Normalized Exponential Tilting: Pricing and Measuring Multivariate Risks

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Abstract

This article discusses methods of risk-neutralizing multivariate probability distributions by applying exponential tilting to the joint probability density function with respect to a set of reference risks. To ensure consistent interpretations of the exponential tilting parameters, a normalization procedure is performed on the reference risks via percentile mapping to standard normal variables. The article establishes links between normalized exponential tilting and multivariate probability distortions. It provides efficient methods for computing risk-neutralized multivariate probability distributions, and it gives illustrative examples in pricing contingent claims on multiple risks.

1. Introduction

Change of probability measure is a common theme in pricing and valuation of risks and contingent claims. In no-arbitrage financial pricing theory, the price of a contingent claim is evaluated as the expected payoff under a risk-neutral probability measure that is different from its statistical counterpart (Harrison and Kreps 1979). In a complete market, the risk-neutral probability measure can be readily inferred from market transaction data. In an incomplete market, however, we do not have sufficient market data to infer a risk-neutral distribution; instead, we have historical data that allow us to estimate statistical distribution of the potential outcomes and their respective likelihoods. The question then arises as to how to construct a risk-neutral density from the estimated statistical density, as a basis for pricing contingent claims written on the underlying risks. This change of measure is referred to as ‘‘risk neutralizing’’ the statistical distribution.

Exponential tilting, as a general method for neutralizing the statistical distribution, has been discussed by many authors (Buhlmann 1980; Gerber and Shiu 1996; Madan and Unal 2004, among many others). Exponential tilting is broadly consistent with much of the current literature on no-arbitrage pricing of contingent claims (Duffie 1992; Heston 1993; Karatzas and Shreve 1991; Gerber and Shiu 1996) and is potentially widely applicable in pricing risks embedded in loan defaults, mortgage refinancing, electricity trading, weather derivatives, and catastrophic insurance.

This article starts by defining *exponential tilting* of the probability density function of X with respect to a reference variable Y. What makes this article unique from earlier ones is by introducing a normalization procedure on the reference Y via percentile mapping to a standard normal variable Z. It shows that *normalized exponential tilting* of the probability density function of X (with respect to Z) is equivalent to applying the Wang transform (Wang 2000) to the cumulative distribution of X, and is an extension of the Capital Asset Pricing Model to risks with general-shaped distributions.

The need for changing multivariate probability measures arises in pricing contingent claims on *multiple* underlying assets or liabilities (and when allocating total company risk capital to various business units). The article extends the normalized exponential tilting to multivariate cases. It gives efficient routines for computing the risk-neutralized multivariate probability distribution and provides examples of pricing contingent claims on multiple risks.

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2. NORMALIZED EXPONENTIAL TILTING AND PROBABILITY DISTORTION

We shall consider *risks* that are random variables in some probability space (Ω, P). For any random variable X, let F_X represent its cumulative distribution function (CDF). Let f_X represent the probability density function (p.d.f) of X (in the discrete case, f_X represents the probability function).

Consider two risks X and Y. Assume that X is absolutely continuous with respect to Y; that is, $f_Y(x) > 0$ for all points x with $f_X(x) > 0$.

DEFINITION 2.1

For each scenario ω in the probability space (Ω, P), the exponential tilting of X with respect to Y is defined by the risk neutralization

$$f_X^*(x(\omega)) = c \cdot f_X(x(\omega)) \cdot \exp(\lambda \cdot y(\omega)), \tag{2.1}$$

where f_X and f_X^* represent the p.d.f. of X, before and after the exponential tilting, respectively, and c is a normalizing coefficient. The real-valued parameter λ in equation (2.1) controls the magnitude of risk adjustment.

In terms of probability density function, the exponential tilting of X with respect to Y can be written as

$$f_X^*(x) = f_X(x) \cdot \frac{\mathrm{E}[\exp(\lambda Y)|X = x]}{\mathrm{E}[\exp(\lambda Y)]}. \tag{2.2}$$

The ratio

$$RN(x) = \frac{f_X^*(x)}{f_X(x)} = \frac{\mathrm{E}[\exp(\lambda Y)|X = x]}{\mathrm{E}[\exp(\lambda Y)]}$$

gives the Radon-Nikodym derivative of f_X^* w.r.t. f_X.

