Determining the Optimum Guarantee Period for a One-Life Retirement Annuity

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Abstract

We consider a risk averse retiree from a defined contribution plan who decides to purchase a one-life annuity with a guarantee period. Given the retiree has a bequest motive, we focus on the problem of determining the optimum length of the guarantee period. Assuming the retiree's bequest function is proportional to his or her utility function, we determine necessary and/or sufficient conditions under which the retiree would choose an annuity with (i) no guarantee period, (ii) the maximum guarantee period, or (iii) an intermediate guarantee period.

1. Introduction

One important and irrevocable decision a retiree from a defined contribution plan must make is to choose the method of withdrawing his or her accumulated assets in the plan. Retirees often have the option of choosing between single and joint life annuities and may also have a choice of guarantee period. Clearly the optimal choice may depend on the retiree's age, gender, health, marital status, wealth, and bequest motive. For example, it may be optimal for a retiree with no bequest motive to choose a one-life annuity with no guarantee period in order to receive the largest monthly benefit while he or she is alive.

Historically, educational institutions have preferred defined contribution plans because of favorable tax treatment granted under Internal Revenue Code section 403(b) (Rosenbloom et al. 1997, p. 48). Plans eligible under section 403(b) include public school systems, colleges and universities, and non-profit research organizations (p. 205). An example of a section 403(b) system is the Teachers Insurance and Annuity Association–College Retirement Equities Fund (TIAA-CREF), which is the world's largest retirement system and provides benefits for the education and nonprofit research communities in the United States. Under TIAA-CREF, retirees have the option of choosing a single or a joint life annuity. In addition, each annuity option allows the retiree to choose a guarantee period.

Using data from a 1988 survey of annuitants from TIAA-CREF, Laitner and Thomas (1996) find evidence of altruism (i.e., a bequest motive) in the behavior of these annuitants. However, they find that altruism toward their children was not a major explanation of the savings in the overall sample studied. Using TIAA-CREF data from 1978 to 1994, King (1996) provides a series of tables of TIAA-CREF data showing the choices made by TIAA-CREF retirees. Tables 1 through 5 in this article are based on King's tables.

Table 1 shows that the proportion of male annuitants under TIAA-CREF retirement annuity contracts who select a one-life option in 1994 is approximately 26%, and the proportion of female annuitants who select this option is around 68%. Reasons for selecting the one-life option at retirement may include (1) the retiree is a single employee or (2) the retiree's spouse has his or her own retirement plan. The total number of individuals choosing a one-life income option has dropped significantly from

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Table 1
Percentage of Total Annuitants Who Select One- and Two-Life Income Options under TIAA-CREF Retirement Annuity Contracts (All Ages, 1978–1994)

Year	Male		Female	
	One-Life	Two-Life	One-Life	Two-Life
1978	43.5	56.5	81.1	18.9
1979	44.5	55.5	79.3	20.7
1980	42.6	57.4	77.5	22.5
1981	42.8	57.2	76.0	24.0
1982	39.7	60.3	74.0	26.0
1983	36.8	63.2	73.1	26.9
1984	36.8	63.2	72.4	27.6
1985	31.1	68.9	73.4	26.6
1986	27.8	72.2	69.2	30.8
1987	27.7	72.3	69.4	30.6
1988	27.7	72.3	68.4	31.6
1989	25.8	74.2	66.3	33.7
1990	25.6	74.4	66.0	34.0
1991	26.0	74.0	66.3	33.7
1992	26.0	74.0	65.9	34.1
1993	25.4	74.6	66.7	33.3
1994	26.0	74.0	67.8	32.2

Source: Table 3 of King (1996, p. 3); used with the permission of the publisher.

1985 to 1986. This may be attributed to the spousal protection provisions of the Retirement Equity Act of 1984.[1]

Among the TIAA-CREF annuitants who choose the one-life option, some choose to take their annuity as a straight life income, while others select a life annuity with a guarantee period of 10, 15, or 20 years. The guarantee period option comes with the price of a smaller benefit. Because each option provides payments that are actuarially equivalent, a fixed amount of accumulated wealth can buy an annuity that provides the highest income by choosing the annuity with no guarantee period. Table 2

Table 2
Selection of Life-Annuity Income Options: Percentage Differences in Benefits According to Income Options Selected under the TIAA Standard Payment Method

Retirement Age	Length of Guarantee Period (Years)		
	0	10	20
60	100% = $1,000/month	98%	94%
65	100% = $1,076/month	97	91
70	100% = $1,185/month	95	85

Source: Table 1 of King (1996, p. 2); used with the permission of the publisher.

[1] The Retirement Equity Act of 1984 contains a spousal consent and protection clause under which all retirement plans regulated under ERISA must pay benefits to an employee married to the spouse (at the time of retirement) for at least one year in the form of a joint and survivor annuity, unless the participant elects otherwise and the spouse consents (McGill and Grubbs 1989, p. 139). The selected joint and survivor annuity must provide at least half of the income payable during the joint life to the surviving spouse.

shows how the percentage differences in benefits across different guarantee periods vary for different ages. For ages higher than 65, the impact is greater because there is a smaller chance that an older individual would outlive the guarantee period. Similarly a retiree under age 65 would see a smaller difference between the options, because it is more likely that he or she will live through the guarantee period.

