# Discussions of Papers Already Published

## "Stochastic Annuities," Daniel Dufresne, January 2007

### Bangwon Ko* and Andrew C. Y. Ng†

Professor Dufresne has obtained many interesting results on annuities when the stochastic future lifetime is approximated by using a combination of exponentials. A main idea behind the approximation is Theorem 3.1 (a), which states that the class of combinations of exponentials is a weakly dense subset of $\mathbb{P}([0, \infty))$, the set of all probability distributions with support $[0, \infty)$. In this discussion we give an alternative proof of this theorem using Erlang distributions, which are gamma distributions with positive integer-valued shape parameters, and hence more familiar to actuaries than logbeta distributions. In the process we shall see that the following classes of distributions are weakly dense subsets of $\mathbb{P}([0, \infty))$:

1. Mixtures of Erlang distributions
2. Phase-type distributions
3. Mixtures of hypoexponentials (Ross 2003, p. 284)
4. Combinations of exponentials.

At the end of the discussion, we shall show how the class of hypoexponentials is related with the class of logbeta distributions and thus see that the class of mixtures of logbeta distribution is weakly dense in $\mathbb{P}([0, \infty))$.

As pointed out in the proof of the theorem, the set of discrete distributions is a weakly dense subset of $\mathbb{P}([0, \infty))$. Hence, so too is the minimal weakly closed convex set containing the degenerate distributions. This means that, if a certain class of distributions can arbitrarily approximate the degenerate distributions, then the mixtures of such distributions will be weakly dense in $\mathbb{P}([0, \infty))$.

Let $\mathrm{Erl}(m, \lambda)$ denote the Erlang distribution with density function

$$\frac{\lambda^m x^{m-1}}{(m-1)!} e^{-\lambda x}, \qquad x > 0.$$

Then $\mathrm{Erl}(m, \lambda)$ has mean $m/\lambda$ and variance $m/\lambda^2$. To approximate a degenerate distribution at $k > 0$, we consider $\{\mathrm{Erl}(m, m/k)\}$ and let $m$ tend to infinity. For the degenerate distribution at 0, we take $\{\mathrm{Erl}(1, \lambda)\}$ and let $\lambda$ tend to infinity. This proves the first case in our claim.

A phase-type random variable is the absorption time of a continuous-time Markov chain with finitely many states, one of which is absorbing and all others are transient. Apart from being a weakly dense subset of $\mathbb{P}([0, \infty))$, the class of phase-type distributions is the smallest subset of $\mathbb{P}([0, \infty))$ with the following three properties. First, it contains the point mass at zero (by allowing the associated Markov chain to start from the absorbing state) and the class of exponential distributions. Second, it is closed under mixture and convolution. Third, each compound geometric sum of phase-type random variables follows a phase-type distribution: Let $N$ be a geometric random variable with support $\{1, 2, \ldots\}$ and probability of failure $q \in [0, 1)$, and $\{X_i\}$ be a sequence of independent random variables, independent of $N$ and following the same phase-type distribution; then

$$\sum_{i=1}^{N} X_i$$

follows a phase-type distribution. Readers interested in this characterization of the class of phase-type distributions can refer to Maier and O'Cinnedie (1992).

From the first two properties mentioned above, the class of mixtures of Erlang distributions is a subclass of the class of phase-type distributions; this proves the second case of our claim. But the class of phase-type distributions is *not* a subclass of the class of combinations of exponentials, nor is the class of combinations of exponentials a subclass of the class of phase-type distributions.

For the third case, we use the well-known result that the distribution of the sum of independent exponentials, each with the same mean, is an

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Erlang distribution. Consider independent exponential random variables, $X_i$, $i = 1, 2, \ldots, m$, with mean $1/\lambda_i$ where $\lambda_i \neq \lambda_j$ for $i \neq j$. Then the random variable

$$\sum_{i=1}^{m} X_i$$

is called a hypoexponential random variable. Now, letting all $\lambda_i$'s tend to $\lambda$, we see that the distribution of the hypoexponential random variable converges weakly to the Erl($m$, $\lambda$) distribution. This proves the third case in our claim. Notice that the class of mixtures of hypoexponentials is a subclass of the class of phase-type distributions.

For the fourth case, we recall a result from Ross (2003, Section 5.2.4), which is a textbook for Exam MLC, that the probability density function of the hypoexponential random variable is given by

$$\sum_{i=1}^{m} C_{i,m}\lambda_i e^{-\lambda_i x},$$

where

$$C_{i,m} = \prod_{j\neq i} \frac{\lambda_j}{\lambda_j - \lambda_i}.$$

Note that some of $C_{i,m}$'s are negative. Because $\sum_{i=1}^{m} C_{i,m} = 1$, this is a combination of exponentials. This implies that the class of mixtures of hypoexponential distributions is a proper subset of the class of combinations of exponentials. This completes the proof of the claim.

With the expression above for the hypoexponential random variable, we can obtain a relationship between the hypoexponential distribution and the logbeta distribution given in Appendix of the paper. If we take

$$\lambda_i = \lambda + (i - 1)\varepsilon,$$

we see that

$$\begin{aligned}
&\sum_{i=1}^{m} C_{i,m}\lambda_i e^{-\lambda_i x} \\
&\quad = \sum_{i=1}^{m} \frac{\lambda(\lambda + \varepsilon)\cdots[\lambda + (m-1)\varepsilon]}{(-1)^{i-1}(i-1)!(m-1)!\varepsilon^{m-1}} e^{-\lambda x-(i-1)\varepsilon x} \\
&\quad = \frac{\lambda(\lambda + \varepsilon)\cdots[\lambda + (m-1)\varepsilon]}{\varepsilon^{m-1}(m-1)!} \\
&\qquad \times e^{-\lambda x}\sum_{i=1}^{m}\binom{m-1}{i-1}(-1)^{i-1}e^{-(i-1)\varepsilon x} \\
&\quad = \frac{\lambda(\lambda + \varepsilon)\cdots[\lambda + (m-1)\varepsilon]}{\varepsilon^{m-1}(m-1)!} \\
&\qquad \times e^{-\lambda x}(1 - e^{-\varepsilon x})^{m-1}.
\end{aligned}$$

Comparing the kernel above with that of LogBeta($a$, $b$, $c$), we see that the class of logbeta distributions, when the parameter $a$ is a positive integer, is a subclass of the hypoexponential distributions with $m = a$, $\lambda = b$ and $\varepsilon = c$. Moreover, because

$$\lim_{\varepsilon\to 0} \frac{1 - e^{-\varepsilon x}}{\varepsilon x} = 1,$$

we see that the hypoexponential density converges pointwise to the density of an Erl($m$, $\lambda$) distribution as $\varepsilon$ tends to zero. This also shows that the class of mixtures of logbeta distributions is weakly dense in $\mathbb{P}([0, \infty))$.

