ for $j = 1, 2, \ldots, r$.

Note that the assumption $A_{i,j} \geq 0$ for $i = 1, 2, \ldots, n_j$ and $j = 1, 2, \ldots, r$ is unnecessary, that is, the entire derivation still holds true when the claim size is distributed as a finite combination of Erlangs. The class of finite combinations of Erlangs includes the class of finite sums of independent Erlangs with Laplace transform

$$\hat{p}(\xi) = \prod_{j=1}^{r} \left(\frac{\beta_j}{\beta_j + \xi} \right)^{n_j} \quad \text{(D.16)}$$

since (D.16) can be resolved into (D.2) using partial fractions with $A_{i,j}$'s being functions of β_j's.

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AUTHORS' REPLY

We thank Mr. Cheung for a stimulating discussion. It is motivated by the fact that claim size distributions can be approximated by mixtures of Erlang distributions. Our preference is to use combinations of exponential distributions because they are also dense in the space of probability distributions on $[0, \infty)$. Indeed, the expression on the right-hand side of formula (D.1) in the discussion is a limit of combinations of exponential distributions. For further discussions on approximation by combinations of exponential distributions, see Dufresne (2006, 2007).

Mr. Cheung used the software Mathematica to numerically maximize $V(x; b)$ with respect to b for different given values of x and found that the optimal level b^* is independent of x. This reinforces a result for threshold strategies in Section 5 of our paper:

$$V(x; b) = \gamma h(x), \qquad 0 \leq x \leq b,$$

where γ is not dependent on x. A related article is Gerber, Lin, and Yang (2006).

We take this opportunity to remark on two previous discussions. Equation (D.17) of Ko (2006) gives a very interesting set of identities. Amazingly, the result can be generalized by (9.13) in Gerber and Shiu (2005).

Let us define a function $g(\rho)$, where $g(\rho_k)$ is expression (12) in Smith (2006). Then, the ratio $\mathcal{N}(x)/\mathcal{N}(0)$ is the nth *divided difference* of the function g with respect to the points of collocation $\rho_0, \rho_1, \ldots, \rho_n$.

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