For those male and female annuitants who select the one-life option, Table 3 displays the percentage of male and female annuitants who select 0-, 10-, 15-, and 20-year guarantee periods. More females take an annuity with no guarantee option than males, possibly indicating that fewer women have a bequest motive, but the gender difference is decreasing.

Table 4 shows the breakdown across ages of the one- and two-life options for various years for both males and females. Note that younger males tend to choose a one-life income option, while the opposite is true for females. Table 5 details the number of one- and two-life annuitants at different levels of accumulation. A lower percentage of both males and females who select one-life income options have high accumulation levels, while the opposite trend is shown among those who select two-life income options.

As seen in Table 3, each of the available one-life retirement options is chosen by some TIAA-CREF retirees. This is not surprising as retirees have different personal and family considerations, and hence have different needs and objectives. Still, there are several unanswered questions, including what bequest motive may induce a choice of a particular guarantee period.

To begin to answer that particular question, we must make assumptions about a retiree's decision-making process. The common approach is to assume retirees are rational in the sense of traditional expected utility theory. Individuals are also assumed to be risk averse, that is, they have utility functions that are increasing but at a decreasing rate.[2]

Table 3 shows that each one-life retirement option is chosen by some TIAA-CREF retirees because retirees have different needs and objectives. Our objective is to develop criteria for finding the length

Table 3

Selection of One-Life Income Options under TIAA-CREF Retirement Annuity Contracts, All Ages, 1978–1994 Percentage of One-Life Annuitants Selecting Option

	Male Primary Annuitants Guarantee Period (Years)				Female Primary Annuitants Guarantee Period (Years)			
Year	0	10	15	20	0	10	15	20
1978	33.6	38.2	0.0	25.1	46.0	30.5	0.0	21.0
1979	35.3	35.1	0.7	25.8	43.5	30.3	0.1	23.6
1980	36.9	31.7	0.7	27.5	44.6	29.8	0.0	23.1
1981	39.3	31.8	0.5	25.7	43.2	29.6	0.1	24.6
1982	38.5	30.5	0.3	27.7	40.9	28.5	0.1	27.6
1983	37.0	31.3	0.3	28.3	40.5	28.7	0.0	27.6
1984	38.9	29.6	0.5	28.0	39.0	28.9	0.0	28.9
1985	34.7	26.0	5.1	30.9	41.0	27.9	4.4	24.0
1986	33.5	27.7	5.8	30.6	39.0	29.0	5.6	24.0
1987	34.3	24.2	9.7	29.2	38.0	26.8	9.1	24.1
1988	33.6	26.0	8.3	30.0	38.3	27.0	7.5	25.3
1989	34.9	25.6	13.2	25.6	37.9	27.8	11.2	22.3
1990	33.6	26.2	14.1	25.4	34.5	29.8	13.8	21.1
1991	32.3	26.2	16.5	24.2	34.5	29.3	14.8	20.7
1992	37.3	25.8	14.6	21.2	37.6	27.2	14.1	20.0
1993	36.6	24.4	14.2	23.6	38.8	28.0	14.1	18.0
1994	33.8	25.8	16.2	23.8	35.0	29.8	15.0	19.5

Source: Table 3 of King (1996, p. 3); used with the permission of the publisher.
Notes: For male and female annuitants, these percentages do not sum to 100% because the TIAA-CREF Installment Refund option has been excluded.

[2] An excellent modern treatise on utility theory is given by Gollier (2001).

Table 4

Selection of One- and Two-Life Annuity Income Options by Age at Retirement: 1978, 1983, 1986, 1990, 1994 Percentage of Total Annuitants

		Male		Female	
Year	Age Annuity Started	One-Life	Two-Life	One-Life	Two-Life
1978	All Ages	43.5	56.5	81.1	18.9
	59 and under	57.1	42.9	71.6	28.4
	60–64	40.9	59.1	77.2	22.8
	65–69	40.7	59.3	84.5	15.5
	70 and over	42.2	57.8	90.1	9.9
1983	All Ages	36.8	63.2	73.1	26.9
	59 and under	51.9	48.1	70.1	29.9
	60–64	35.0	65.0	67.6	32.4
	65–69	32.8	67.2	76.9	23.1
	70 and over	31.6	68.4	85.6	14.4
1986	All Ages	27.8	72.2	69.2	30.8
	59 and under	43.8	56.2	65.1	34.9
	60–64	26.0	74.0	65.2	34.8
	65–69	24.2	75.8	73.4	26.6
	70 and over	22.2	77.8	77.5	22.5
1990	All Ages	25.6	74.4	66.0	34.0
	59 and under	43.0	57.0	62.7	37.3
	60–64	23.8	76.2	61.1	38.9
	65–69	23.0	77.0	67.1	32.9
	70 and over	20.4	79.6	75.9	24.1
1994	All Ages	26.0	74.0	67.8	32.2
	59 and under	40.9	59.1	68.4	31.6
	60–64	24.6	75.4	62.2	37.8
	65–69	23.1	76.9	71.1	28.9
	70 and over	25.0	75.0	71.8	28.2

Source: Table 4 of King (1996, p. 6); used with the permission of the publisher.

of the guarantee period of a one-life annuity that maximizes the retiree's expected utility including bequests. In addition, we will explore the influence of the bequest motive on the choice of a particular guarantee period, and determine the conditions under which a retiree would choose a life annuity without a guarantee or choose the longest guarantee period.