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# ''The Impact of DC Pension Systems on Population Dynamics,'' Bonnie-Jeanne MacDonald and Andrew J. G. Cairns, January 2007

## MARK MALNATI*

By the authors' reasoning, transferring ownership of all houses and apartments from the government to individuals would cause large unknown variances in individual wealth and ''could lead to personal hardship and anxiety'' and thus a bad societal result. This ignores the tremendous benefits to society of having more people own a stake in the community and care deeply about schools, the crime rate, taxes, etc. Most homeowners also buy casualty insurance to avoid losses they can't afford. They can even take out a reverse mortgage to secure retirement income.

The conclusion that transferring retirement funding risk to individuals creates an undesirable result ignores many benefits that would result. At a young age many people who don't know about investing would start to get educated. They would become wealthier because investing in stocks and bonds would easily outperform the government treasuries that Social Security trust funds currently are invested in. If the private sector were allowed to offer solutions like inflation-indexed income annuities or target retirement age funds with automatic asset reallocation, then individuals could get rid of any risk they didn't want to take. They would feel like owners of the economy. This would also greatly reduce class warfare/envy as many low-income workers would become owners too. Imagine a world where auto workers cheer a factory closing because they think it is a good move to become profitable again. If most people can see how their favorite football player is worth $50 million a year to the team, why not their favorite CEO?

If the thought of a small percentage of the population so mismanaging their money by jumping in and out of stocks at exactly the wrong time for 40 years straight seems bothersome, why not have some threshold of retirement income at which the government takes money from all accounts to provide income to retirees who have exhausted theirs or workers who have become disabled?

I'm certain there are countries the authors can relocate themselves to where the government owns everything and inequality and anxiety are greatly reduced.

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## AUTHORS' REPLY

We would like to thank Mr. Malnati for his discussion. His comments suggest that the psychological benefits of holding a DC versus a DB pension plans in terms of quality of life and community involvement would be an interesting discussion, and would likely have strong arguments in both directions. The focus of our paper, however, was the impact on the *retirement dynamics* of an entire population of DC pension plan members who maintain static investment strategies (and not jumping in and out of stocks). Specifically, we concentrated on the retirement patterns of the participants. Furthermore, Mr. Malnati was quite right to suggest that there are a number of ways in which the governments could set up a DC environment with some element of protection. This could be the subject of future research. We were considering, however, a totally unregulated DC environment with no government involvement to assess the impact on the workforce demographics.

# "An Extreme Value Analysis of Advanced Age Mortality Data," Kathryn A. Watts, Debbie J. Dupuis, and Bruce L. Jones, October 2006

## MARK BEBBINGTON,* CHIN-DIEW LAI,† AND RIČARDAS ZITIKIS‡

We have read with great interest the thought-provoking contribution by Watts, Dupuis, and Jones on the application of extreme value theory to the analysis of senescent mortality. Needless to say, it is a research area of interest and importance to policy makers, public health professionals, social scientists, and actuaries.

In addition to senescent mortality, it is also of interest to have a perspective on the entire human lifespan, for the purposes of creating appropriate insurance premiums, pension plans, etc. Researchers and policy makers may also be interested in understanding and thus finding ways to rectify health and other inequities between (sub-)populations.

Bebbington, Lai, and Zitikis (2006) introduced a new statistical methodology for modeling human mortality. The technique was formulated, and interpreted, from a biological viewpoint by Bebbington, Lai, and Zitikis (2007b). This utilizes the flexible Weibull (FW) distribution (Bebbington, Lai, and Zitikis 2007a), with survival function

$$S_{\mathrm{FW}}(t) = \exp\{-e^{\alpha t - \beta/t} - \gamma t\},$$

which is a generalization of an extreme value distribution. (All distributions in this discussion are defined for $t > 0$.) Note the close relationship between the FW survival function and the Gompertz-Makeham survival function

$$S_{\mathrm{GM}}(t) = \exp\{-a(e^{bt} - 1) - ct\},$$

as well as the generalized and limiting extreme value survival functions in the paper under discussion,

$$S_{\mathrm{GEV}}(t) = 1 - \exp\left\{-\left[1 + \xi\left(\frac{t-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

and

$$S_{\mathrm{GEV}\infty}(t) = 1 - \exp\left\{-\exp\left[-\left(\frac{t-\mu}{\sigma}\right)\right]\right\}.$$

Bebbington, Lai, and Zitikis (2006, 2007b) use the mixture

$$S_{\mathrm{MIX}}(t) = pS_{\mathrm{FW}}(t) + (1-p)S_{\mathrm{RAW}}(t)$$

of the FW distribution and the reduced additive Weibull (RAW) distribution,

$$S_{\mathrm{RAW}}(t) = \exp\{-(at)^b - (at)^{1/b} - ct\},$$

to model endogenous and exogenous mortality, respectively, as illustrated on data sets of Canadian and Indonesian mortality. Note that this formulation implicitly does not set a finite upper bound on the maximum age. Rather, the hazard rate function (or instantaneous mortality rate) increases with age. An observation of interest in the analysis of advanced age mortality data is that this method identifies the *late life mortality deceleration* (see Mueller and Rose 1996), where the mortality rate, while still increasing, decelerates somewhat among the very old.

Bebbington, Lai, and Zitikis (2007c) show, in a comparative examination of Maori and non-Maori mortality in New Zealand, that the methodology can be used to understand the dynamics of, and inequities between, subpopulations. Inequalities between non-Maori and Maori health are clearly identified and explained, particularly the exogenous effects revealed in the analysis of the Maori data. One can, of course, similarly analyze health dynamics and related issues in other countries, and possibly between countries.

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# An Actuarial Premium Pricing Model for Nonnormal Insurance and Financial Risks in Incomplete Markets, Zinoviy Landsman and Michael Sherris, January 2007


## EDWARD FURMAN* AND RIČARDAS ZITIKIS†


There are many actuarial and other reasons to adopt various premium calculation principles. As a result, numerous risk measures have been developed in the literature. The paper by Landsman and Sherris, which is an extension of Landsman (2004), provides a powerful model for actuarial pricing in incomplete markets whose distribution tails might have various degrees of heaviness, ranging from the multivariate Gaussian to multivariate Student-*t* tails (e.g., Kotz and Nadarajah 2004). These two classes are covered by the elliptical family. The flexibility of the family allows for good fit to a variety of data sets, and its parametric form helps to develop statistical inference.

