ABSOLUTE RUIN PROBABILITIES IN A JUMP DIFFUSION RISK MODEL WITH INVESTMENT

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ABSTRACT

This article considers the compound Poisson insurance risk model perturbed by diffusion with investment. We assume that the insurance company can invest its surplus in both a risky asset and the risk-free asset according to a fixed proportion. If the surplus is negative, a constant debit interest rate is applied. The absolute ruin probability function satisfies a certain integro-differential equation. In various special cases, closed-form solutions are obtained, and numerical illustrations are provided.

1. INTRODUCTION

For over a century, ruin theory has been of major interest in actuarial science. The early work can be traced back to Lundberg (1903, 1926). In the classical model, the premium rate is constant, and the surplus does not yield any interest. As a consequence, the security loading is constant. It is assumed to be positive, so that the surplus drifts to infinity in the long run.

The situation changes when investment income is introduced in the model. The instantaneous security loading depends on the surplus, which drifts to infinity in the long run without any assumption about the premium rate. If a positive surplus earns interest, the next step is to assume that a negative surplus can be covered by a loan. Typically the applicable interest rate on the loan is much higher. This leads to a new definition of ruin: Absolute ruin occurs if the premiums received are not suffcient to make the interest payments on the debt. Note that if the interest rate applicable to the loan goes to infinity, absolute ruin is identical to the traditional event ruin.

Gerber (1971) considers the probability of absolute ruin in the compound Poisson model when the debt and credit interest rates are the same. A closed-form solution is given in the case of an exponential claim amount distribution. Substantial refinements and generalizations are given by Dassios and Embrechts (1989). Embrechts and Schmidli (1994) discuss the absolute ruin probability when the surplus process is a piecewise-deterministic Markov process. Cai (2007) studies the Gerber-Shiu function in the classical insurance risk model with absolute ruin. While Gerber (1979) and Rolski et al. (1999) consider models with investment income, the only monograph to treat the absolute ruin problem is Asmussen (2000).

In this article, the compound Poisson model is enriched by an independent Brownian motion. If the surplus is positive, it is allocated in a fixed proportion between a risky and a risk-free asset. Consequently the surplus is a jump diffusion in this model. The definition of absolute ruin has to be adopted. A company can go on as long as the deterministic drift term of the jump diffusion remains positive. Otherwise, absolute ruin occurs.

The article is structured as follows. In Section 2 the jump diffusion model for the surplus of a company is explained, and the probability of absolute ruin is introduced. In Section 3 it is seen that a

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certain family of risk models with investment leads in a natural way to the jump diffusion model. Sections 4–6 discuss three particular cases. For each, closed-form formulas for the probability of absolute ruin are developed and illustrated with numerical examples.

2. THE JUMP DIFFUSION MODEL

In the subsequent sections we shall consider particular models for the surplus of a company. In each, the surplus process turns out to be a jump diffusion. The purpose of this section is to explain in intuitive terms what a jump diffusion is and to set the general framework.

A jump diffusion is given by a Poisson parameter λ, a probability density function of individual claim sizes $p(x)$, $x > 0$, a drift function $\mu(u)$, and a variance per unit time function $\sigma^2(u)$. A jump diffusion is a stationary Markov process. Let U_t denote the surplus of the company at time t. If $U_0 = u$, the change of surplus between times 0 and dt has three sources:

1. The diffusion component: a normal random variable with mean 0 and variance $\sigma^2(u)\,dt$
2. A deterministic component $\mu(u)\,dt$
3. With probability $\lambda\,dt$ a claim occurs. Then the surplus jumps from u to $u - X$, where X is distributed according to the probability density function $p(x)$.