Table 5

Selection of One-Life and Two-Life TIAA-CREF Annuity Income Options by Combined Size of Annuity Accumulations Converted to Life Annuities, 1994

	Male Primary Annuitants				Female Primary Annuitants			
	One-Life Annuity		Two-Life Annuity		One-Life Annuity		Two-Life Annuity	
Accumulation Range	Number	Percent	Number	Percent	Number	Percent	Number	Percent
Under $50,000	2,654	32.7	5,463	67.3	5,116	68.3	2,374	31.7
50,000–99,999	738	24.2	2,315	75.8	1,193	64.5	658	35.5
100,000–149,999	379	20.0	1,519	80.0	512	67.6	245	32.4
150,000–199,999	248	20.5	964	79.5	283	69.7	123	30.3
200,000–249,999	148	17.3	708	82.7	140	73.7	50	26.3
250,000–299,999	122	18.9	522	81.1	74	70.5	31	29.5
300,000–349,999	61	14.6	358	85.4	46	68.7	21	31.3
350,000–399,999	50	17.4	238	82.6	21	65.6	11	34.4
400,000–449,999	41	17.6	192	82.4	10	71.4	4	28.6
450,000–499,999	31	16.1	162	83.9	6	46.2	7	53.8
$500,000 and over	47	10.2	414	89.8	17	50.0	17	50.0
Total	4,519	26.0	12,855	74.0	7,418	67.8	3,541	32.2

Source: From Table 5, page 7, of King (1996); used with the permission of the publisher.

2. THE OPTIMUM GUARANTEE PERIOD

2.1 The Model

Consider a newly retired participant age x with accumulated wealth $w_0 > 0$ in his or her individual account in a defined contribution plan eligible under section 403(b). To provide for his or her retirement income until death, the retiree voluntarily purchases a single life annuity with level continuous annual payments paid by an insurance company. The retiree has the option of choosing a whole life annuity with a guarantee period of n years or a basic whole life annuity with no guarantee period (i.e., $n = 0$ years). We assume all benefits are consumed, there are no expenses or taxes, and there is a constant annual force of interest of δ per year. The retiree has a utility function[3] $u(y)$ that measures the utility gained from the retiree's own consumption of y units of wealth, and a bequest function $r(y)$ that measures the utility the retiree gains from an altruistic transfer of y units of wealth to an heir upon the retiree's death.[4] In addition, we assume

A.1: The retiree is risk averse so that (1) $u'(y) > 0$ and $r'(y) > 0$ and (2) $u''(y) < 0$ and $r''(y) < 0$ for $-\infty < y < \infty$.

A.2: $n^{\max}$ $(0 < n^{\max} \leq \infty)$ is the maximum available guarantee period.

A.3: The retiree has accumulated wealth w_0 available to purchase the annuity, and, without loss of generality, the unit of wealth is such that $w_0 = 1$.

A.4: The retiree's expected mortality follows that assumed by the insurance company in the calculation of the annual retirement benefit.[5]

These assumptions imply that annuities are sold at an actuarially fair price.

Let B_n denote the actuarially fair annual benefit paid by the one-life annuity with a guarantee period of n years, and purchased with unit accumulated wealth ($w_0 = 1$). By the actuarial equivalence principle, B_n satisfies the equation

$$B_n = \frac{w_0}{\bar{a}_{\overline{x:\overline{n}|}}}, \tag{2.1}$$

where $\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x\bar{a}_{x+n}$ using standard actuarial notation. Note that

$$\frac{\partial B_n}{\partial n} = \frac{-B_n v^n {}_nq_x}{\bar{a}_{\overline{x:\overline{n}|}}} \geq 0 \tag{2.2}$$

and

$$\frac{\partial B_n\bar{a}_{\overline{n}|}}{\partial n} = B_n v^n \left(1 - \frac{{}_nq_x\bar{a}_{\overline{n}|}}{\bar{a}_{\overline{x:\overline{n}|}}}\right) \leq 0, \tag{2.3}$$

where $v = e^{-\delta}$. Equation (2.2) confirms that the benefit is decreasing in the guarantee period while equation (2.3) can be interpreted as saying that the total present value of certain benefits is increasing in the guarantee period.

[3] In practice individuals typically do not know their own utility function. However, it is possible to elicit or construct an individual's utility function (e.g., Tsetlin 2006 and Blavatskyy 2006 and references therein).

[4] Following Hurd (1987), a person is said to have a bequest motive if he or she cares about the welfare of the recipients of his or her estate. For a person with a given wealth, the desire to leave a bequest depends on the person's value of his or her own consumption to the welfare of his or her estate. Economists have questioned the existence of bequest motives among retirees; see, for example, Hurd (1987, 1989) and Bernheim (1991) and references therein. For example, Hurd (1987) shows that there is no evidence for a bequest motive among the elderly.