It frequently happens in the mathematical sciences that generalizing a special object highlights the essence of the matter and in this way helps to understand it better. This has happened, for example, with the classical central limit theorem for real-valued random variables when the theorem was generalized to random elements taking values in general abstract spaces (e.g., Ledoux and Talagrand 1991).

Likewise, we believe that several aspects of the paper by Landsman and Sherris, as well as by Landsman (2004), can also be seen more clearly when generalized appropriately. For this reason, we next define a "weighted tilting" of a random variable $X$ induced by a random vector $\mathbf{Y} = (Y_1, \ldots, Y_K)$:

$$H[X, \mathbf{Y}] = \frac{\mathbf{E}[Xw(\mathbf{Y})]}{\mathbf{E}[w(\mathbf{Y})]},$$

where $w(\mathbf{y})$ is a nonnegative function such that the expectation $\mathbf{E}[w(\mathbf{Y})]$ is nonzero and finite. Note that the weighted tilting $H[X, \mathbf{Y}]$ is a special case of expectation $\mathbf{E}[XZ]$, which is more general and appears frequently in the literature (e.g., the last equation on p. 3 of Landsman and Sherris) where one of the random variables, say, $Z$, usually satisfies the property $\mathbf{E}[Z] = 1$. However, the special structure of the random variable $Z$ as the ratio $w(\mathbf{Y})/\mathbf{E}[w(\mathbf{Y})]$ used in the present discussion connects our research with the notion of weighted distribution, and in this way leads to particularly fruitful research in the area. Indeed, we easily check the equation

$$H[X, \mathbf{Y}] = \int_0^\infty (1 - F_w(x))\, dx,$$

where $F_w(x)$ is the weighted distribution function defined by the equation

$$F_w(x) = \frac{\mathbf{E}[\mathbf{1}\{X \leq x\}w(\mathbf{Y})]}{\mathbf{E}[w(\mathbf{Y})]}.$$


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Note that when $w(\mathbf{Y}) \equiv 1$, then $F_w = F$, and so $H[X, \mathbf{Y}]$ is the net premium $\mathbf{E}[X]$.

Furman and Zitikis (2007) suggest using the concept of weighted distributions in an actuarial context. The concept helps to unify and derive desired properties of a number of premium calculation principles. We note, however, that the aforementioned paper is mainly concerned with the case $\mathbf{Y} = X$, since the main focus of the paper is on premium calculation principles and not on capital allocation rules. It should also be noted that weighted distributions, their properties, and manifold applications have been thoroughly discussed by Patil and Rao (1978) and Rao (1997); see also references therein.

To develop a better understanding of the weighted tilting $H[X, \mathbf{Y}]$, we next offer several examples. Let $\mathbf{Y} = X$, in which case $H[X, \mathbf{Y}]$ becomes the weighted premium $H[X]$ investigated by Furman and Zitikis (2007). For example, when $w(y) = \exp\{\lambda y\}$ for some $\lambda > 0$, then the weighted premium $H[X]$ is the classical Esscher's premium. For a discussion of statistical inference for Esscher's premium, we refer to, for example, Furman and Zitikis (2007). When $w(y) = \mathbf{1}\{y > F^{-1}(t)\}$ for some $t \in (0, 1)$, where $F$ is the distribution function of $X$ and $\mathbf{1}\{\cdot\}$ is the indicator function, then the weighted premium $H[X]$ is the conditional tail expectation (CTE). For statistical inference concerning the CTE, see Brazauskas et al. (2007), as well as to references therein.

Next, let $\mathbf{Y}$ again be one-dimensional, say, $Y$, but now it is not necessarily $X$, unlike in the discussion above. The case when $Y$ and $X$ are dependent is particularly interesting, since the independent case is trivial in the sense that it always yields the net premium $\mathbf{E}[X]$. For example, we may think about $Y$ as the sum $Y_1 + \cdots + Y_K$ or, more generally, a linear combination of the assets $Y_k$ of the earlier noted "portfolio" $\mathbf{Y}$ (e.g., Panjer and Jing 2001; Denault 2001; Furman and Landsman 2006). Continuing our discussion with a generic random variable $Y$, we note that if $w(y) = \exp\{\lambda y\}$ for some $\lambda > 0$, then we obtain the exponential tilting of $X$ induced by $Y$ (Wang 2002). For a discussion of statistical inference for Wang's exponential tilting, see a discussion in Furman and Zitikis (2007) related to the case $Y = X$; only slight adjustments are required in that discussion to cover the current case. When $w(y) = \mathbf{1}\{y \le G^{-1}(t)\}$ for some $t \in (0, 1)$ where $G$ is the distribution function of $Y$, we arrive at the absolute concentration curve (ACC) in econometrics, as well as at the notion of marginal conditional stochastic dominance (MCSD), which appears in, and has been thoroughly discussed by, Shalit and Yitzhaki (1994). For related statistical inferential results, see, for example, Schechtman et al. (2007), as well as references therein.

Our next example concerns the exponential tilting investigated by Landsman (2004) and Landsman and Sherris, which can be written as the above introduced weighted tilting $H[X, \mathbf{Y}]$ by taking the weight function

$$w(\mathbf{y}) = \frac{g(2^{-1}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}) - \lambda \mathbf{e}_k^T \mathbf{y})}{g(2^{-1}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}))},$$

where $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^K$ with 1 in the $k$th place, $\boldsymbol{\mu}$ is a $K$-dimensional vector, and $\Sigma$ is a $K \times K$ positive-definite matrix. The above weight function $w(\mathbf{y})$ is a special case of

$$w_h(\mathbf{y}) = \frac{g(2^{-1}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}) - h(\mathbf{y}))}{g(2^{-1}(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu}))},$$

where $h : \mathbf{R}^K \to \mathbf{R}$ is an appropriately chosen function, for example, $h(\mathbf{y}) = \lambda \mathbf{e}_k^T \mathbf{y}$, in which case we have the earlier defined $w(\mathbf{y})$. Note that when $g(x) = e^{-x}$, which is the Gaussian characteristic generator, then the weight function $w_h(\mathbf{y})$ yields

$$H[X, \mathbf{Y}] = \frac{\mathbf{E}[Xe^{h(\mathbf{Y})}]}{\mathbf{E}[e^{h(\mathbf{Y})}]};$$

compare the right-hand side with the Esscher premium and the corresponding exponential tilting.

