

# THE DISCOUNTED JOINT DISTRIBUTION OF THE SURPLUS PRIOR TO RUIN AND THE DEFICIT AT RUIN IN A SPARRE ANDERSEN MODEL

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## ABSTRACT

In this article, we consider the risk model with phase-type interclaim times. We first derive a simple matrix-form expression for the discounted joint density of the surplus prior to ruin and the deficit at ruin when the initial surplus is zero. Then we express the discounted joint density in terms of certain expected discounted penalty functions when the initial surplus is greater than zero. This result generalizes that in Gerber and Shiu (1997).

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## 1. INTRODUCTION

Consider the Sparre Andersen risk process defined by

$$U(t) = u + ct - \sum_{k=1}^{N_t} Z_k, \quad t \geq 0, \quad (1.1)$$

where  $\{Z_k, k \geq 1\}$  is a sequence of i.i.d. claim random variables with probability distribution function  $P$ , density function  $p$ , and Laplace transform  $\hat{p}(s) = \int_0^\infty e^{-sx} p(x) dx$ . The process  $\{N_t, t \geq 0\}$  counts the number of claims up to time  $t$  and is assumed to be an ordinary renewal process, so that the last claim occurs at time  $T_0 = 0$ . For  $k = 1, 2, \dots$ , let  $T_k$  denote the time when the  $k$ th claim occurs. Then  $V_k = T_k - T_{k-1}$ ,  $k = 1, 2, \dots$ , are the interclaim random variables. They are independent and assumed to have common distribution function  $A$ , density function  $a$ , and Laplace transform  $\hat{a}(s) = \int_0^\infty e^{-sx} a(x) dx$ . The constant  $u \geq 0$  denotes the insurer's surplus after the claim at time 0 has been paid. The constant  $c > 0$  is the constant premium rate. It satisfies  $c\mathbb{E}(V_k) > \mathbb{E}(Z_k)$ , so ruin is not certain.

Commonly studied quantities related to this model are the time of ruin  $T = \inf\{t : U(t) < 0\}$ , the probability of ultimate ruin  $\psi(u) = \mathbb{P}(T < \infty | U(0) = u)$ , and more generally the expected discounted penalty

$$\phi(u) = \mathbb{E}[e^{-\delta T} \varpi(U(T-), |U(T)|) I(T < \infty) | U(0) = u], \quad (1.2)$$

where  $\varpi(x_1, x_2)$  is a nonnegative penalty function of the surplus prior to ruin and the deficit at ruin and  $I$  is an indicator function.

With  $\varpi(x_1, x_2)$  being a bivariate Dirac delta function concentrated on  $x$  and  $y$ ,  $\phi(u)$  becomes the discounted joint probability density function of  $U(T-)$  and  $|U(T)|$ , defined by

$$f(x, y|u) = \int_0^\infty e^{-\delta t} f(x, y, t|u) dt, \quad (1.3)$$

where  $f(x, y, t|u)$  denotes the (defective) joint probability density function of  $U(T-)$ ,  $|U(T)|$ , and  $T$  given initial surplus  $u$ .

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With  $\varpi(x_2, x_2) = \varpi(x_2)$  being a univariate Dirac delta function concentrated on  $y$ ,  $\phi(u)$  becomes the discounted probability density function of  $|U(T)|$ , defined by

$$f(y|u) = \int_0^\infty f(x, y|u) dx. \tag{1.4}$$

With  $\varpi(x_1, x_2) = 1$ ,  $\phi(u)$  becomes the discounted probability of ruin, defined by

$$\psi_\delta(u) = \mathbb{E}[e^{-\delta T} I(T < \infty) | U(0) = u]. \tag{1.5}$$

Obviously,  $\psi_\delta(u) = \int_0^\infty f(y|u) dy$ .

Most current studies start from the fact that  $\phi(u)$  satisfies a certain integral-differential equation. Under some assumptions for the interclaim time distribution, the integral-differential equation may be solved explicitly. For example, explicit solutions for  $f(x, y|u)$  and  $f(y|u)$  were studied when the distribution function of interclaim waiting times is (a) Erlang(2) (Dickson and Hipp 1998; Cheng and Tang 2003; Lin 2003); (b) Erlang ( $n$ ) (Li and Garrido 2004); (c) phase-type (2) (Dickson and Hipp 2001; Dickson and Drekić 2004); (d) convolution of  $n$  exponential distribution (Gerber and Shiu 2005a); (e) phase-type (Schmidli 2005; Albrecher and Boxma 2005); or (f)  $K_n$  (Li and Garrido 2005). The explicit formulas obtained usually involve the positive roots of Lundberg’s fundamental equation (see, e.g., Gerber and Shiu 2005a).