For the exponential tilting in equation (2.1), we do not have a consistent interpretation of the λ parameter; the scale and shape of the reference variable Y can significantly impact the result of the exponential tilting. In order to get a consistent interpretation of λ, a normalization procedure of the reference variable Y through percentile-matching to a standard normal variable Z is proposed. In other words, $Y = F_Y^{-1}(\Phi(Z))$, with Φ being the CDF of standard normal Z, and $F_Y^{-1}(p) = \inf\{y|F_Y(y) \geq p\}$. We shall call Z a normalized variable of Y. Next let us use Z to replace the reference variable Y in the exponential tilting.

DEFINITION 2.2

Let Z be a normalized variable of the reference Y. We can define a *normalized exponential tilting* of X with respect to reference Y as

$$f_X^*(x) = f_X(x) \cdot \frac{\mathrm{E}[\exp(\lambda Z)|X = x]}{\mathrm{E}[\exp(\lambda Z)]}. \tag{2.3}$$

Note the differences in Definitions 2.1 and 2.2:

- In Definition 2.1, we apply exponential tilting of X w.r.t. Y
- In Definition 2.2, we first perform normalization via percentile mapping from Y to standard normal variable Z, then apply exponential tilting of X w.r.t. Z.

Now let us introduce probability distortions, as another method of changing probability measures, and as a general class of coherent measures of risk (including the conditional tail expectation measure as advocated by Artzner et al. 1999).

DEFINITION 2.3

Let g:[0, 1] $\rightarrow$ [0, 1] be a differentiable function with $g(0) = 0$ and $g(1) = 1$. Given the CDF $F(x)$ for a random variable X, the transformed CDF

$$F_X^*(x) = g(F(x)) \tag{2.4}$$

defines a *probability distortion,* representing a change of probability measure.

The probability distortion in equation (2.4) yields the following Radon-Nikodym derivative:

- In the discrete case where X takes on values $\{x_1, \ldots, x_{i-1}, x_i, \ldots\}$:

$$RN_g(x_i) = \frac{f_X^*(x_i)}{f_X(x_i)} = \frac{g(F_X(x_i)) - g(F_X(x_{i-1}))}{F_X(x_i) - F_X(x_{i-1})};$$

- In the continuous case where X has a positive probability density at x:

$$RN_g(x) = g'(F_X(x)).$$

The following distortion is the Wang transform (Wang 2000; Dowd 2005):

$$F_X^*(x) = g(F_X(x)) = \Phi\lfloor\Phi^{-1}(F_X(x)) - \lambda\rfloor. \tag{2.5}$$

In the case that X is a continuous variable, the Wang transform corresponds to the following Radon-Nikodym derivative:

$$RN_g(x) = g'(F_X(x)) = \exp(\lambda \cdot \Phi^{-1}(F_X(x))) \cdot \exp\left(-\frac{\lambda^2}{2}\right).$$

The Wang transform in equation (2.5) preserves both normal and lognormal distributions (this property reveals connections with the Black-Scholes (1973) formula for options):

- If F has a Normal(μ,σ^2) distribution, F^* is also a normal distribution with $\mu^* = \mu + \lambda\sigma$ and $\sigma^* = \sigma$
- If F has a log-normal(μ,σ^2) distribution such that $\ln(X) \sim$ Normal(μ,σ^2), F^* is another log-normal distribution with $\mu^* = \mu + \lambda\sigma$ and $\sigma^* = \sigma$.

Theorem 2.1

Assume that X and Y have bivariate normal copula with a correlation coefficient of $\rho_{X,Y}$. The normalized exponential tilting in equation (2.3) is equivalent to applying the Wang transform

$$F_X^*(x) = g(F_X(x)) = \Phi\lfloor\Phi^{-1}(F_X(x)) - \beta\rfloor, \text{ with } \beta = \rho_{X,Y} \cdot \lambda.$$

A proof can be found in Wang (2003).