Expressions for $H[X, \mathbf{Y}]$ with the above weight function $w(\mathbf{y})$ in terms of the population and elliptical parameters (e.g., Landsman 2004; Landsman and Sherris) play a crucial role in developing the corresponding (parametric) statistical inferential results. These include, for example, constructing confidence intervals for $H[X, \mathbf{Y}]$, testing hypotheses about one $H[X, \mathbf{Y}]$ or several $H[X_1, \mathbf{Y}], \ldots, H[X_k, \mathbf{Y}]$. In the latter case we may think, for example, about the situation when $\{X_1, \ldots, X_k\} \subseteq \{Y_1, \ldots, Y_K\}$. For details and references on testing hypotheses about several risk measure values, see, for example, Jones and Zitikis (2005) and Jones, Puri, and Zitikis (2006), and references therein.

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# A Risk Model with Multilayer Dividend Strategy, Hansjorg Albrecher and Jürgen Hartinger, April 2007

## ERIC C. K. CHEUNG*

Professors Albrecher and Hartinger are to be congratulated for providing a comprehensive paper that generalizes previous work on the Gerber-Shiu discounted penalty function and the discounted dividend payments for threshold strategies. In Section 3.3 a brilliant recursive approach (in terms of the number of layers $k$) is suggested for the evaluation of the expected discounted dividends so as to avoid numerical evaluation of a large number of constants, which are needed if the approach in Section 2.2 is used. However, it is not clear whether the approach in Section 3.3 can be applied to evaluate higher moments of the discounted dividends. The purpose of this discussion is to provide a recursive scheme (in terms of the order of moments) to compute these higher moments assuming the number of layers $k$ is fixed, using the approach described in Section 2.2. For illustration purposes, throughout the discussion it will be assumed the claim size is distributed as a combination of exponentials, that is, the p.d.f. is given by

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$$p(y) = P'(y) = \sum_{l=1}^{r} p_l \beta_l e^{-\beta_l y}, \qquad y > 0. \tag{D.1}$$

A combination of exponentials is used because it is mathematically tractable, and it is a good modeling tool. See Dufresne (2007) for the fitting of combinations of exponentials.

For each fixed $n \in \mathbb{N}$ and each fixed $i = 1, 2, \ldots, k$, $\rho_{n,i,j}$'s ($j = 0, 1, \ldots, r$) are the (assumed to be) distinct roots of the Lundberg fundamental equation (in $\xi$)

$$\lambda + n\delta - (c_i - a_i)\xi = \lambda \sum_{l=1}^{r} \frac{p_l \beta_l}{\beta_l + \xi},$$

and we know from Gerber and Shiu (1998) that for $\delta > 0$ there must be a unique positive root, which shall be denoted by $\rho_{n,i,0} > 0$.

Using identical notation as in Example 2.3, define, for $m = 0, 1, \ldots,$

$$W_{k,m}(u) = \sum_{i=1}^{k} W_{k,m}^{(i)}(u) I_{\{b_{i-1} \le u < b_i\}}. \tag{D.2}$$

As an illustration, two key simplifying assumptions will be made, namely, $k = 3$ and $a_1 = 0$. Later it will become clear that these assumptions can be removed.

### Analysis with Assumptions $k = 3$ and $a_1 = 0$

Equation (2.18) of the paper gives, for $m \in \mathbb{N}$,

$$\left\{ c_1 \frac{\partial}{\partial u} - \lambda - \delta m \right\} W_{3,m}^{(1)}(u) + \lambda \int_0^u W_{3,m}^{(1)}(u - z) p(z)\, dz = 0, \qquad u \in [0, b_1),$$

which, after application of the operator (see, e.g., Gerber and Shiu 2006b, Appendix A)

$$\prod_{l=1}^{r} \left( \frac{d}{du} + \beta_l \right), \tag{D.3}$$

yields

$$W_{3,m}^{(1)}(u) = \sum_{j=0}^{r} C_{m,j}^{(1)} e^{\rho_{m,1,j} u}, \qquad u \in [0, b_1), \tag{D.4}$$

where $C_{m,j}^{(1)}$'s ($j = 0, 1, \ldots, r$) satisfy

$$\sum_{j=0}^{r} C_{m,j}^{(1)} \left( \frac{1}{\beta_l + \rho_{m,1,j}} \right) = 0, \qquad l = 1, 2, \ldots, r. \tag{D.5}$$

### Proposition D.1

For all $m = 0, 1, \ldots,$

$$W_{3,m}(u) = \begin{cases} W_{3,m}^{(2)}(u) = \left( \dfrac{a_2}{\delta} \right)^m + \sum_{j=0}^{r} \sum_{n=1}^{m} C_{m,n,j}^{(2)} e^{\rho_{n,2,j} u}, & u \in [b_1, b_2) \\ W_{3,m}^{(3)}(u) = \left( \dfrac{a_3}{\delta} \right)^m + \sum_{j=1}^{r} \sum_{n=1}^{m} C_{m,n,j}^{(3)} e^{\rho_{n,3,j} u}, & u \in [b_2, \infty) \end{cases} \tag{D.6}$$

for some constants $C_{m,n,j}^{(2)}$'s ($j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m$) and $C_{m,n,j}^{(3)}$'s ($j = 1, 2, \ldots, r$; $n = 1, 2, \ldots, m$).

**PROOF**

This will be proved by induction on $m$. Clearly, (D.6) holds true for $m = 0$ since the zero-th moment is 1. Assume (D.6) is true with $m$ replaced by $m - 1$, that is,

$$W_{3,m-1}(u) = \begin{cases} W_{3,m-1}^{(2)}(u) = \left(\dfrac{a_2}{\delta}\right)^{m-1} + \sum_{j=0}^{r} \sum_{n=1}^{m-1} C_{m-1,n,j}^{(2)} e^{\rho_{n,2,j}u}, & u \in [b_1, b_2) \\ W_{3,m-1}^{(3)}(u) = \left(\dfrac{a_3}{\delta}\right)^{m-1} + \sum_{j=1}^{r} \sum_{n=1}^{m-1} C_{m-1,n,j}^{(3)} e^{\rho_{n,3,j}u}, & u \in [b_2, \infty) \end{cases}. \tag{D.7}$$

Substituting the claim size distribution assumption (D.1) and the first equation of the induction assumption (D.7) into (2.18) (with $i = 2$), applying the operator (D.3), it is found that $W_{3,m}^{(2)}(u)$ satisfies an ordinary differential equation of the form

$$\sum_{j=0}^{r+1} S_{m,j}^{(2)} \frac{\partial^j}{\partial u^j} W_{3,m}^{(2)}(u) = Q_m^{(2)} + \sum_{j=0}^{r} \sum_{n=1}^{m-1} Q_{m,n,j}^{(2)} e^{\rho_{n,2,j}u}, \qquad u \in [b_1, b_2), \tag{D.8}$$

for some constants $S_{m,j}^{(2)}$'s ($j = 0, 1, \ldots, r + 1$), $Q_m^{(2)}$ and $Q_{m,n,j}^{(2)}$'s ($j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m - 1$).