[5] It has been observed by several studies (Finkelstein and Poterba 1999; Brown et al. 2001; Mitchell and McCarthy 2002; Balls 2006) that people who *voluntarily* purchase annuities have lower mortality rates than the population as a whole because they may have private information about their own good health and they tend to be wealthier than the general population. To deal with adverse selection in annuities, insurers use mortality tables with favorable mortality projections embedded into the mortality rates. In the case of *mandatory or compulsory* annuity purchases, Brown (2003) has observed a more heterogeneous mortality distribution.

For a given guarantee period n, let Y_n denote the present value of the future retirement benefit and U_n denote the utility gained from these payments. Let $T(x)$ denote the retiree's future lifetime at age x. Then the random variables Y_n and U_n can be written as

$$Y_n = \begin{cases} B_n \bar{a}_{\overline{T(x)}|} + B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{T(x)}|}) & \text{if } T(x) \le n \\ B_n \bar{a}_{\overline{T(x)}|} & \text{otherwise;} \end{cases} \tag{2.4}$$

while the total utility, including bequests, is not $u(Y_n)$, rather it is

$$U_n = \begin{cases} u(B_n \bar{a}_{\overline{T(x)}|}) + r(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{T(x)}|})) & \text{if } T(x) \le n \\ u(B_n \bar{a}_{\overline{T(x)}|}) & \text{otherwise.} \end{cases} \tag{2.5}$$

The expected utility function, $\mathcal{M}(n) = E[U_n]$, is given by

$$\mathcal{M}(n) = \int_0^\infty u(B_n \bar{a}_{\overline{t}|})\,{}_tp_x \mu_{x+t}\,dt + \int_0^n r(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))\,{}_tp_x \mu_{x+t}\,dt, \tag{2.6}$$

where ${}_tp_x\mu_{x+t}$ represents the probability density function of $T(x)$. From equation (2.6) it is clear that if the retiree has no bequest motive, that is, $r \equiv 0$, then the whole life annuity without a guarantee period is optimum because $B_0 > B_n \Rightarrow u(B_0\bar{a}_{\overline{t}|}) > u(B_n\bar{a}_{\overline{t}|})$ for n, $t > 0$. Thus we have established the following result:

Result 1

If the retiree has no bequest motive, that is, $r \equiv 0$, *then the whole life annuity without a guarantee period is optimum.*

Though Result 1 is intuitively obvious, it is not supported by data. For example, using the RHS data,[6] economists have observed that the amount of privately purchased annuities was small even among persons with no identifiable heirs (Hurd 1987). One possible reason for this result is adverse selection in the market for private annuities, which may cause prices to be higher than those determined by the actuarial equivalence principle.

2.2 The Utility Maximization Equation

As $\mathcal{M}(n)$ satisfies the conditions for applying Leibnitz's integral rule, it can easily be shown that

$$\frac{\partial \mathcal{M}(n)}{\partial n} = \frac{B_n v^n}{\bar{a}_{x:\overline{n}|}} \left[\int_0^n Q_x(n, t) r'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))\,{}_tp_x\mu_{x+t}\,dt \right.$$
$$\left. - {}_nq_x \int_0^\infty \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{t}|})\,{}_tp_x\mu_{x+t}\,dt \right], \tag{2.7}$$

where

$$Q_x(n, t) = \bar{a}_{\overline{x:\overline{n}|}} - {}_nq_x(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}) \ge 0. \tag{2.8}$$

To find the guarantee period that maximizes expected utility requires more information on the relationship between the utility and bequest functions. Following Campbell (1980), Abel (1987), and Cropper and Sussman (1988), we assume

[6] The Longitudinal Retirement History Survey (RHS) was commissioned by the Social Security Administration and conducted by the Census Bureau. This data collection represents part of a longitudinal study of the nature of retirement and of the transition to a retirement lifestyle. About 11,000 households whose heads were born between 1906 and 1911 were interviewed every two years from 1969 to 1979. Data on the retirees or their surviving spouses include level of educational attainment, present health condition, current employment status, work history, industry and occupational classification, wages, retirement plans, attitudes toward retirement, pension benefits, insurance coverage, contacts with relatives, living expenses, assets, debts, income, and basic demographic information. For more on this study see Irelan (1973).

$$r(y) = \beta u(y), \tag{2.9}$$

where $\beta > 0$ is called the *bequest constant*. When $0 < \beta < 1$, the beneficiary's utility for receiving the certain payments after the death of the annuitant is less than the utility that would have been gained if the retiree had lived, while if $\beta > 1$, the retiree places a higher value to her beneficiary receiving benefits than the retiree.