Typically $\mu(u)$ is an increasing function of u. Let u_0 denote the zero of the function $\mu(u)$. Absolute ruin occurs when the surplus drops below the critical level u_0. Let $\psi(u)$ denote the probability of absolute ruin, considered a function of the initial surplus u. Thus

$$\psi(u) = 1 \quad \text{if} \quad u < u_0. \tag{2.1}$$

For $u > u_0$, it follows from the law of total probability that

$$\psi(u) = E[\psi(U_{\Delta t})|U_0 = u] + o(\Delta t). \tag{2.2}$$

Hence

$$\lim_{\Delta t \to 0} \frac{E[\psi(U_{\Delta t})|U_0 = u] - \psi(u)}{\Delta t} = 0. \tag{2.3}$$

This limit can be determined componentwise, according to the three sources for the change of the surplus. Hence we see that

$$\frac{\sigma^2(u)}{2}\psi''(u) + \mu(u)\psi'(u) + \lambda \int_0^\infty [\psi(u - x) - \psi(u)]p(x)\,dx = 0, \tag{2.4}$$

where (2.1) should be kept in mind. This is an integro-differential equation for the function $\psi(u)$, $u > u_0$.

Equation (2.4) together with one or two boundary conditions determines the function $\psi(u)$. If there is no diffusion term, $\sigma^2(u) = 0$, the unique condition is that

$$\psi(\infty) = 0. \tag{2.5}$$

Note that in this case the surplus can never go beyond u_0 once it is below u_0. This explains the expression *absolute ruin*. If there is a diffusion component, in particular $\sigma^2(u_0) > 0$, the additional condition is that

$$\psi(u_0+) = 1. \tag{2.6}$$

In this case the surplus could bounce back from below u_0 because of oscillation. Nevertheless, we also use the term *absolute ruin*.

REMARK

If there is no diffusion term, condition (2.6) may or may not be true. One can show that it is true, if and only if the function $\mu(u)$ is suffciently small in the neighborhood of u_0 in the sense that

$$\int_{u_0}^{u_0+\varepsilon} \frac{1}{\mu(u)}\, du = \infty$$

for any $\varepsilon > 0$.

3. INSURANCE RISK MODEL WITH INVESTMENT INCOME

In this section we present a model that is a special case of the jump diffusion model in the last section. Assume that the surplus at time t of an insurance company before investment is given by

$$u + ct + \sigma_1 W_t - L_t, \tag{3.1}$$

where u is the initial surplus, $c > 0$ is the premium rate, $\{L_t\}$, the aggregate claims process, is a compound Poisson process given by λ and $p(x)$, and $\{W_t\}$ is a standard Brownian motion independent of the aggregate claims process. We assume that the surplus can be invested in a risk-free bond or a risky asset. If the company invests money in the bank or borrows money from the bank, the rate of return can be described by

$$dR_t = \begin{cases} rR_t\, dt & \text{if the surplus is positive} \\ \tau R_t\, dt & \text{if the surplus is negative.} \end{cases} \tag{3.2}$$

That is, we assume that the company's lending rate is r and the borrowing rate is $\tau \geq r > 0$. The price dynamic of the risky asset is given by

$$dS_t = \mu S_t\, dt + \sigma_2 S_t\, dB_t,$$

where B_t is a standard Brownian motion independent of the aggregate claims process. We assume that W_t and B_t are correlated with $dW_t\, dB_t = \rho\, dt$.

Assume that the insurance company invests a fixed proportion α of its surplus in the risky asset when its surplus is positive. That is, the insurance company will rebalance its portfolio continuously to maintain that proportion. When the surplus is negative, the company has to borrow money from the bank using the debt rate τ. Under this investment strategy, the surplus of the company is a jump diffusion with

$$\mu(u) = \begin{cases} [\alpha\mu + (1 - \alpha)r)]u + c & \text{if } u \geq 0, \\ \tau u + c & \text{if } u < 0, \end{cases} \tag{3.3}$$

$$\sigma^2(u) = \begin{cases} \alpha^2\sigma_2^2 u^2 + \sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u & \text{if } u \geq 0, \\ \sigma_1^2 & \text{if } u < 0. \end{cases} \tag{3.4}$$

In this article, we assume that the company is allowed to continue its business as long as its surplus is above the critical value $u_0 = -c/\tau$, that is, as long as c exceeds $\tau|u_0|$, the instantaneous interest payment on the debt.