REMARK

Wang (2003) discussed the normalized exponential tilting in the context of Buhlmann's economic model. In another line of independent research, Goovaerts and Lauven (2004) defined a transformation that is very similar to the normalized exponential tilting; they also gave an axiomatic characterization of the arbitrage-free price of financial derivatives.

Next we will extend the normalized exponential tilting to multivariate risks.

3. MULTIVARIATE NORMALIZED EXPONENTIAL TILTING

Consider n variables $\{X_1, X_2, \ldots, X_n\}$ and k references $\{Y_1, Y_2, \ldots, Y_k\}$ on a probability space (Ω, P).

DEFINITION 3.1

For each scenario ω in the probability space (Ω, P), the exponential tilting of $\{X_1, X_2, \ldots, X_n\}$ with respect to references $\{Y_1, Y_2, \ldots, Y_k\}$ is defined by the following Radon-Nikodym derivative:

$$\frac{f^*(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))}{f(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))} = c \cdot \mathrm{E}\left[\exp\left(\sum_{j=1}^{k} \lambda_j Y_j(\omega)\right)\right], \tag{3.1}$$

where $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are real-valued parameters that control the magnitude of risk adjustment, and c is a normalizing coefficient.

In terms of joint probability density function, we can reformulate equation (3.1) as

$$\frac{f^*(x_1, x_2, \ldots, x_n)}{f(x_1, x_2, \ldots, x_n)} = c \cdot \mathrm{E}\left[\exp\left(\sum_{j=1}^{k} \lambda_j Y_j\right) | X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n\right].$$

In Definition 3.1 much flexibility is left in the choice of the references $\{Y_1, Y_2, \ldots, Y_k\}$. For instance, one can choose the references $\{Y_1, Y_2, \ldots, Y_k\}$ to be the risks $\{X_1, X_2, \ldots, X_n\}$ themselves, the company aggregate, or some industry indices.

Note that the scale and shape of the reference variables $\{Y_1, Y_2, \ldots, Y_k\}$ can significantly impact the result of the exponential tilting. To get consistent interpretations of the parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, we need to apply a normalization procedure to all references $\{Y_1, Y_2, \ldots, Y_k\}$.

DEFINITION 3.2

Assume that there exist standard normal variables $\{Z_1, Z_2, \ldots, Z_k\}$ such that

$$Y_1 = F_{Y_1}^{-1}(\Phi(Z_1)), Y_2 = F_{Y_2}^{-1}(\Phi(Z_2)), \ldots, Y_k = F_{Y_k}^{-1}(\Phi(Z_k)).$$

We define the normalized multivariate exponential tilting of $\{X_1, X_2, \ldots, X_n\}$ with respect to references $\{Y_1, Y_2, \ldots, Y_k\}$ as the following:

$$\frac{f^*(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))}{f(x_1(\omega), x_2(\omega), \ldots, x_n(\omega))} = c \cdot \mathrm{E}\left[\exp\left(\sum_{j=1}^{k} \lambda_j Z_j(\omega)\right)\right]. \tag{3.2}$$

4. CHANGE OF MEASURE VIA MULTIVARIATE DISTORTION

As another way of changing probability measure, we will now extend probability distortion to multivariate distributions. In doing so, the correlation structure comes into play.

Consider multivariate risks $\{X_1, X_2, \ldots, X_n\}$ that have marginal CDFs

$$\{F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)\}, \text{ respectively.}$$

Assume that $\{X_1, X_2, \ldots, X_n\}$ have a joint CDF specified by

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)),$$

where $C(\ldots)$ is a multivariate uniform distribution or a copula function (see Embrechts, McNeil, and Straumann 2002). In general, we can describe a change of multivariate probability distribution in terms of marginal distributions and their new copula function:

$$F^*_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = C^*(F^*_{X_1}(x_1), F^*_{X_2}(x_2), \ldots, F^*_{X_n}(x_n)).$$