In this article, motivated by Asmussen (1992), we define a matrix  $\mathbf{Q}$  whose eigenvalues are negatives of the positive solutions to Lundberg’s fundamental equation. This enables us to present a simple explicit expression for the discounted joint density of the surplus prior to ruin and the deficit at ruin when the initial surplus is zero. Further, we express the discounted joint density function in terms of certain expected discounted penalty functions when the initial surplus is greater than zero. This generalizes equation (5.1) in Gerber and Shiu (1997).

The organization of the article is as follows. Section 2 shows a matrix form of Lundberg’s fundamental equation. Section 3 derives a formula for the discounted joint density of the surplus prior to ruin and the deficit at ruin when the initial surplus is zero. Section 4 provides an expression for the discounted joint density when the initial surplus is greater than zero.

## 2. MATRIX FORM OF LUNDBERG’S FUNDAMENTAL EQUATION

We assume that the distribution of the interclaim time  $A$  is phase-type with representation  $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b})$ , where  $\boldsymbol{\alpha}$  and  $\mathbf{b}$  are row vectors of length  $n$  and  $\mathbf{B}$  is an  $n \times n$  matrix. That is, each of the random variables  $V_k$ ,  $k = 1, 2, \dots$  corresponds to the time to absorption in a terminating continuous-time Markov chain  $J_t^{(k)}$  with  $n$  transient states  $\{E_1, E_2, \dots, E_n\}$  and one absorbing state  $E_0$ . Let  $\mathbf{e}$  denote a row vector of length  $n$  with all elements being one. Then  $\mathbf{b}^\top = -\mathbf{B}\mathbf{e}^\top$ . Following Theorem 1.5 of Asmussen (2000, p. 218),

$$\begin{aligned} A(t) &= 1 - \boldsymbol{\alpha}e^{t\mathbf{B}}\mathbf{e}^\top, & t \geq 0, \\ \alpha(t) &= \boldsymbol{\alpha}e^{t\mathbf{B}}\mathbf{b}^\top, & t \geq 0, \end{aligned}$$

and

$$\hat{a}(s) = \int_0^\infty e^{-sx} \alpha(x) dx = \boldsymbol{\alpha}(s\mathbf{I} - \mathbf{B})^{-1}\mathbf{b}^\top. \tag{2.1}$$

For  $j = 1, 2, \dots, n$ , let  $\phi_j(u)$  denote the expected discounted penalty given  $U(0) = u$  and  $J_0^{(1)} = E_j$ , that is,

$$\phi_j(u) = \mathbb{E}[e^{-\delta T} \varpi(U(T-), |U(T)|) I(T < \infty) | U(0) = u, J_0^{(1)} = E_j], \quad j = 1, 2, \dots, n. \tag{2.2}$$

Then the expected discounted penalty at ruin  $\phi(u)$  may be computed by

$$\phi(u) = \boldsymbol{\alpha}\phi(u), \quad (2.3)$$

where  $\boldsymbol{\phi}(u) = (\phi_1(u), \dots, \phi_n(u))^T$  is a column vector of functions.

Schmidli (2005) showed that the vector  $\boldsymbol{\phi}(u)$  satisfies the equation

$$\boldsymbol{\phi}'(u) = \frac{\delta}{c} \boldsymbol{\phi}(u) - \frac{1}{c} \mathbf{B}\boldsymbol{\phi}(u) - \frac{1}{c} \left( \int_0^u \boldsymbol{\alpha}\boldsymbol{\phi}(u-x)p(x) dx + \boldsymbol{\omega}(u) \right) \mathbf{b}^T, \quad (2.4)$$

where  $\boldsymbol{\omega}(u) = \int_u^\infty \boldsymbol{\omega}(u, x-u)p(x) dx$ .