REMARK

The classical probability of ruin can be treated as the limiting case $\tau \to \infty$.

4. THE PURE DIFFUSION MODEL

In this section we consider the pure diffusion model ($\lambda = 0$). Thus the surplus process before investment is a Brownian motion with a drift. Such a model can be used to approximate the jump diffusion model.

Equation (2.4) reduces to an ordinary differential equation,

$$\frac{1}{2}\sigma^2(u)\psi''(u) + \mu(u)\psi'(u) = 0, \quad u > u_0, \tag{4.1}$$

where $u_0 = -c/\tau$ and $\mu(u)$ and $\sigma(u)$ are given in equations (3.3) and (3.4). To solve (4.1), we must distinguish whether $u > 0$ or $u_0 < u < 0$. Let $F(x)$ be an integral of the function $-2\mu(x)/\sigma^2(x)$ for $x > 0$, and let $G(x)$ be an integral of $-2\mu(x)/\sigma^2(x)$ for $u_0 < x < 0$. From (4.1) and (2.5) it follows that

$$\psi(u) = \psi(0) \frac{\displaystyle\int_u^\infty e^{F(x)}\,dx}{\displaystyle\int_0^\infty e^{F(x)}\,dx}, \qquad u \geq 0. \tag{4.2}$$

From (4.1) and (2.6), it follows that

$$\psi(u) = 1 - [1 - \psi(0)] \frac{\displaystyle\int_{u_0}^u e^{G(x)}\,dx}{\displaystyle\int_{u_0}^0 e^{G(x)}\,dx}, \qquad u_0 \leq u \leq 0. \tag{4.3}$$

The smooth junction condition $\psi'(0+) = \psi'(0-)$ yields a linear equation for $\psi(0)$. We find that

$$\psi(0) = \frac{e^{G(0)} \displaystyle\int_0^\infty e^{F(x)}\,dx}{e^{G(0)} \displaystyle\int_0^\infty e^{F(x)}\,dx + e^{F(0)} \displaystyle\int_{u_0}^0 e^{G(x)}\,dx}, \tag{4.4}$$

which is needed in (4.2) and (4.3). Luckily, for $F(x)$ and $G(x)$, closed-form expressions are available,

$$\begin{aligned} F(x) &= -2 \int \frac{[\alpha\mu + (1-\alpha)r]u + c}{\alpha^2\sigma_2^2 u^2 + \sigma_1^2 + 2\rho\alpha\sigma_1\sigma_2 u}\,du \\ &= -2\,\frac{\alpha\mu + (1-\alpha)r}{\alpha^2\sigma_2^2} \ln[\sigma(x)] - \frac{2c\alpha\sigma_2 - \rho(\alpha\mu + (1-\alpha)r)\sigma_1}{\alpha^2\sigma_1\sigma_2^2\sqrt{1-\rho^2}} \arctan \frac{\alpha\sigma_2 x + \rho\sigma_1}{\sigma_1\sqrt{1-\rho^2}} + K_1, \end{aligned} \tag{4.5}$$

where K_1 is a constant. This result can be verified with Mathematica. More easily, we have

$$G(x) = -2 \int \frac{\tau u + c}{\sigma_1^2}\,du = -\frac{\tau x^2 + 2cx}{\sigma_1^2} + K_2, \tag{4.6}$$

where K_2 is a constant. Note that the constants K_1 and K_2 drop out of the expressions in the right-hand side of (4.2)–(4.4).

EXAMPLE 4.1

We assume that the risky asset provides a return with rate $\mu = 0.2$ and volatility $\sigma_2 = 0.3$. We assume $\sigma_1 = 10$ with $\rho = 0.1$, and a premium rate $c = 2$. The proportion invested in the risky asset is $\alpha = 0.5$. Table 1 shows the numerical values of the absolute ruin probability when $\tau = 0.1$ for various r and u.