Using the definition of the bequest function in equation (2.9) leads to

$$\frac{\partial \mathcal{M}(n)}{\partial n} = \frac{B_n v^n}{\bar{a}_{\overline{x:n}|}} H(n, \beta), \tag{2.10}$$

where

$$H(n, \beta) = \beta R(n) - S(n)$$

and R and S are nonnegative functions defined, for $n \geq 0$, as

$$R(n) = \int_0^n Q_x(n, t) u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))_t p_x \mu_{x+t}\, dt \tag{2.11}$$

and

$$S(n) = {}_n q_x \int_0^\infty \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{t}|})_t P_x \mu_{x+t}\, dt. \tag{2.12}$$

The derivatives of R and S are

$$\begin{aligned} \frac{\partial R(n)}{\partial n} &= {}_n p_x \mu_{x+n} \left[\bar{a}_{\overline{x:n}|} u'(0) - \int_0^n (\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}) u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))_t p_x \mu_{x+t}\, dt \right] \\ &\quad - \frac{B_n v^n}{\bar{a}_{\overline{x:n}|}} \int_0^n (Q_x(n, t))^2 (-u''(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})))_t p_x \mu_{x+t}\, dt, \end{aligned} \tag{2.13}$$

which can be positive or negative, and

$$\begin{aligned} \frac{\partial S(n)}{\partial n} &= {}_n p_x \mu_{x+n} \int_0^\infty \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{t}|})_t p_x \mu_{x+t}\, dt \\ &\quad + \frac{B_n v^n ({}_n q_x)^2}{\bar{a}_{\overline{x:n}|}} \int_0^\infty (\bar{a}_{\overline{t}|})^2 (-u''(B_n \bar{a}_{\overline{t}|}))_t p_x \mu_{x+t}\, dt, \end{aligned} \tag{2.14}$$

which is nonnegative. Notice that

$$\begin{aligned} {}_n q_x \frac{\partial H(n, \beta)}{\partial n} &= {}_n p_x \mu_{x+n} H(n, \beta) \\ &\quad + \beta\ {}_n p_x \mu_{x+n} \bar{a}_{\overline{x:n}|} \int_0^n (u'(0) - u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})))_t p_x \mu_{x+t}\, dt \\ &\quad - \beta \frac{B_n v^n {}_n q_x}{\bar{a}_{\overline{x:n}|}} \int_0^n (Q_x(n, t))^2 (-u''(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})))_t p_x \mu_{x+t}\, dt \\ &\quad - \frac{B_n v^n ({}_n q_x)^3}{\bar{a}_{\overline{x:n}|}} \int_0^\infty (\bar{a}_{\overline{t}|})^2 (-u''(B_n \bar{a}_{\overline{t}|}))_t p_x \mu_{x+t}\, dt. \end{aligned} \tag{2.15}$$

Next we define a function β_n, for $n > 0$. To this end, let $f_e(t)$ denote an annuity-weighted equilibrium distribution[7] with respect to the pdf of $T(x)$, that is,

[7] The equilibrium distribution of a positive random variable X with cdf $F_X(x)$ is the pdf $f_e(x) = (1 - F_X(x))/\mathbb{E}[X]$ for $x > 0$.

$$f_e(t) = \frac{\bar{a}_{\overline{t}|\,t}p_x\mu_{x+t}}{\bar{a}_x}, \tag{2.16}$$

and let T_e be a random variable with pdf $f_e(t)$. We now define β_n as

$$\beta_n = \frac{1}{\bar{a}_x u'(B_n\bar{a}_{\overline{n}|})}\int_0^\infty \bar{a}_{\overline{t}|}u'(B_n\bar{a}_{\overline{t}|})\,{}_tp_x\mu_{x+t}\,\mathrm{d}t \tag{2.17}$$

$$= \frac{\mathbb{E}[u'(B_n\bar{a}_{\overline{T_e}|})]}{u'(B_n\bar{a}_{\overline{n}|})}. \tag{2.18}$$

In other words, β_n is the marginal ratio of the expected marginal utility of discounted annuity consumption at the end of the ''equilibrium lifetime'' to the marginal utility of discounted annuity consumption at the end of the guarantee period.

Because $\partial B_n/\partial n \leq 0$ and $\partial(B_n\bar{a}_{\overline{n}|})/\partial n \geq 0$ as noted earlier, it follows that

$$\frac{\partial\beta_n}{\partial n} = \frac{1}{\bar{a}_x u'(B_n\bar{a}_{\overline{n}|})}\left[\left|\frac{\partial B_n}{\partial n}\right|\int_0^\infty (\bar{a}_{\overline{t}|})^2(-u''(B_n\bar{a}_{\overline{t}|}))\,{}_tp_x\mu_{x+t}\,\mathrm{d}t + \frac{\partial B_n\bar{a}_{\overline{n}|}}{\partial n}(-u''(B_n\bar{a}_{\overline{n}|}))\beta_n\right] \geq 0 \tag{2.19}$$

for $n \geq 0$.

Two important constants used in our analysis are the lower bequest threshold, $\beta_{\min} = \beta_0 < 1$, and the upper bequest threshold, $\beta_{\max} = \beta_{n^{\max}}$. Note that $\beta_{\min}$ satisfies $\lim_{n\to 0+} \partial H(n, \beta_{\min})/\partial n = 0$. As we will prove later, $\beta \leq \beta_{\min}$ implies no guarantee period is chosen, while $\beta \geq \beta_{\max}$ ensures that the maximum guarantee period is chosen.

2.3 Properties of the Optimum Guarantee Period

Let n^* denote the optimum choice of a guarantee period. To determine n^* three types of maximizing solutions are possible:

- The lower corner solution $n^* = 0$, which occurs if β is relatively small
- An upper corner solution $n^* = n^{\max}$, which occurs if β is relatively large and
- An interior maximizing solution $0 < n^* < n^{\max}$.