DEFINITION 4.1

For a given multivariate cumulative distribution

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = C(F_{X_1}(x_1), F_{X_2}(x_2), \ldots, F_{X_n}(x_n)),$$

we define a new multivariate cumulative distribution

$$F^*_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = C^*(g_1(F_{X_1}(x_1)), g_2(F_{X_2}(x_2)), \ldots, g_n(F_{X_n}(x_n)))$$

with the following marginal distributions:

$$F^*_{X_j}(x_j) = g_j[F_{X_j}(x_j)],$$

and a new copula function C^*. We will call this change of measure a multivariate distortion induced by functions $\{g_1, g_2, \ldots, g_n\}$ and the mapping $C \to C^*$. In the special case that $C = C^*$, we will call the multivariate distortion *marginal* distortions induced by $\{g_1, g_2, \ldots, g_n\}$.

Example 4.1

When $C = C^*$ and $g_j(u) = \Phi[\Phi^{-1}(u) - \lambda_j]$, for $j = 1, 2, \ldots, n$, the multivariate distortion in Definition 4.1 is *marginal Wang transforms* with parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

Example 4.2

Consider a multivariate standard normal distribution with correlation matrix Σ. Suppose that, after a change of measure, the new multivariate distribution is a t-distribution with k degrees of freedom and the same correlation matrix $\Sigma^* = \Sigma$. This change of measure is induced by distortions $g_j(u) = t_k[\Phi^{-1}(u)]$ and a mapping from normal copula C to t-copula C^*.

DEFINITION 4.2

We define *joint* distortions $\{g_1, g_2, \ldots, g_n\}$ in terms of the joint Radon-Nikodym derivative:

$$f^*_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = RN_{g_1,g_2,\ldots,g_n}(x_1, x_2, \ldots, x_n) \cdot f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n).$$

The Radon-Nikodym derivative is specified jointly by the distortions $\{g_1, g_2, \ldots, g_n\}$:

In the discrete case for the point $\bar{x}_i = (x_{1;i}, x_{2;i}, \ldots, x_{n;i})$ we have

$$RN_{g_1,g_2,\ldots,g_n}(x_{1;i}, x_{2;i}, \ldots, x_{n;i}) = c \cdot \prod_{j=1}^{n} \frac{g_j(F_{X_j}(x_{j;i})) - g_j(F_{X_j}(x_{j;i-1}))}{F_{X_j}(x_{j;i}) - F_{X_j}(x_{j;i-1})};$$

In the continuous case for the point $\bar{x} = (x_1, x_2, \ldots, x_n)$ we have

$$RN_{g_1,g_2,\ldots,g_n}(x_1, x_2, \ldots, x_n) = c \cdot \prod_{j=1}^{n} g_j'(F_{X_j}(x_j)).$$

For instance, when, $g_j(u) = \Phi[\Phi^{-1}(u) - \lambda_j]$, for $j = 1, 2, \ldots, n$, the joint distortions in Definition 4.2 represent *joint Wang transforms* with parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.

When $\{X_1, X_2\}$ have uncorrelated marginal distributions, both the marginal distortions and the joint distortions yield the same risk-neutralized multivariate probability distribution with uncorrelated marginal distributions

$$f^*_{X_1,X_2}(x_1, x_2) = f^*_{X_1}(x_1) \cdot f^*_{X_2}(x_2) \text{ and}$$
$$F^*_{X_j}(x_j) = g_j[F_{X_j}(x_j)], \text{ with } j = 1, 2.$$

When $\{X_1, X_2\}$ are correlated, however, the *marginal* distortions and the *joint* distortions can yield different results. Joint distortions reflect the interactions between X_1 and X_2 in the probability adjustment, whereas marginal distortions do not. Consider the special case that $X_1 = X_2$, and $g_1 = g_2$ are a Wang transform with the same parameter λ. Under the joint distortions $\{g_1, g_2\}$, the risk-neutralized distribution for X_1 is equivalent to applying the Wang transform to X_1 with the parameter 2λ. In contrast, under the marginal distortions $\{g_1, g_2\}$, the risk-neutralized distribution for X_1 is equivalent to applying the Wang transform to X_1 with the parameter λ.