5. RISK-FREE INVESTMENT ONLY AND EXPONENTIAL CLAIMS

In this section we assume that the insurance company invests all its surplus in the risk-free asset ($\alpha = 0$) and that the claim size distribution is exponential,

Table 1

Absolute Ruin Probability When $\mu = 0.2$, $\sigma_2 = 0.3$, $\sigma_1 = 10$, $\rho = 0.1$, and $c = 2$

u	$r = 0.01$	$r = 0.02$	$r = 0.04$	$r = 0.06$	$r = 0.08$	$r = \tau = 0.1$
∞	0	0	0	0	0	0
50	0.00596541	0.00519839	0.00396166	0.00303177	0.00232848	0.00179392
20	0.0803612	0.00766294	0.0698315	0.0642215	0.0584253	0.0536081
10	0.179585	0.175005	0.166458	0.158628	0.151415	0.144741
5	0.260086	0.255591	0.247147	0.239341	0.232088	0.225316
2	0.320698	0.316473	0.308522	0.301155	0.294291	0.287866
1	0.343099	0.339	0.331284	0.324132	0.317467	0.311226
0	0.366623	0.362666	0.355219	0.348314	0.341879	0.335852
−1	0.391191	0.387389	0.38023	0.373593	0.367407	0.361615
−2	0.416711	0.413068	0.406209	0.399851	0.393924	0.388375
−5	0.498797	0.495667	0.489773	0.484309	0.479217	0.474448
−10	0.652266	0.650094	0.646005	0.642215	0.638681	0.635373
−15	0.821805	0.820692	0.818597	0.816655	0.814844	0.813149
−20	1	1	1	1	1	1

$$p(x) = \beta e^{-\beta x}, \qquad x > 0,\ \beta > 0. \tag{5.1}$$

In this case, equation (2.4) becomes

$$\begin{aligned}
&\lambda e^{-\beta(u+c/\tau)} + \lambda\beta \int_0^{u+c/\tau} \psi(u-x)e^{-\beta x}\,dx - \lambda\psi(u) + (ru+c)\psi'(u) + \frac{1}{2}\sigma_1^2\psi''(u) = 0,\\
&\qquad u \ge 0,\\
&\lambda e^{-\beta(u+c/\tau)} + \lambda\beta \int_0^{u+c/\tau} \psi(u-x)e^{-\beta x}\,dx - \lambda\psi(u) + (\tau u+c)\psi'(u) + \frac{1}{2}\sigma_1^2\psi''(u) = 0,\\
&\qquad -\frac{c}{\tau} \le u < 0.
\end{aligned} \tag{5.2}$$

After some calculations, we obtain ordinary differential equations,

$$\begin{aligned}
&[(ru+c)\beta - \lambda + r]\psi'(u) + \left(ru + c + \frac{1}{2}\beta\sigma_1^2\right)\psi''(u) + \frac{1}{2}\sigma_1^2\psi'''(u) = 0, \qquad u \ge 0,\\
&[(\tau u+c)\beta - \lambda + \tau]\psi'(u) + \left(\tau u + c + \frac{1}{2}\beta\sigma_1^2\right)\psi''(u) + \frac{1}{2}\sigma_1^2\psi'''(u) = 0, \qquad -\frac{c}{\tau} \le u < 0.
\end{aligned} \tag{5.3}$$

The boundary conditions are

$$\psi(\infty) = 0, \qquad \psi\left(-\frac{c}{\tau}\right) = 1, \qquad \psi''\left(-\frac{c}{\tau}+\right) = 0. \tag{5.4}$$

Equations similar to (5.3) can be found in Paulsen and Gjessing (1997) and Cai and Yang (2005). By introducing the new variable

$$\begin{aligned}
x &= -\frac{r}{\sigma_1^2}\left(u + \frac{c}{r} - \frac{\beta\sigma_1^2}{2r}\right)^2, \qquad u \ge 0,\\
x &= -\frac{\tau}{\sigma_1^2}\left(u + \frac{c}{\tau} - \frac{\beta\sigma_1^2}{2\tau}\right)^2, \qquad -\frac{c}{\tau} \le u < 0,
\end{aligned} \tag{5.5}$$