Applying Laplace transforms to both sides of (2.4) and rearranging, we have

$$\mathbf{L}_\delta(s)\hat{\boldsymbol{\phi}}(s) = \boldsymbol{\phi}(0) - \frac{1}{c} \hat{\boldsymbol{\omega}}(s)\mathbf{b}^T, \quad (2.5)$$

where

$$\mathbf{L}_\delta(s) = \left( s - \frac{\delta}{c} \right) \mathbf{I} + \frac{1}{c} \mathbf{B} + \frac{1}{c} \mathbf{b}^T \boldsymbol{\alpha} \hat{p}(s) \quad (2.6)$$

and

$$\hat{\boldsymbol{\omega}}(s) = \int_0^\infty e^{-sy} \boldsymbol{\omega}(y) dy.$$

As discussed in Albrecher and Boxma (2005), the matrix  $\mathbf{L}_\delta(s)$  contains much information about the discounted penalty function  $\phi(u)$ . In particular, for values of  $s$  such that the matrix  $(s - \delta/c) \mathbf{I} + 1/c \mathbf{B}$  is invertible,

$$\begin{aligned} \det(\mathbf{L}_\delta(s)) &= \frac{1}{c} \det((cs - \delta)\mathbf{I} + \mathbf{B}) \det(\mathbf{I} + ((cs - \delta)\mathbf{I} + \mathbf{B})^{-1} \mathbf{b}^T \boldsymbol{\alpha} \hat{p}(s)) \\ &= \frac{1}{c} \det((cs - \delta)\mathbf{I} + \mathbf{B}) (1 + \boldsymbol{\alpha} ((cs - \delta)\mathbf{I} + \mathbf{B})^{-1} \mathbf{b}^T \hat{p}(s)) \\ &= \frac{1}{c} \det((cs - \delta)\mathbf{I} + \mathbf{B}) (1 - \hat{a}(\delta - cs) \hat{p}(s)). \end{aligned} \quad (2.7)$$

In the second step, we utilized the identity<sup>1</sup>

$$\det(\mathbf{I} + \mathbf{XY}) = \det(\mathbf{I} + \mathbf{YX}).$$

Since the matrix  $(cs - \delta)\mathbf{I} + \mathbf{B}$  is assumed to be invertible, (2.7) indicates that the solutions for

$$\det(\mathbf{L}_\delta(s)) = 0 \quad (2.8)$$

and the solutions for Lundberg's fundamental equation

$$\hat{a}(\delta - cs) \hat{p}(s) = 1 \quad (2.9)$$

as defined in Gerber and Shiu (2005a) are identical.

Albrecher and Boxma (2005) calls (2.8) the generalized Lundberg's fundamental equation and show that it has exactly  $n$  solutions in the right half of the complex plane using matrix theory.

It is also possible to show that (2.9) has exactly  $n$  solutions in the right half of the complex plane as follows. Let  $\gamma(s) = 1/\hat{a}(\delta - cs)$ . Then, as pointed out by Gerber and Shiu (2005b), its zeros occur

<sup>1</sup> This method was kindly pointed out by Professor Elias Shiu. Notice that the matrix  $\mathbf{X}$  and  $\mathbf{Y}$  must have correct dimensions and  $I$  has different dimensions on the left- and right-hand sides.

at  $s = (\delta - \zeta)/c$ , where  $\zeta$  ranges over all  $n$  eigenvalues of  $\mathbf{B}$ . However, since  $\mathbf{B}$  is an intensity matrix of all transient states in a continuous Markov chain, it is nonsingular matrix with all its eigenvalues having negative real parts (see Corollary 8.2.1 in Rolski et al. 1999). Therefore, all zeros of  $\gamma(s)$  must be positive. Consequently, similar to Gerber and Shiu (2005a), Rouché’s theorem guarantees that (2.9) has exactly  $n$  solutions in the right half of the complex plane.

In the following, we assume that the  $n$  positive solutions of Lundberg’s fundamental equation have distinct values  $\rho_1, \rho_2, \dots, \rho_n$ .

### 3. A FORMULA FOR $\phi(\mathbf{0})$

Define

$$\mathbf{v}_i = \boldsymbol{\alpha}((\delta - c\rho_i)\mathbf{I} - \mathbf{B})^{-1}, \quad i = 1, 2, \dots, n. \tag{3.1}$$

Then by (2.9),

$$\mathbf{v}_i \mathbf{b}^\top \hat{p}(\rho_i) = 1$$

and thus

$$\mathbf{v}_i \mathbf{L}_\delta(\rho_i) = -\frac{1}{c} \boldsymbol{\alpha} + \frac{1}{c} \mathbf{v}_i \mathbf{b}^\top \hat{p}(\rho_i) \boldsymbol{\alpha} = 0, \quad i = 1, 2, \dots, n.$$