Example 4.3

Consider the following multivariate extension of the Wang transform by Kijima (2006, eq. 3.7):

$$F^*_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \Phi_n\left(\Phi^{-1}(F_{X_1}(x_1)) - \sum_{j=1}^{n} \lambda_j\rho_{1j}, \ldots, \Phi^{-1}(F_{X_n}(x_n)) - \sum_{j=1}^{n} \lambda_j\rho_{nj}\right),$$

where Φ_n represents the cumulative distribution for a multivariate standard normal with $\Sigma = (\rho_{ij})_{n\times n}$. This change of measure is induced by distortions $g_i(u) = \Phi[\Phi^{-1}(u_i) - \Sigma_{j=1}^{n} \lambda_j\rho_{ij}]$, $i = 1, 2, \ldots, n$, and a mapping from a normal copula C onto itself with $C^* = C$. One can verify that this change of multivariate probability measure is equivalent to applying *joint Wang transforms* with parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and is equivalent to applying *marginal Wang transforms* with parameters $\{\sum_{j=1}^n \lambda_j\rho_{1j}, \ldots, \sum_{j=1}^n \lambda_j\rho_{nj}\}$.

5. LINK BETWEEN MULTIVARIATE EXPONENTIAL TILTING AND MULTIVARIATE DISTORTION

As stated in Theorem 2.1 for the univariate case, there is a close connection between the normalized exponential tilting and the probability distortion. Likewise, we can extend this connection to the multivariate cases.

Theorem 5.1

Assume that we have n variables and k references

$$\{X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_k\},$$

which follow a normal copula correlation structure. The normalized multivariate exponential tilting (3.2) of $\{X_1, X_2, \ldots, X_n\}$ w.r.t. $\{Y_1, Y_2, \ldots, Y_k\}$ is equivalent to applying *marginal* Wang transforms to X_i with

$$g_i(u) = \Phi\lfloor\Phi^{-1}(u) - \beta_i\rfloor \text{ and } \beta_i = \sum_{j=1}^{k} \rho_{X_i,Y_j} \cdot \lambda_j \text{ (for } i = 1, 2, \ldots, n).$$

The correlation matrix between $\{X_1, X_2, \ldots, X_n\}$ is unchanged after the normalized multivariate exponential tilting, $\Sigma^* = \Sigma$.

REMARK

This result coincides with the multivariate extension of the Wang transform by Kijima (2006).

Example 5.1

Assume that $\{X_1, X_2\}$ the risks have a bivariate normal(0, 1) with correlation coefficients

$$\Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

According to Theorem 5.1, under the bivariate normalized exponential tilting (3.2) with references $Y_1 = X_1$ and $Y_2 = X_2$, the risk-neutralized joint distribution for $\{X_1^*, X_2^*\}$ is also bivariate normal with correlation coefficients

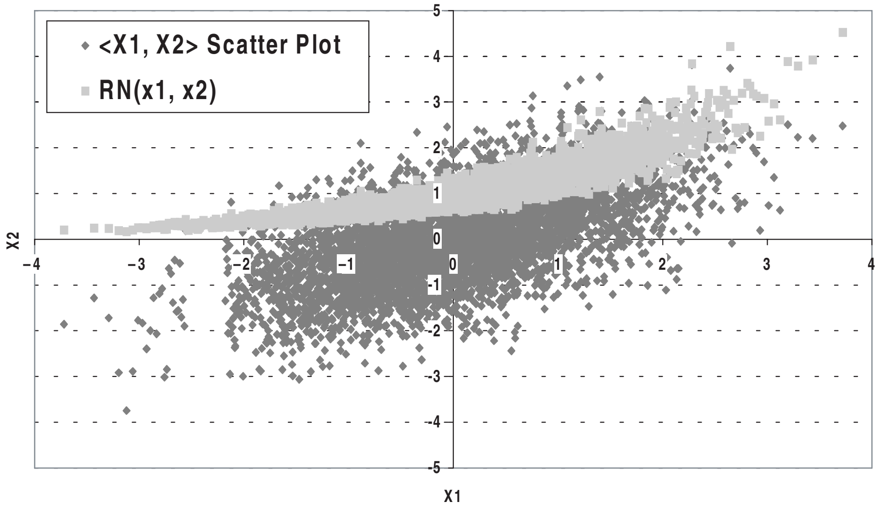
$$\Sigma^* = \Sigma = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0.6 \\ 0.6 & 1 \end{pmatrix}.$$