We now provide necessary and/or sufficient conditions for these solutions.

Result 2

For a risk-averse retiree, a necessary and sufficient condition for the existence of lower corner solution $n^* = 0$ *is* $\beta \leq \beta_{\min}$.

PROOF

To prove the necessary part of Result 2, we must prove that if $\beta > \beta_{\min}$ then there exists an n such that $\mathcal{M}(n) > \mathcal{M}(0)$. However, if $\beta > \beta_{\min}$ then $\lim_{n\to 0+} \partial H(n, \beta_{\min})/\partial n > 0$. This implies there is a neighborhood of 0+, which we call $(0, \varepsilon(\beta))$, where $H(n, \beta) > 0$ for some $n \in (0,\varepsilon(\beta))$, which in turn implies $\mathcal{M}(n) > \mathcal{M}(0)$, and hence the necessary condition is proved. To prove the sufficient part of result 2, we must prove that if $\beta \leq \beta_{\min}$, then $H(n, \beta) \leq 0$ for $n \geq 0$. Let $\beta = \kappa\beta_{\min}$, where $0 \leq \kappa \leq 1$. Expanding $u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))$ about 0 yields

$$\beta R(n) = \kappa\beta_{\min}\,{}_nq_x\bar{a}_x u'(0) - \beta\int_0^n B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})Q_x(n, t)(-u''(\xi_1(t)))\,{}_tp_x\mu_{x+t}\,\mathrm{d}t,$$

where $0 \le \xi_1(t) \le B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})$. Similarly, expanding $u'\ (B_n\bar{a}_{\overline{t}|})$ about $B_0\bar{a}_{\overline{t}|}$ yields

$$S(n) = \beta_{\min}\ {}_nq_x\bar{a}_xu'(0) + {}_nq_x \int_0^\infty (B_0 - B_n)(\bar{a}_{\overline{t}|})^2(-u''(\xi_2(t)))_tp_x\mu_{x+t}\ dt,$$

where $B_n\bar{a}_{\overline{t}|} \le \xi_2(t) \le B_0\bar{a}_{\overline{t}|}$. As $-u''\ (\cdot)) \ge 0$, then, for $0 \le \kappa \le 1$ we have $H(n, \beta) \le 0$, which completes the proof.

Next, we specify the conditions under which the upper corner solution is optimal. Clearly, for $n^* = n^{\max}$ to be a solution for *any* $n^{\max} > 0$, we must have $\ge \beta_{\max}$ and $H(n, \beta) \ge 0$ for $n > 0$.

Result 3

For a risk-averse retiree, a sufficient condition for $n^* = n^{\max}$ to be optimal is $\beta \ge \beta_{\max}$.

PROOF

To prove Result 3 we must prove that if $\beta \ge \beta_{\max}$, then $H(n, \beta) \ge 0$ for $0 \le n \le n^{\max}$. Let $\beta > \beta_{\max}$. Expanding $u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))$ about $B_n\bar{a}_{\overline{n}|}$ yields

$$\beta R(n) = \beta\ {}_nq_x\bar{a}_xu'(B_n\bar{a}_{\overline{n}|}) + \beta \int_0^n B_n\bar{a}_{\overline{t}|}Q_x(n, t)(-u''(\xi_3(t)))_tp_x\mu_{x+t}\ dt$$
$$\ge \beta\ {}_nq_x\bar{a}_xu'(B_n\bar{a}_{\overline{n}|}),$$

where $\xi_3(t)$ lies between $B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})$ and $B_n\bar{a}_{\overline{n}|}$. As

$$S(n) = \beta_n\ {}_nq_x\bar{a}_xu'(B_n\bar{a}_{\overline{n}|}) \tag{2.20}$$

and $\beta \ge \beta_n$, it follows that $H(n, \beta) \ge 0$ for $0 \le n \le n^{\max}$, and the result is proved.

In the case of the interior maximizing solution, β must be such that there exists an $n^* \in (0, n^{\max})$ satisfying $\partial\mathcal{M}(n^*)/\partial n = 0$, that is,

$$H(n^*, \beta) = 0. \tag{2.21}$$

From the definition of n^* as an interior *maximum,* we must have

$$\frac{\partial H(n^*, \beta)}{\partial n^*} < 0. \tag{2.22}$$

It is interesting to consider the behavior of this interior solution n^*, when n^* is viewed as a function of β. Taking the derivative of both sides of equation (2.21) leads to

$$R(n^*) + \frac{\partial H(n^*, \beta)}{\partial n^*}\frac{\partial n^*}{\partial \beta} = 0, \tag{2.23}$$

which implies

$$\frac{\partial n^*}{\partial \beta} = \frac{-R(n^*)}{\dfrac{\partial H(n^*, \beta)}{\partial n^*}} > 0. \tag{2.24}$$

Now, for the lower corner solution at $n^*(\beta) = 0$, increasing β cannot decrease this n^* because n^* must be nonnegative. On the other hand, for the upper corner solution at $n^*(\beta) = n^{\max}$, $\beta R(n^{\max}) - S(n^{\max}) \ge 0$, so increasing β cannot decrease n^*. Thus we have established the following result:

Result 4

When the bequest function is proportional to the utility function, the optimum guarantee period is a nondecreasing function of the bequest constant.