The general solution of (D.8) is given by

$$W_{3,m}^{(2)}(u) = B_m^{(2)} + \sum_{j=0}^{r} \sum_{n=1}^{m-1} C_{m,n,j}^{(2)} e^{\rho_{n,2,j}u} + \sum_{j=0}^{r} C_{m,m,j}^{(2)} e^{\gamma_{m,2,j}u}, \qquad u \in [b_1, b_2) \tag{D.9}$$

for some constants $B_m^{(2)}$, $C_{m,n,j}^{(2)}$'s ($j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m - 1$), $C_{m,m,j}^{(2)}$'s ($j = 0, 1, \ldots, r$), and $\gamma_{m,2,j}$'s ($j = 0, 1, \ldots, r$), which are to be determined.

By writing (2.18) (with $i = 2$) in the form

$$\left\{(c_2 - a_2) \frac{\partial}{\partial u} - \lambda - \delta m\right\} W_{3,m}^{(2)}(u) + \lambda \int_{u-b_1}^{u} W_{3,m}^{(1)}(u - z)p(z)\, dz$$
$$+ \lambda \int_{0}^{u-b_1} W_{3,m}^{(2)}(u - z)p(z)\, dz + a_2 m W_{3,m-1}^{(2)}(u) = 0, \qquad u \in [b_1, b_2),$$

it is found after some algebra that substitution of (D.1), (D.4), and (D.9) and the first equation of (D.7) together with application of (D.5) gives

$$\left\{-m\delta B_m^{(2)} + m a_2 \left(\frac{a_2}{\delta}\right)^{m-1}\right\}$$
$$+ \sum_{j=0}^{r} \sum_{n=1}^{m-1} C_{m,n,j}^{(2)} \left\{(c_2 - a_2)\rho_{n,2,j} - (\lambda + m\delta) + m a_2 \frac{C_{m-1,n,j}^{(2)}}{C_{m,n,j}^{(2)}} + \lambda \sum_{l=1}^{r} \frac{p_l \beta_l}{\beta_l + \rho_{n,2,j}}\right\} e^{\rho_{n,2,j}u}$$
$$+ \sum_{j=0}^{r} C_{m,m,j}^{(2)} \left\{(c_2 - a_2)\gamma_{m,2,j} - (\lambda + m\delta) + \lambda \sum_{l=1}^{r} \frac{p_l \beta_l}{\beta_l + \gamma_{m,2,j}}\right\} e^{\gamma_{m,2,j}u}$$
$$+ \lambda \sum_{l=1}^{r} p_l \beta_l \left\{\sum_{j=0}^{r} \frac{C_{m,j}^{(1)}}{\beta_l + \rho_{m,1,j}} e^{(\beta_l + \rho_{m,1,j})b_1} - \left(\frac{B_m^{(2)}}{\beta_l}\right) e^{\beta_l b_1} - \sum_{j=0}^{r} \sum_{n=1}^{m-1} \frac{C_{m,n,j}^{(2)}}{\beta_l + \rho_{n,2,j}} e^{(\beta_l + \rho_{n,2,j})b_1}\right.$$
$$\left. - \sum_{j=0}^{r} \frac{C_{m,m,j}^{(2)}}{\beta_l + \gamma_{m,2,j}} e^{(\beta_l + \gamma_{m,2,j})b_1}\right\} e^{-\beta_l u} = 0, \qquad u \in [b_1, b_2). \tag{D.10}$$

Equating the constant term with 0 yields

$$B_m^{(2)} = \left(\frac{a_2}{\delta}\right)^m. \tag{D.11}$$

Equating the coefficients of $e^{\rho_{n,2,j}u}$ ($j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m-1$) in (D.10) with 0 and using the definition of the corresponding $\rho_{n,2,j}$'s, we obtain

$$C^{(2)}_{m,n,j} = \frac{m}{m-n}\left(\frac{a_2}{\delta}\right) C^{(2)}_{m-1,n,j}, \qquad j = 0, 1, \ldots, r;\ n = 1, 2, \ldots, m-1, \tag{D.12}$$

which says that $C^{(2)}_{m,n,j}$'s can be easily obtained from the corresponding $C^{(2)}_{m-1,n,j}$'s for $j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m-1$. It is interesting to note that Gerber and Shiu (2006a, eq. (8.17)) obtain a recursive formula very similar to (D.12) to compute the moments of discounted dividends in a Wiener process model.

By looking at the coefficients of $e^{\gamma_{m,2,j}u}$ ($j = 0, 1, \ldots, r$) in (D.10), it is clear from the definition of the corresponding $\rho_{m,2,j}$'s that we can set

$$\gamma_{m,2,j} = \rho_{m,2,j}, \qquad j = 0, 1, \ldots, r. \tag{D.13}$$

Finally, the coefficients of $e^{\beta_l u}$ ($l = 1, 2, \ldots, r$) in (D.10) together with (D.11) and (D.13) imply

$$\sum_{j=0}^{r} C^{(1)}_{m,j}\left(\frac{1}{\beta_l + \rho_{m,1,j}}\right) e^{\rho_{m,1,j}b_1} = \frac{1}{\beta_l}\left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C^{(2)}_{m,n,j}\left(\frac{1}{\beta_l + \rho_{n,2,j}}\right) e^{\rho_{n,2,j}b_1}, \qquad l = 1, 2, \ldots, r. \tag{D.14}$$

With (D.11) and (D.13), we observe that (D.9) is indeed the first equation of (D.6), and therefore we have completed half of the induction. The remaining task is to prove the second equation of (D.6) and to obtain enough number of linear equations to solve for the coefficients.