For illustration, let us choose $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$. The risk-neutralized marginal distributions are equivalent to applying *marginal* Wang transforms $F_{X_j}^*(x) = \Phi\lfloor\Phi^{-1}(F_{X_j}(x)) - \beta_j\rfloor$ for $j = 1, 2$, with

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 & \rho_{X_1,X_2} \\ \rho_{X_1,X_2} & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \rho_{X_1,X_2}\lambda_2 \\ \rho_{X_1,X_2}\lambda_1 + \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.38 \end{pmatrix}.$$

Figure 1 shows a scatter plot of $\{X_1, X_2\}$ and their corresponding Radon-Nikodym derivatives. The right-most diamond is a scatter plot of $(x_1 = 3.195, x_2 = 2.505)$. The right-most square gives its value of Radon-Nikodym derivative $RN(3.195, 2.505) = 3.884$. One can see that the Radon-Nikodym derivatives increase exponentially when the point (x_1, x_2) moves from the lower left quadrant to the upper right quadrant.

Figure 1
Scatter Plot Bivariate Variables $\{X_1, X_2\}$ and Radon-Nikodym Derivatives $RN(x_1, x_2)$



6. VALUING CONTINGENT CLAIMS ON MULTIPLE UNDERLYING RISKS

Consider contingent claims on multiple underlying variables:

$$X_i = h_i(Y_1, Y_2, \ldots, Y_k),\ i = 1, 2, \ldots, n.$$

When valuing the contingent claim $X_i = h_i(Y_1, Y_2, \ldots, Y_k)$, the market price of risk should be specified through the underlying risks $\{Y_1, Y_2, \ldots, Y_k\}$.

Theoretically we should first adjust the multivariate probability measure for the underlying risks, and then value contingent claims as expected payoff under the risk-neutralized probability measure. Accordingly we should first apply normalized exponential tilting of $\{Y_1, Y_2, \ldots, Y_k\}$ w.r.t. themselves, and calculate the expected value of $X_i = h_i(Y_1, Y_2, \ldots, Y_k)$ under the risk-neutralized distribution of the underlying risks $\{Y_1, Y_2, \ldots, Y_k\}$.

Theorem 6.1

If we let $X_j = Y_j$ be the underlying risks themselves, for $j = 1, 2, \ldots, k$, the normalized multivariate exponential tilting (3.2) of $\{Y_1, Y_2, \ldots, Y_k\}$ w.r.t. themselves is equivalent to applying joint Wang transforms with parameters $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$.

Theorem 6.1 provides an efficient numerical method for computing the risk-neutralized probabilities through joint Wang transforms. This enables rather easy implementation of the multivariate normalized exponential tilting in pricing contingent claims on multiple underlying risks $\{Y_1, Y_2, \ldots, Y_k\}$.

Example 6.1

Consider the bivariate distribution (that does not follow a normal copula) in Table 1. We want to compute the adjusted joint distribution for the normalized multivariate exponential tilting of (X_1, X_2), with reference to themselves, and with $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$.

We first apply the Wang transform to X_1 with $\lambda_1 = 0.3$ in Table 2. We then apply the Wang transform to X_2 with $\lambda_2 = 0.2$ (Table 3).