For  $i = 1, 2, \dots, n$ , letting  $s = \rho_i$  and left-multiplying both sides of (2.5) by vector  $\mathbf{v}_i$  yields  $n$  equations

$$0 = \mathbf{v}_i \mathbf{L}_\delta(\rho_i) \hat{\boldsymbol{\Phi}}(s) = \mathbf{v}_i \left( \boldsymbol{\Phi}(0) - \frac{1}{c} \hat{\omega}(\rho_i) \mathbf{b}^\top \right), \quad i = 1, 2, \dots, n. \tag{3.2}$$

In matrix form,

$$\mathbf{V} \boldsymbol{\Phi}(0) = \frac{1}{c} \begin{pmatrix} \hat{\omega}(\rho_1) & & & \\ & \hat{\omega}(\rho_2) & & \\ & & \dots & \\ & & & \hat{\omega}(\rho_n) \end{pmatrix} \mathbf{V} \mathbf{b}^\top, \tag{3.3}$$

where  $\mathbf{V} = [\mathbf{v}_1^\top, \mathbf{v}_2^\top, \dots, \mathbf{v}_n^\top]^\top$  denotes the matrix with  $\mathbf{v}_i$  in its  $i$ th row.

As in Albrecher and Boxma (2005), we further assume that  $\mathbf{v}_i, i = 1, 2, \dots, n$ , are linearly independent. Then  $\mathbf{V}$  is invertible and

$$\boldsymbol{\Phi}(0) = \frac{1}{c} \mathbf{V}^{-1} \begin{pmatrix} \hat{\omega}(\rho_1) & & & \\ & \hat{\omega}(\rho_2) & & \\ & & \dots & \\ & & & \hat{\omega}(\rho_n) \end{pmatrix} \mathbf{V} \mathbf{b}^\top. \tag{3.4}$$

Let

$$\mathbf{Q} = \mathbf{V}^{-1} \begin{pmatrix} -\rho_1 & & & \\ & -\rho_2 & & \\ & & \dots & \\ & & & -\rho_n \end{pmatrix} \mathbf{V} \tag{3.5}$$

be a  $n \times n$  matrix with negative eigenvalues  $\{-\rho_1, -\rho_2, \dots, -\rho_n\}$  and the corresponding left eigenvectors  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ . Then  $\boldsymbol{\Phi}(0)$  has the more concise form

$$\boldsymbol{\phi}(0) = \frac{1}{c} \hat{\omega}(-\mathbf{Q})\mathbf{b}^\top. \quad (3.6)$$

Proposition (3.1) in Albrecher and Boxma (2005) gives a matrix formula for  $\boldsymbol{\phi}(0)$ , but (3.6) is considerably simpler.

### 3.1 The Discounted Joint Density When $u = 0$

We next derive a formula for  $f(x, y|0)$ . To this end, we choose the penalty function  $\varpi(x_1, y_1)$  to be the Dirac delta function with respect to  $x_1 = x$  and  $y_1 = y$ . As a result  $\hat{\omega}(s) = e^{-sx}p(x + y)$  and

$$\boldsymbol{\phi}(0) = \mathbf{f}(x, y|0) = [f_1(x, y|0), f_2(x, y|0), \dots, f_n(x, y|0)]^\top,$$

where  $f_i(x, y|0)$ ,  $i = 1, 2, \dots, n$ , denotes the (defective) joint probability density function of  $U(T-)$  and  $|U(T)|$  given initial surplus  $u = 0$  and  $J_0^{(1)} = E_i$ .

With (3.6),

$$\begin{aligned} \mathbf{f}(x, y|0) &= \frac{1}{c} \mathbf{V}^{-1} \begin{pmatrix} p(x + y)e^{-\rho_1 x} & & & & \\ & p(x + y)e^{-\rho_2 x} & & & \\ & & \dots & & \\ & & & & p(x + y)e^{-\rho_n x} \end{pmatrix} \mathbf{V} \mathbf{b}^\top \\ &= \frac{1}{c} p(x + y)e^{\mathbf{Q}x}\mathbf{b}^\top. \end{aligned} \quad (3.7)$$

Consequently,

$$f(x, y|0) = \boldsymbol{\alpha}\mathbf{f}(x, y|0) = \frac{1}{c} \boldsymbol{\alpha}e^{\mathbf{Q}x}\mathbf{b}^\top p(x + y), \quad x > 0, y > 0. \quad (3.8)$$

This simple formula generalizes equation (8.3) in Gerber and Shiu (2005a).