Now, an identical procedure can be applied to the quantity $W^{(3)}_{3,m}(u)$ to show that the second equation of (D.6) holds, and therefore the details are skipped. The only additional thing we need is the asymptotic formula (2.19), which together with the fact that $\rho_{m,3,0} > 0$ implies the coefficient of $e^{\rho_{m,3,0}u}$ must be 0. Analogous to (D.12) and (D.14), we arrive at

$$C^{(3)}_{m,n,j} = \frac{m}{m-n}\left(\frac{a_3}{\delta}\right) C^{(3)}_{m-1,n,j}, \qquad j = 1, 2, \ldots, r;\ n = 1, 2, \ldots, m-1, \tag{D.15}$$

and

$$\frac{1}{\beta_l}\left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C^{(2)}_{m,n,j}\left(\frac{1}{\beta_l + \rho_{n,2,j}}\right) e^{\rho_{n,2,j}b_2} = \frac{1}{\beta_l}\left(\frac{a_3}{\delta}\right)^m + \sum_{j=1}^{r}\sum_{n=1}^{m} C^{(3)}_{m,n,j}\left(\frac{1}{\beta_l + \rho_{n,3,j}}\right) e^{\rho_{n,3,j}b_2}, \qquad l = 1, 2, \ldots, r. \tag{D.16}$$

The continuity condition (2.20) gives

$$\sum_{j=0}^{r} C^{(1)}_{m,j} e^{\rho_{m,1,j}b_1} = \left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C^{(2)}_{m,n,j} e^{\rho_{n,2,j}b_1} \tag{D.17}$$

and

$$\left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C^{(2)}_{m,n,j} e^{\rho_{n,2,j}b_2} = \left(\frac{a_3}{\delta}\right)^m + \sum_{j=1}^{r}\sum_{n=1}^{m} C^{(3)}_{m,n,j} e^{\rho_{n,3,j}b_2}. \tag{D.18}$$

Now it becomes clear that given we know a lower moment of order $m-1$, $C^{(2)}_{m,n,j}$'s ($j = 0, 1, \ldots, r$; $n = 1, 2, \ldots, m-1$) and $C^{(3)}_{m,n,j}$'s ($j = 1, 2, \ldots, r$; $n = 1, 2, \ldots, m-1$) can be found directly from (D.12) and (D.15), while the remaining $3r + 2$ coefficients $C^{(1)}_{m,j}$'s ($j = 0, 1, \ldots, r$), $C^{(2)}_{m,m,j}$'s ($j = 0, 1, \ldots, r$), and $C^{(3)}_{m,m,j}$'s ($j = 1, 2, \ldots, r$) can be solved from the system of $3r + 2$ equations, which consists of (D.5), (D.14), (D.16), (D.17), and (D.18). □

Note that the above proof of Proposition D.1 indeed gives a recursion scheme (with respect to the order of moment) for the evaluation of the moments of discounted dividends in a three-layer model where $a_1 = 0$.

## Analysis with Assumption $a_1 = 0$

The above results can be generalized to any $k \geq 2$ along the same lines. Therefore the following proposition can be stated without proof.

### Proposition D.2

When $a_1 = 0$, for any integer $k \geq 2$, under the claim size distribution assumption (D.1), the $m$th moment ($m \in \mathbb{N}$) of the discounted dividends is given by (D.2) with

$$W_{k,m}(u) = \begin{cases} W_{k,m}^{(1)}(u) = \sum_{j=0}^{r} C_{k,m,j}^{(1)} e^{\rho_{m,1,j}u}, & u \in [0, b_1) \\ W_{k,m}^{(i)}(u) = \left(\dfrac{a_i}{\delta}\right)^m + \sum_{j=0}^{r} \sum_{n=1}^{m} C_{k,m,n,j}^{(i)} e^{\rho_{n,i,j}u}, & u \in [b_{i-1}, b_i),\ i = 2, 3, \ldots, k \end{cases}.$$

For each $m \in \mathbb{N}$, the $(r + 1)\{1 + m(k - 1)\}$ coefficients can be determined using the following algorithm. First, $(r + 1)(k - 1)(m - 1) + 1$ of them can be computed directly as follows:

$$C_{k,m,n,j}^{(i)} = \frac{m}{m-n}\left(\frac{a_i}{\delta}\right) C_{k,m-1,n,j}^{(i)}, \qquad i = 2, 3, \ldots, k-1;\ j = 0, 1, \ldots, r;\ n = 1, 2, \ldots, m-1,$$

$$C_{k,m,n,j}^{(k)} = \frac{m}{m-n}\left(\frac{a_k}{\delta}\right) C_{k,m-1,n,j}^{(k)}, \qquad j = 1, 2, \ldots, r;\ n = 1, 2, \ldots, m-1,$$

and

$$C_{k,m,n,0}^{(k)} = 0, \quad n = 1, 2, \ldots, m.$$

Then the remaining $(r + 1)k - 1$ coefficients can be solved from the following system of $(r + 1)k - 1$ linear equations:

$$\sum_{j=0}^{r} C_{k,m,j}^{(1)} \left(\frac{1}{\beta_l + \rho_{m,1,j}}\right) = 0, \qquad l = 1, 2, \ldots, r.$$

$$\sum_{j=0}^{r} C_{k,m,j}^{(1)} \left(\frac{1}{\beta_l + \rho_{m,1,j}}\right) e^{\rho_{m,1,j}b_1} = \frac{1}{\beta_l}\left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(2)} \left(\frac{1}{\beta_l + \rho_{n,2,j}}\right) e^{\rho_{n,2,j}b_1}, \qquad l = 1, 2, \ldots, r,$$

$$\frac{1}{\beta_l}\left(\frac{a_i}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(i)} \left(\frac{1}{\beta_l + \rho_{n,i,j}}\right) e^{\rho_{n,i,j}b_i}$$
$$= \frac{1}{\beta_l}\left(\frac{a_{i+1}}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(i+1)} \left(\frac{1}{\beta_l + \rho_{n,i+1,j}}\right) e^{\rho_{n,i+1,j}b_i}, \qquad i = 2, 3, \ldots, k-1;\ l = 1, 2, \ldots, r,$$

$$\sum_{j=0}^{r} C_{k,m,j}^{(1)} e^{\rho_{m,1,j}b_1} = \left(\frac{a_2}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(2)} e^{\rho_{n,2,j}b_1},$$

and

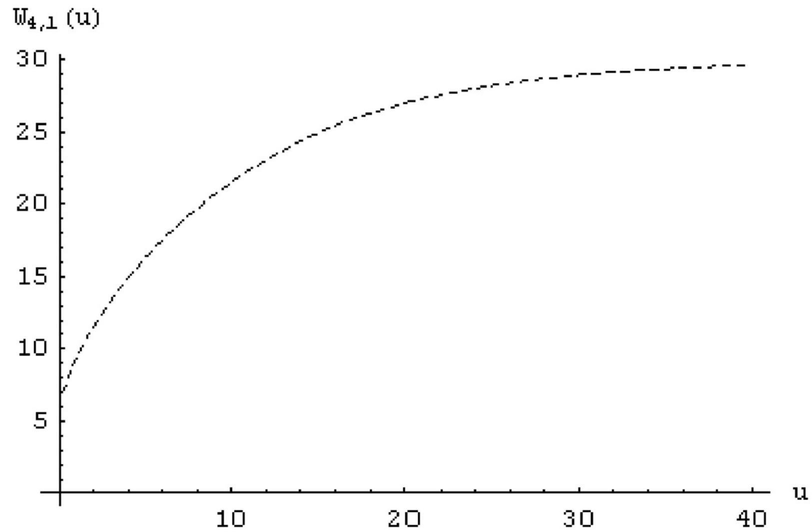
$$\left(\frac{a_i}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(i)} e^{\rho_{n,i,j}b_i} = \left(\frac{a_{i+1}}{\delta}\right)^m + \sum_{j=0}^{r}\sum_{n=1}^{m} C_{k,m,n,j}^{(i+1)} e^{\rho_{n,i+1,j}b_i}, \qquad i = 2, 3, \ldots, k-1.$$