According to Theorem 6.1, the bivariate Radon-Nikodym derivatives are

$$RN_g(x_1, x_2) = \frac{f^*_{X_1,X_2}(x_1, x_2)}{f_{X_1,X_2}(x_1, x_2)} = c \cdot \frac{f^*_{X_1}(x_1)}{f_{X_2}(x_1)} \cdot \frac{f^*_{X_2}(x_2)}{f_{X_2}(x_2)}.$$

The final risk-neutralized joint probability (density) function is shown in Table 4.

Table 1
Joint Probability Function for a Bivariate Distribution

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	$X_2 = 4$	$X_2 = 5$
$X_1 = 1$	0.20	0.07	0.06	0.05	0.04
$X_1 = 2$	0.06	0.05	0.04	0.03	0.03
$X_1 = 3$	0.05	0.04	0.03	0.03	0.02
$X_1 = 4$	0.03	0.03	0.02	0.02	0.01
$X_1 = 5$	0.03	0.02	0.01	0.02	0.01

Table 2
Wang Transform to X_1, $\lambda_1 = 0.3$

$X_1 = x_1$	$f(x_1)$	$F(x_1)$	$F^*(x_1)$	$f^*(x_1)$
1	0.42	0.42	0.30787	0.30787
2	0.21	0.63	0.51271	0.20483
3	0.17	0.80	0.70596	0.19325
4	0.11	0.91	0.85101	0.14505
5	0.09	1.00	1.00000	0.14899

Table 3
Wang Transform to X_2, $\lambda_2 = 0$

$X_2 = x_2$	$f(x_2)$	$F(x_2)$	$F^*(x_2)$	$f^*(x_2)$
1	0.37	0.37	0.29741	0.29741
2	0.21	0.58	0.50076	0.20334
3	0.16	0.74	0.67124	0.17049
4	0.15	0.89	0.84768	0.17644
5	0.11	1.00	1.00000	0.15232

Table 4
Final Risk-Neutralized Joint Probability (Density) Function

	$X_2 = 1$	$X_2 = 2$	$X_2 = 3$	$X_2 = 4$	$X_2 = 5$
$X_1 = 1$	0.1178	0.0497	0.0469	0.0431	0.0406
$X_1 = 2$	0.0470	0.0472	0.0416	0.0344	0.0405
$X_1 = 3$	0.0457	0.0440	0.0363	0.0401	0.0315
$X_1 = 4$	0.0318	0.0383	0.0281	0.0310	0.0183
$X_1 = 5$	0.0399	0.0321	0.0176	0.0389	0.0229

As shown in Figure 2, the Radon-Nikodym derivative increases to its highest value at $\{X_1 = 5, X_2 = 5\}$, indicating the largest relative risk adjustment at the joint tail of the bivariate variables.

Example 6.2

Let (Y_1, Y_2) represent the loss variable and the loss adjustment expenses for a liability insurance contract. The variables (Y_1, Y_2) have an empirical bivariate distribution as in Table 5. Consider the following three insurance contracts:

Figure 2
Bivariate Radon-Nikodym Derivatives

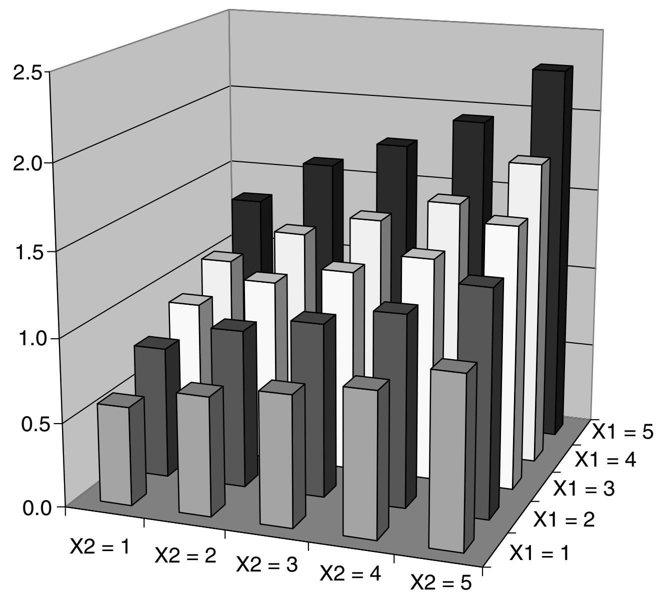


Table 5
Empirical Bivariate Distribution for (Y_1, Y_2)