and letting

$$\begin{aligned} h(x) &= e^{\beta(u+c/r-\beta\sigma_1^2/(2r))}\psi'(u), \qquad u \geq 0, \\ h(x) &= e^{\beta(u+c/\tau-\beta\sigma_1^2/(2\tau))}\psi'(u), \qquad -\frac{c}{\tau} \leq u < 0, \end{aligned} \tag{5.6}$$

then equation (5.3) can be converted into Kummer's confluent hypergeometric equation for the function $h(x)$:

$$\begin{aligned} xh''(x) + \left(\frac{1}{2} - x\right) h'(x) - \left(\frac{1}{2} - \frac{\lambda}{2r}\right) h(x) &= 0, \qquad u \geq 0, \\ xh''(x) + \left(\frac{1}{2} - x\right) h'(x) - \left(\frac{1}{2} - \frac{\lambda}{2\tau}\right) h(x) &= 0, \qquad -\frac{c}{\tau} \leq u < 0. \end{aligned} \tag{5.7}$$

The solution of Kummer's equation can be expressed in terms of the confluent hypergeometric function; see Slater (1960) and Seaborn (1991) for details. Using the solution of (5.7) and equations (5.5) and (5.6), we conclude that

$$\begin{aligned} \psi'(u) &= C_1 f_1(u, r) + C_2 f_2(u, r), \qquad u \geq 0 \\ \psi'(u) &= D_1 f_1(u, \tau) + D_2 f_2(u, \tau), \qquad -\frac{c}{\tau} < u < 0, \end{aligned} \tag{5.8}$$

where C_1, C_2, D_1, and D_2 are constants and

$$\begin{aligned} f_1(u, s) &= \exp\left\{-\left(\beta u + \frac{s(u + c/s - (\beta\sigma_1^2)/(2s))^2}{\sigma_1^2}\right)\right\} \\ &\quad \times U\left(\frac{\lambda}{2s}, \frac{1}{2}; \frac{s(u + c/s - (\beta\sigma_1^2)/(2s))^2}{\sigma_1^2}\right) \\ f_2(u, s) &= \left|u + \frac{c}{s} - \frac{\beta\sigma_1^2}{2s}\right| \exp\left\{-\left(\beta u + \frac{s(u + c/s - (\beta\sigma_1^2)/(2s))^2}{\sigma_1^2}\right)\right\} \\ &\quad \times M\left(\frac{1}{2} + \frac{\lambda}{2s}, \frac{3}{2}; \frac{s(u + c/s - (\beta\sigma_1^2)/(2s))^2}{\sigma_1^2}\right). \end{aligned} \tag{5.9}$$

Here $M(a, b; x)$ and $U(a, b; x)$ are the confluent hypergeometric functions of the first and second kind, respectively. By using the first two boundary conditions in (5.4), we have

$$\begin{aligned} \psi(u) &= \psi(0) - \psi(0)\,\frac{\int_0^u f_1(y, r)\, dy + C\int_0^u f_2(y, r)\, dy}{\int_0^\infty f_1(y, r)\, dy + C\int_0^\infty f_2(y, r)\, dy}, \qquad u \geq 0, \\ \psi(u) &= 1 - [1 - \psi(0)]\,\frac{\int_{-c/\tau}^u f_1(y, \tau)\, dy + D\int_{-c/\tau}^u f_2(y, \tau)\, dy}{\int_{-c/\tau}^0 f_1(y, \tau)\, dy + D\int_{-c/\tau}^0 f_2(y, \tau)\, dy}, \qquad -\frac{c}{\tau} < u < 0, \end{aligned} \tag{5.10}$$

where C and D are two constants that need to be determined. From the third boundary condition in (5.4) we obtain

$$D = \frac{f_1'\left(-\frac{c}{\tau}, \tau\right)}{f_2'\left(-\frac{c}{\tau}, \tau\right)}, \tag{5.11}$$