## Analysis without Both Simplifying Assumptions

The assumption $a_1 = 0$ can be further removed by noting that for $k \geq 2$, if we let $b_1$ go to 0 in a $k$-layer model with $a_1 = 0$, the model is equivalent to a $(k - 1)$–layer model with $a_1 > 0$. Since this is a trivial extension of Proposition D.2, the details are omitted.

Figure D.1

**$W_{4,1}(u)$ under Bohman's Distribution for Parameter Set (2.13)**




## Example

Assuming the claim size follows a Bohman's distribution (see, e.g., Dickson and Wong 2004) with p.d.f.

$$p(y) = \frac{1}{3}\left(\frac{1}{2}\, e^{-(1/2)x}\right) + \frac{2}{3}\,(2e^{-2x}), \qquad y > 0,$$

which has mean 1, we use the parameters (2.13) (except $\beta = 1$) and $\delta = 0.01$, and we fix $k = 4$. Figure D.1 shows $W_{4,1}(u)$ as a function of $u$. Furthermore, if $\mathbb{D}_u$ denotes the random variable representing the total discounted dividend payments starting with initial surplus $u$, then we are also interested in the higher standardized moments of $\mathbb{D}_u$, namely, the coefficient of variation, the coefficient of skewness, and the coefficient of kurtosis, which are defined respectively as follows (see, e.g., Stuart and Ord 1998):

$$CV(u) = \frac{\sqrt{\mathrm{E}[(\mathbb{D}_u - \mathrm{E}[\mathbb{D}_u])^2]}}{\mathrm{E}[\mathbb{D}_u]} = \frac{\sqrt{W_{4,2}(u) - \{W_{4,1}(u)\}^2}}{W_{4,1}(u)},$$

$$CS(u) = \frac{\mathrm{E}[(\mathbb{D}_u - \mathrm{E}[\mathbb{D}_u])^3]}{\{\mathrm{E}[(\mathbb{D}_u - \mathrm{E}[\mathbb{D}_u])^2]\}^{3/2}} = \frac{W_{4,3}(u) - 3W_{4,2}(u)W_{4,1}(u) + 2\{W_{4,1}(u)\}^3}{(W_{4,2}(u) - \{W_{4,1}(u)\}^2)^{3/2}},$$

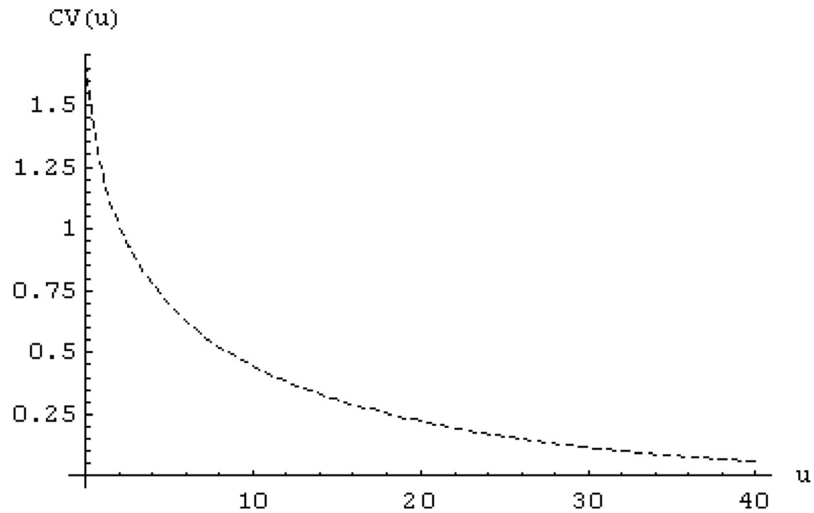
and

$$CK(u) = \frac{\mathrm{E}[(\mathbb{D}_u - \mathrm{E}[\mathbb{D}_u])^4]}{\{\mathrm{E}[(\mathbb{D}_u - \mathrm{E}[\mathbb{D}_u])^2]\}^2}$$

$$= \frac{W_{4,4}(u) - 4W_{4,3}(u)W_{4,1}(u) + 6W_{4,2}(u)\{W_{4,1}(u)\}^2 - 3\{W_{4,1}(u)\}^4}{(W_{4,2}(u) - \{W_{4,1}(u)\}^2)^2}.$$

Figures D.2, D.3, and D.4 show, respectively, $CV(u)$, $CS(u)$, and $CK(u)$ as a function of $u$.

Figure D.2
**$CV(u)$ under Bohman's Distribution for Parameter Set (2.13)**




**REMARK**

It is clear from Proposition D.2 that as the number of layers and the order of the moments increase, the procedure becomes more and more computationally involved. However, it is not obvious that the approach in Section 3.3 can be used to set up a "global" recursion to find the higher moments of the discounted dividends. Therefore, the technique used in Section 2.2 is still of importance. Furthermore, when the claim size is distributed as a finite scale and shape mixture of Erlangs, the above analysis can still be carried out if we combine the procedures given here and those given by Cheung (2007).