Scenario	Y_1	Y_2
1	—	1,954.08
2	—	2,239.22
3	—	2,974.21
4	—	3,275.38
5	—	3,351.93
6	—	6,526.96
7	—	9,542.63
8	—	13,999.95
9	—	14,279.63
10	—	14,519.32
11	—	16,179.92
12	—	19,134.14
13	—	35,071.98
14	—	57,591.43
15	—	62,967.38
16	—	82,638.17
17	—	248,909.05
18	638.80	3,331.31
19	1,533.11	2,047.14
20	5,110.36	1,159.07
21	6,387.95	2,152.74
22	6,387.95	8,940.58
23	8,943.13	4,949.35
24	11,498.32	—
25	15,331.09	—
26	27,279.12	—
27	35,772.54	5,634.79
28	93,264.12	24,115.73
29	102,207.25	6,287.06
30	191,638.60	34,096.74
31	246,010.43	232,641.59
32	511,036.26	39,161.24
33	511,036.26	150,301.30
34	662,650.50	73,140.35

Table 6
Expected Contract Payoffs

	Expected Payoff of Contract 1	Expected Payoff of Contract 2	Expected Payoff of Contract 3
No risk adjustment	$33,257	$17,399	$50,656
With risk adjustment	68,240	24,847	93,087
Loading	105%	43%	84%

1. Contract 1 has a contingent payoff in the amount of Y_1 in excess of 200,000. That is, the payoff $X_1 = \max\{Y_1 - 200{,}000, 0\}$.
2. Contract 2 has a contingent payoff of 50% of the amount of Y_2. That is, $X_2 = 0.5Y_2$
3. Contract 3 has a contingent payoff in the amount of Y_1 in excess of 200,000, plus 50% of the amount of Y_2. Technically Contract 3 is simply the combination of Contract 1 and Contract 2: $X_3 = X_1 + X_2$.

Without risk adjustment, the expected payoffs of Contract 1, 2, 3 are $33,257, $17,399, and $50,656, respectively.

We can easily calculate the linear correlation coefficient between (Y_1, Y_2), which is 0.38. However, the correlation structure between (Y_1, Y_2) is far from a normal copula. In fact, 17 out of the 34 cases have zero loss payments, but nonzero loss adjustment (defense) expenses. Nevertheless, we can still apply multivariate normalized exponential tilting.

Suppose that the market price of risk for the underlying risks Y_1 and Y_2 are $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$, respectively. We can derive a risk-neutralized bivariate distribution by applying bivariate normalized exponential tilting of Y_1 and Y_2 with respect to themselves, using $\lambda_1 = 0.3$ and $\lambda_2 = 0.2$.

Theorem 6.1 facilitates a numerical method for calculating the risk-neutralized probabilities for each of the 34 scenarios. Based on the risk-neutralized probabilities for each scenario, we can calculate the prices for Contracts 1, 2, and 3 being $68,240, $24,847, and $93,087, respectively (Table 6).

Note that the obtained prices are additive. Indeed, the only way to ensure price additivity is through a change of bivariate probability measure.

7. Conclusion and Future Research

This article has discussed changes of probability measure for multivariate variables. One method is through the normalized multivariate exponential tilting, and the other method is by the multivariate probability distortion. A link has been established between the normalized multivariate exponential tilting and the multivariate probability distortion, and efficient numerical routines for risk-neutralizing multivariate probability distributions have been identified. They are useful methods for pricing risks with respect to given reference risks, and for valuing contingent claims on given underlying risks.

One area of future research is to investigate other plausible normalization procedures on the reference risk from an economic point of view, and to perform empirical tests based on market data. Another area of future research is to investigate to what extent the interactions among multiple reference risks can impact the normalized multivariate exponential tilting.

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