where

$$\begin{aligned} f_1'\left(-\frac{c}{\tau}, \tau\right) &= \frac{\lambda\beta}{2\tau} \exp\left\{\frac{\beta}{\tau}\left(c - \frac{\beta\sigma_1^2}{4}\right)\right\} U\left(\frac{\lambda}{2\tau} + 1, \frac{3}{2}; \frac{\beta^2\sigma_1^2}{4\tau}\right), \\ f_2'\left(-\frac{c}{\tau}, \tau\right) &= -\exp\left\{\frac{\beta}{\tau}\left(c - \frac{\beta\sigma_1^2}{4}\right)\right\} M\left(\frac{1}{2} + \frac{\lambda}{2\tau}, \frac{3}{2}; \frac{\beta^2\sigma_1^2}{4\tau}\right) \\ &\quad - \frac{\beta^2\sigma_1^2(\tau + \lambda)}{6\tau^2} \exp\left\{\frac{\beta}{\tau}\left(c - \frac{\beta\sigma_1^2}{4}\right)\right\} M\left(\frac{3}{2} + \frac{\lambda}{2\tau}, \frac{5}{2}; \frac{\beta^2\sigma_1^2}{4\tau}\right). \end{aligned}$$

Note that, in the calculations above, we have used the following properties of the confluent hypergeometric functions:

$$\frac{d}{dx} U(a, b; x) = -aU(a + 1, b + 1; x), \tag{5.12}$$

$$\frac{d}{dx} M(a, b; x) = \frac{a}{b} M(a + 1, b + 1; x). \tag{5.13}$$

From the smooth junction conditions we have

$$\psi(0) \frac{f_1(0, r) + Cf_2(0, r)}{\int_0^\infty f_1(y, r)\, dy + C \int_0^\infty f_2(y, r)\, dy} = [1 - \psi(0)] \frac{f_1(0, \tau) + Df_2(0, \tau)}{\int_{-c/\tau}^0 f_1(y, \tau)\, dy + D \int_{c/\tau}^0 f_2(y, \tau)\, dy},$$

$$\psi(0) \frac{f_1'(0, r) + Cf_2'(0, r)}{\int_0^\infty f_1(y, r)\, dy + C \int_0^\infty f_2(y, r)\, dy} = [1 - \psi(0)] \frac{f_1'(0, \tau) + Df_2'(0, \tau)}{\int_{-c/\tau}^0 f_1(y, \tau)\, dy + D \int_{c/\tau}^0 f_2(y, \tau)\, dy}.$$

After some calculations, we obtain that

$$C = \frac{f_1(0, r)[f_1'(0, \tau) + Df_2'(0, \tau)] - f_1'(0, r)[f_1(0, \tau) + Df_2(0, \tau)]}{f_2'(0, r)[f_1(0, \tau) + Df_2(0, \tau)] - f_2\,(0, r)[f_1'(0, \tau) + Df_2'(0, \tau)]}, \tag{5.14}$$

and

$$\psi(0) = \frac{A}{A + B}, \tag{5.15}$$

where

$$A = [f_1(0, \tau) + Df_2(0, \tau)] \left[\int_0^\infty f_1(x, r)\, dx + C \int_0^\infty f_2(x, r)\, dx\right],$$

$$B = [f_1(0, r) + Cf_2(0, r)] \left[\int_{c/\tau}^0 f_1(x, \tau)\, dx + D \int_{c/\tau}^0 f_2(x, \tau)\, dx\right],$$

Table 2

Absolute Ruin Probability When $\lambda = 2$, $\beta = 1$, $\sigma_1 = 2$, and $c = 2.1$

u	$r = 0.01$	$r = 0.05$	$r = 0.1$	$r = 0.15$	$r = \tau = 0.2$
∞	0	0	0	0	0
20	0.00267377	0.000105153	5.85886×10^{-6}	4.53731×10^{-7}	1.7198×10^{-13}
10	0.00622598	0.00108483	0.000246231	0.0000564725	1.67287×10^{-11}
5	0.00885331	0.00253215	0.000923896	0.000317063	5.75863×10^{-7}
2	0.0112098	0.00430052	0.0021719	0.00115616	0.000461682
1	0.0152602	0.00815699	0.00584252	0.0046682	0.00380817
0	0.0416864	0.0343809	0.0319047	0.0306494	0.0297143
−1	0.173409	0.167108	0.164972	0.163889	0.163083
−2	0.333626	0.328546	0.326824	0.325951	0.325301
−5	0.735424	0.733407	0.732723	0.732377	0.732118
−10	0.992452	0.992394	0.992374	0.992365	0.992357
−10.5	1	1	1	1	1