3. BOUNDS ON *R*(*n*) AND *S*(*n*)

To develop bounds on R and S, we will make the following assumption about $u'(y)$:

A.5: $u'(y)$ is a convex function of y, that is, $u'''(y) > 0$ for $y \geq 0$.

Bounds on R and S can be found via Jensen's inequality and the covariance rule. Note that, from assumption A.5, $u'(B_n\bar{a}_{\overline{t}|})$ is a decreasing convex function of t, while $u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))$ is an increasing concave function of t. First we construct the pdf $g_x(n, t)$ given by

$$g_x(n, t) = \frac{Q_x(n, t)_t p_x \mu_{x+t}\, dt}{{}_nq_x\bar{a}_x} \quad \text{for } 0 \leq t \leq n. \tag{3.1}$$

It follows that we can write

$$R(n) = {}_nq_x\bar{a}_x u'(B_n\bar{a}_{\overline{n}|})\, \mathbb{E}\left[\frac{u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{T_g}|}))}{u'(B_n\bar{a}_{\overline{n}|})}\right], \tag{3.2}$$

where T_g is a random variable with pdf $g_x(n, t)$. Jensen's inequality now can be used to give the upper bound:

$$R(n) \leq {}_nq_x\bar{a}_x u'(B_n\bar{a}_{\overline{n}|})\, \frac{u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{\bar{t}_g}|}))}{u'(B_n\bar{a}_{\overline{n}|})}, \tag{3.3}$$

where

$$\bar{t}_g = \mathbb{E}[T_g] = \int_0^n t g_x(n, t)\, dt.$$

As both $Q_x(n, t)$ and $u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))$ are increasing functions,

$$\begin{aligned} R(n) &= {}_nq_x \int_0^n Q_x(n, t) u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))\, \frac{{}_tp_x\mu_{x+t}}{{}_nq_x}\, dt \\ &\geq {}_nq_x\bar{a}_x \int_0^n u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))\, \frac{{}_tp_x\mu_{x+t}}{{}_nq_x}\, dt \end{aligned}$$

from the covariance rule. Further, the concavity of $u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|}))$ implies it lies on or above the chord connecting the points $(0, u'(B_n\bar{a}_{\overline{n}|}))$ and $(n, u'(0))$, that is,

$$u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})) \geq u'(B_n\bar{a}_{\overline{n}|}) + (u'(0) - u'(B_n\bar{a}_{\overline{n}|}))\, \frac{t}{n}$$

for $0 \leq t \leq n$. As

$$\int_0^n \left(\frac{t}{n}\right) \frac{{}_tp_x\mu_{x+t}}{{}_nq_x}\, dt = \frac{1}{{}_nq_x}\left(\frac{\mathring{e}_{x:\overline{n}|}}{n} - {}_np_x\right),$$

we have

$$R(n) \geq {}_nq_x\bar{a}_x u'(B_n\bar{a}_{\overline{n}|})\left[1 + \frac{(u'(0) - u'(B_n\bar{a}_{\overline{n}|}))}{{}_nq_x u'(B_n\bar{a}_{\overline{n}|})}\left(\frac{\mathring{e}_{x:\overline{n}|}}{n} - {}_np_x\right)\right]. \tag{3.4}$$

From equation (2.20), it is clear that

$$\beta_{\min}\, {}_nq_x\bar{a}_{\overline{n}|} u'(B_n\bar{a}_{\overline{n}|}) \leq S(n) \leq \beta_{\max}\, {}_nq_x\bar{a}_{\overline{n}|} u'(B_n\bar{a}_{\overline{n}|}) \tag{3.5}$$

for $n \le n^{\max}$. Another upper bound can be placed on $S(n)$ as follows: from equation (2.12)

$$S(n) \le {}_nq_x \int_0^n \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{t}|})\, {}_tp_x \mu_{x+t}\, dt + \int_n^\infty \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{n}|})\, {}_tp_x \mu_{x+t}\, dt$$

$$= {}_nq_x \left[\int_0^n \bar{a}_{\overline{t}|} u'(B_n \bar{a}_{\overline{t}|})\, {}_tp_x \mu_{x+t}\, dt + {}_np_x (\bar{a}_{\overline{n}|} + v^n \bar{a}_{x+n}) u'(B_n \bar{a}_{\overline{n}|}) \right],$$

which implies

$$S(n) \le \int_0^n [{}_nq_x \bar{a}_{\overline{t}|} (u'(B_n \bar{a}_{\overline{t}|}) - u'(B_n \bar{a}_{\overline{n}|})) + Q(n, t) u'(B_n \bar{a}_{\overline{n}|})]\, {}_tp_x \mu_{x+t}\, dt. \tag{3.6}$$