Figure D.3
**$CS(u)$ under Bohman's Distribution for Parameter Set (2.13)**

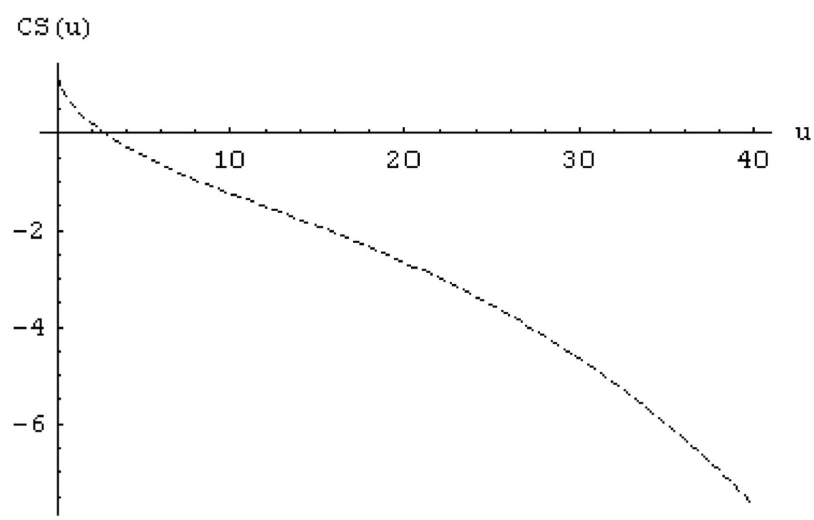
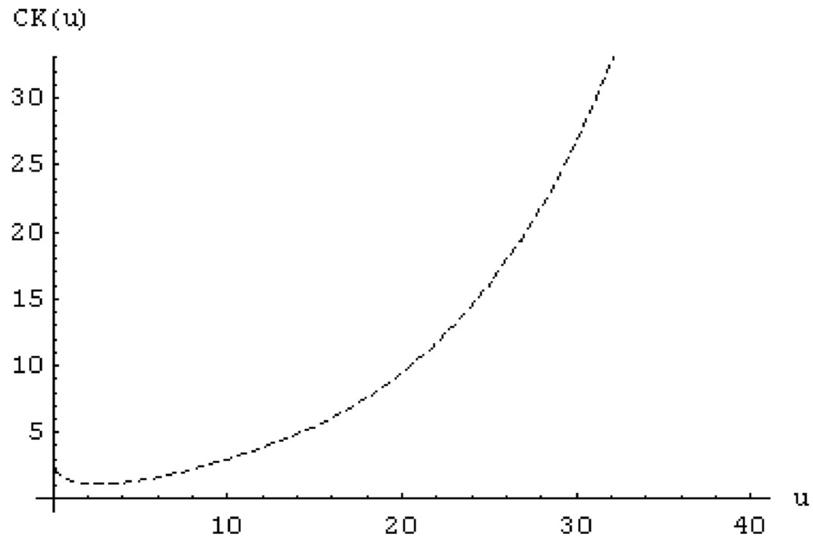


Figure D.4
**$CK(u)$ under Bohman's Distribution for Parameter Set (2.13)**

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# Call for Papers
# Risk Theory Society Annual Seminar

**April 25–27, 2008**
**Colorado State University**
**Fort Collins, Colorado, USA**

The Risk Theory Society will hold its annual seminar on April 25–27, 2008 at Colorado State University in Fort Collins, Colorado.

The Risk Theory Society is a group of economists, financial economists, and actuaries who undertake theoretical and applied research in the areas of insurance economics, financial economics related to insurance markets, actuarial science, and, more generally, in the economic analysis of risk and uncertainty. Membership in the Society is earned by presenting a paper at the yearly Risk Theory Society Seminar.

The Risk Theory Society invites interested parties to submit a paper for the 2008 meeting. Authors of selected papers present their research to the members in a seminar setting. Historically, the seminar has been thought provoking and the no-holds-barred approach has allowed for rigorous discussion of an author's findings.

Five-page abstracts or a rough draft of a paper on any relevant topic should be e-mailed by December 17th, 2007 to:

Professor John M. Marshall
University of California, Santa Barbara
Email: marshall@econ.ucsb.edu

The program committee will notify successful authors by January 16th, 2008. Papers must be completed and sent to all seminar participants by March 14th. Financial support for travel (North American portion only) for one author per paper may be available.

For further information, please contact Professor Marshall or visit the Risk Theory Society web site at http://aria.org/rts. For details regarding local arrangements for the 2008 Seminar, please visit the web site or contact Professor Vickie Bajtelsmit at vickie.bajtelsmit@business.colostate.edu.

THE ACTUARIAL FOUNDATION

## 2008 INDIVIDUAL GRANTS COMPETITION
## *** Announcement of The Actuarial Foundation's AERF Committee and Society of Actuaries' CKER Competitions ***

**Purpose**

The Actuarial Foundation's (TAF) AERF Committee and the Society of Actuaries' Committee on Knowledge Extension Research (CKER) announce the 2008 Individual Grants Competition to support the advancement of knowledge in actuarial science. Grants will be funded through TAF and the SOA.

*Note*: This is the only solicitation for grant proposals; one application will be used to submit proposals to both organizations. Each organization may award grants.

**Who may apply**

Individuals and groups may apply. Letters of intent are expected to come from:

*a) practitioners, usually where the research project is not part of their employment*
*b) industry and university researchers embarking on a collaborative project*
*c) academics, individual or as a team*

Graduate and undergraduate students are not eligible to apply individually, but may be part of a group in one of the above categories.

**Selection criteria**

The project may be either theoretical or empirical in nature. A key criterion for both organizations is that the project should have the potential to contribute significantly to the advancement of knowledge in actuarial science.

TAF's AERF Committee gives preference to projects relating to current policy issues or having direct applications, and those that further the basic or continuing education of actuaries. Proposals for innovative developments in actuarial education also are invited for consideration.

**Publication of research**

The result of each research project should be a manuscript suitable for publication in a scholarly journal or as a monograph. The sponsoring organization(s) reserve the right to publish the results of any research they have funded. If this right is not exercised, suitable credit must be given the sponsoring organization(s) at the time of publication. Information on previously published manuscripts is available on TAF's website at http://www.actuarialfoundation.org/ under "Completed Research."

**Research contract**

The selected researcher and the sponsoring organization(s) will enter into a formal contractual arrangement. Oversight will be coordinated by the funding organization(s).

**Intent to propose**

Interested parties must submit a brief letter of intent (1-2 pages) outlining the nature of the research project, a rough estimate of the funding requirements, the anticipated distribution of results and the qualifications of the applicant(s). Letters should be submitted to Sheree Baker (sbaker@soa.org) by **October 15, 2007**.

**Application form**

Application forms will be distributed by **November 21, 2007**, to researchers being invited to submit proposals for funding consideration. Completed application forms are due by **January 4, 2008**.

**Awards**

Grant awards will be announced by **March 17, 2008**.

**Questions to**

Curtis E. Huntington, Chairperson of TAF's AERF Committee and CKER
phone: 734-763-0293 fax: 734-764-7048 E-mail: chunt@umich.edu