and

$$
\begin{aligned}
f_1'(0, s) = & -\frac{2c}{\sigma_1^2} \exp\left\{-\frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right\} U\left(\frac{\lambda}{2s}, \frac{1}{2}; \frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right) \\
& - \frac{\lambda}{s\sigma_1^2} (c - \beta\sigma_1^2/2) \exp\left\{-\frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right\} U\left(\frac{\lambda}{2s} + 1, \frac{3}{2}; \frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right), \\
f_2'(0, s) = & \exp\left\{-\frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right\} M\left(\frac{1}{2} + \frac{\lambda}{2s}, \frac{3}{2}; \frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right) \\
& - \frac{2c}{s\sigma_1^2} (c - \beta\sigma_1^2/2) \exp\left\{-\frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right\} M\left(\frac{1}{2} + \frac{\lambda}{2s}, \frac{3}{2}; \frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right) \\
& + \frac{s + \lambda}{3s^2} (c - \beta\sigma_1^2/2)^2 \exp\left\{-\frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right\} M\left(\frac{3}{2} + \frac{\lambda}{2s}, \frac{5}{2}; \frac{(c - \beta\sigma_1^2/2)^2}{s\sigma_1^2}\right).
\end{aligned}
$$

Note that we have assumed that $2c > \beta\sigma_1^2$, otherwise, the expression for $f_2'(0, s)$ should times -1, and we have used (5.12) and (5.13) again.

EXAMPLE 5.1

Suppose the claim numbers constitute a Poisson process with $\lambda = 2$, the claim size is exponentially distributed with $\beta = 1$, the premium rate $c = 2.1$, and $\sigma_1 = 2$. Table 2 provides numerical results for the absolute ruin probability when $\tau = 0.2$ and for various r and u. The results show that absolute ruin probability is a decreasing function of both u and r.

6. THE CLASSICAL MODEL AND EXPONENTIAL CLAIMS

In this section we consider the classical compound Poisson model with constant interest rates, that is, $\alpha = 0$ and $\sigma_1^2 = 0$. Again we assume that the claim size is exponential with density given by (5.1). Then equation (5.3) reduces to

$$
\begin{aligned}
&(ru + c)\psi''(u) + [(ru + c)\beta - \lambda + r]\psi'(u) = 0, \qquad u \geq 0, \\
&(\tau u + c)\psi''(u) + [(\tau u + c)\beta - \lambda + \tau]\psi'(u) = 0, \qquad -\frac{c}{\tau} \leq u < 0. \qquad (6.1)
\end{aligned}
$$

Note that this equation has the same form as equation (4.1). Hence, the probability of absolute ruin is given by formulas (4.2)–(4.4), with

$$\begin{aligned} F(x) &= -\int \frac{(ru + c)\beta - \lambda + r}{ru + c}\,du \\ &= -\beta x + \left(\frac{\lambda}{r} - 1\right)\ln(rx + c) + K_1 \end{aligned} \tag{6.2}$$

and

$$\begin{aligned} G(x) &= -\int \frac{(\tau u + c)\beta - \lambda + \tau}{\tau u + c}\,du \\ &= -\beta x + \left(\frac{\lambda}{\tau} - 1\right)\ln(\tau x + c) + K_2. \end{aligned} \tag{6.3}$$

In particular, we find from (4.4) that

$$\psi(0) = \frac{c^{\lambda/\tau}\displaystyle\int_0^\infty e^{-\beta x}(rx + c)^{(\lambda/r)-1}\,dx}{c^{\lambda/\tau}\displaystyle\int_0^\infty e^{-\beta x}(rx + c)^{(\lambda/r)-1}\,dx + c^{\lambda/r}\displaystyle\int_{-c/\tau}^0 e^{-\beta x}(\tau x + c)^{(\lambda/\tau)-1}\,dx}. \tag{6.4}$$