Inequality (3.6) implies

$$\beta_n \le 1 + {}_nq_x \int_0^n \frac{\bar{a}_{\overline{t}|}}{\bar{a}_x} \left(\frac{(u'(B_n \bar{a}_{\overline{t}|}) - u'(B_n \bar{a}_{\overline{n}|}))}{u'(B_n \bar{a}_{\overline{n}|})} \right) \frac{{}_tp_x \mu_{x+t}}{{}_nq_x}\, dt.$$

The convexity of $u'(B_n \bar{a}_{\overline{t}|}) - u'(B_n \bar{a}_{\overline{n}|})$ implies it lies on or below the chord connecting the points $(0, u'(0) - u'(B_n \bar{a}_{\overline{n}|}))$ and $(n, 0)$, that is,

$$u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t}|})) \le (u'(0) - u'(B_n \bar{a}_{\overline{n}|})) \left(1 - \frac{t}{n}\right).$$

As

$$\int_0^n \left(1 - \frac{t}{n}\right) \frac{{}_tp_x \mu_{x+t}}{{}_nq_x}\, dt = \frac{1}{{}_nq_x} \left(1 - \frac{\mathring{e}_{x:\overline{n}|}}{n}\right),$$

we have

$$\beta_n \le 1 + \frac{(\bar{a}_{x:\overline{n}|} - {}_np_x \bar{a}_{\overline{n}|})}{\bar{a}_x} \left(\frac{(u'(0) - u'(B_n \bar{a}_{\overline{n}|}))}{{}_nq_x u'(B_n \bar{a}_{\overline{n}|})} \right) \left(1 - \frac{\mathring{e}_{x:\overline{n}|}}{n}\right). \tag{3.7}$$

Let

$$\mu_n = \mathbb{E}\left[\frac{u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{T_g}|}))}{u'(B_n \bar{a}_{\overline{n}|})} \right].$$

Inequalities (3.3) and (3.4) imply

$$1 + \frac{(u'(0) - u'(B_n \bar{a}_{\overline{n}|}))}{{}_nq_x u'(B_n \bar{a}_{\overline{n}|})} \left(\frac{\mathring{e}_{x:\overline{n}|}}{n} - {}_np_x \right) \le \mu_n \le \frac{u'(B_n(\bar{a}_{\overline{n}|} - \bar{a}_{\overline{t_n}|}))}{u'(B_n \bar{a}_{\overline{n}|})}. \tag{3.8}$$

From inequalities (3.7) and (3.8) we notice that

$$\mu_n - \beta_n \ge 0 \quad \text{if} \quad \left(\frac{\mathring{e}_{x:\overline{n}|}}{n} - {}_np_x \right) \ge \frac{(\bar{a}_{x:\overline{n}|} - {}_np_x \bar{a}_{\overline{n}|})}{\bar{a}_x} \left(1 - \frac{\mathring{e}_{x:\overline{n}|}}{n}\right).$$

However, as

$$\mathring{e}_{x:\overline{n}|} - n\, {}_np_x \ge \bar{a}_{x:\overline{n}|} - {}_np_x \bar{a}_{\overline{n}|},$$

it follows that $\mu_n - \beta_n \ge 0$ if $n \le \mathring{e}_{x:\overline{n}|} + \bar{a}_x$. Recalling equations (3.2) and (2.20) gives the following result.

Result 5

If a risk-averse retiree's utility function is such that u' is convex, and $n^{\max}$ is such that $n^{\max} \leq \mathring{e}_{x:\overline{n^{\max}}|} + \bar{a}_x$, then the maximum guarantee period is the optimum choice for the retiree when $\beta \geq 1$.

4. CLOSING COMMENTS

For a single-life annuity option, we proved that if the retiree's bequest constant $\beta \leq \beta_{\min}$, the bequest threshold, then it is optimal to choose a straight life annuity without the guarantee period, while if $\beta > \beta_{\max}$, then the maximum guarantee period is the best choice. The determination of the intermediate optimum guarantee period ($0 < n^* < n^{\max}$) is obtained as a solution to equation (2.21). We proved that n^* is nondecreasing in β.

In practice, however, the retirement age x is usually between ages 55 and 75, and the maximum guarantee period $n^{\max}$ is an integer that is at most 30 years. Thus it may be better to determine the optimum n^* by simply finding the expected utility obtained for each $n = 0, 1, \ldots, n^{\max}$, then choose the n^* that yields the maximum expected utility. This approach avoids using derivatives altogether.

Note that we did not deal with the problem of whether an employee elects to purchase a retirement annuity or not (see Milevsky 1998, 2001 and references therein), nor do we deal with the type of joint-life annuity choices a couple must make (see Brown and Poterba 2000 and references therein), nor do we consider retirees who seek to minimize their probabilities of financial ruin, as in Moore and Young (2006).

In addition, marginal tax rates were not taken into consideration because the maximum differential between benefit amounts is less than 10% for a retiree age 65 (see Table 2) and because annuity payments under section 403(b) are generally treated as ordinary income. Thus it seems reasonable to assume that marginal tax rates are not influential in a typical retiree's decision making.

The lump-sum option was not examined, as tax considerations make this option optimal only under special circumstances because the five-year averaging provision available for lump-sum distributions that allowed the marginal tax rate to be computed from one-fifth of the lump sum is no longer available (Slesnick and Suttle 1999, p. 2/18).

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