REMARK

As noted earlier, the classical probability of ruin can be retrieved as the limit $\tau \to \infty$. For example, if $u = 0$, it follows from (6.4) that

$$\psi(0) = \frac{\displaystyle\int_0^\infty e^{-\beta x}(rx + c)^{(\lambda/r)-1}\,dx}{\displaystyle\int_0^\infty e^{-\beta x}(rx + c)^{(\lambda/r)-1}\,dx + c^{\lambda/r}\frac{1}{\lambda}} \tag{6.5}$$

in the limit. This formula agrees with a result that is due to Segerdahl; see Gerber (1979, formulas (1.10), (2.9), and (2.10) of Chapter 9) and Sundt and Teugels (1995).

EXAMPLE 6.1

Suppose the claim number is Poisson distributed with $\lambda = 1$, the claim size is exponentially distributed with $\beta = 0.5$, and the premium rate $c = 2$ (note that $c = \lambda/\beta$). Table 3 provides numerical results for the absolute ruin probability when $\tau = 0.1$ and for various r. The results show that absolute ruin probability is a decreasing function of both u and r.

EXAMPLE 6.2

Consider Example 6.1, but a different premium rate $c = 2.4$; that is, the security loading in this example is positive. From Tables 3 and 4, we see that the absolute ruin probabilities for the case of $c = 2.4$ are much smaller than the corresponding ones for $c = 2$.

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Table 3
Absolute Ruin Probability When $\lambda = 1$, $\beta = 0.5$, and $c = 2$

u	$r = 0.02$	$r = 0.04$	$r = 0.06$	$r = 0.08$	$r = \tau = 0.1$
∞	0	0	0	0	0
50	0.0012464	0.0000422388	4.09217×10^{-6}	7.15624×10^{-7}	1.82137×10^{-7}
10	0.320179	0.192239	0.12949	0.093039	0.0698537
5	0.477772	0.355779	0.284721	0.236586	0.201431
1	0.624891	0.530709	0.472185	0.429945	0.397133
0.5	0.644135	0.554586	0.498821	0.458481	0.427068
0.1	0.659617	0.573905	0.520491	0.481824	0.45169
0	0.663499	0.578761	0.525954	0.487724	0.45793
−0.1	0.667392	0.583034	0.531438	0.49365	0.464201
−0.5	0.683141	0.60335	0.553625	0.517626	0.489571
−1	0.703164	0.628414	0.581832	0.548108	0.521826
−5	0.8612	0.826248	0.804466	0.788697	0.776408
−10	0.980242	0.975267	0.972166	0.969921	0.968172
−20	1	1	1	1	1

Table 4
Absolute Ruin Probability When $\lambda = 1$, $\beta = 0.5$, and $c = 2.4$

u	$r = 0.02$	$r = 0.04$	$r = 0.06$	$r = 0.08$	$r = \tau = 0.1$
∞	0	0	0	0	0
50	0.000062902	4.015129×10^{-6}	5.77154×10^{-7}	1.30601×10^{-7}	3.99619×10^{-8}
10	0.0930903	0.0614744	0.0441736	0.0333804	0.0261247
5	0.182083	0.1431	0.118442	0.100954	0.0877594
1	0.29752	0.260138	0.235099	0.216343	0.201431
0.5	0.315424	0.278864	0.254326	0.235908	0.221233
0.1	0.330361	0.294559	0.270513	0.252452	0.238051
0	0.334183	0.298583	0.274672	0.256712	0.242392
−0.1	0.338046	0.302652	0.27888	0.261025	0.246788
−0.5	0.353977	0.319435	0.296235	0.278809	0.264915
−1	0.374963	0.341544	0.319097	0.302237	0.288795
−5	0.579761	0.557292	0.5422	0.530864	0.521826
−10	0.851033	0.843068	0.837718	0.8337	0.830496
−20	1	1	1	1	1

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