### 3.2 Examples

#### Example 1

When the interclaim times are exponentially distributed with rate  $\lambda$ , (2.8) becomes

$$\mathbf{L}_\delta(s) = s - \frac{\delta}{c} - \frac{\lambda}{c} + \frac{\lambda}{c} \hat{p}(s) = 0. \quad (3.9)$$

This is equation (3.3) in Gerber and Shiu (1997) and was shown to have a root  $\rho$  with positive real part.

In this case,  $\mathbf{V}$  is a scalar, and so  $\mathbf{Q} = -\rho$ . Thus, (3.8) becomes

$$f(x, y|0) = e^{-\rho x} p(x + y) \frac{\lambda}{c}, \quad (3.10)$$

which is formula (3.12) in Gerber and Shiu (1997).

#### Example 2

When the distribution of the interclaim time is a convolution of  $n$  exponential distributions with rate  $\lambda_1, \lambda_2, \dots, \lambda_n$ , respectively,

$$\boldsymbol{\alpha} = (1, 0, \dots, 0), \mathbf{B} = \begin{pmatrix} -\lambda_1 & \lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda_n \end{pmatrix}, \mathbf{b}^\top = \begin{pmatrix} 0 \\ 0 \\ \dots \\ \lambda_n \end{pmatrix}. \quad (3.11)$$

For  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} \mathbf{v}_i &= \boldsymbol{\alpha}((\delta - c\rho_i)\mathbf{I} - \mathbf{B})^{-1} \\ &= \frac{1}{c} \left( \frac{1}{\lambda_1 + \delta - c\rho_i}, \frac{\lambda_1}{\prod_{j=1}^2 (\lambda_j + \delta - c\rho_j)}, \dots, \frac{\prod_{j=1}^{n-1} \lambda_j}{\prod_{j=1}^n (\lambda_j + \delta - c\rho_j)} \right), \end{aligned}$$

which for the purpose of calculation, may be normalized to

$$\mathbf{v}_i = \left( \prod_{j=2}^n \frac{\lambda_j + \delta - c\rho_j}{\lambda_{j-1}}, \dots, \frac{\lambda_n + \delta - c\rho_n}{\lambda_{n-1}}, 1 \right).$$

Finally,

$$\begin{aligned} f(x, y|u = 0) &= \frac{1}{c} \boldsymbol{\alpha} \mathbf{V}^{-1} \begin{pmatrix} e^{-\rho_1} & & & \\ & e^{-\rho_2} & & \\ & & \dots & \\ & & & e^{-\rho_n} \end{pmatrix} \mathbf{V} \mathbf{b}^\top p(x + y) \\ &= \frac{\lambda_1, \lambda_2, \dots, \lambda_n}{c^n} p(x + y) \sum_{j=1}^n e^{-\rho_j x} \prod_{k=1, k \neq j}^n \frac{1}{\rho_k - \rho_j}, \end{aligned} \tag{3.12}$$

which is formula (8.3) in Gerber and Shiu (2005a).

We omit some details of calculations because the matrix  $\mathbf{V}$  used here turns out to be exactly the same as the matrix  $\mathbf{K}$  defined in Albrecher and Boxma (2005) for solving the same problem. One may therefore just follow their calculations to get the inverse of matrix  $\mathbf{V}$  and consequently obtain (3.12).

## 4. THE DISCOUNTED JOINT DENSITY WHEN $u > 0$

### 4.1 The Size of Discontinuity in $f(x|u)$

Let  $g(y) = \int_0^\infty f(x, y|0) dx$  denote the discounted density function of the deficit at ruin when  $u = 0$ . Then as in Gerber and Shiu (1997),

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z) dz + f(x - u, u + y|0)I(u < x). \tag{4.1}$$

With formula (3.8), (4.1) can be written as

$$f(x, y|u) = \int_0^u f(x, y|u - z)g(z) dz + \frac{1}{c} \boldsymbol{\alpha} e^{\mathbf{Q}(x-u)} \mathbf{b}^\top p(x + y)I(u < x). \tag{4.2}$$

Integrating (4.2) with respect to  $y$ , we obtain

$$f(x|u) = \int_0^u f(x|u - z)g(z) dz + \frac{1}{c} \boldsymbol{\alpha} e^{\mathbf{Q}(x-u)} \mathbf{b}^\top (1 - P(x))I(u < x). \tag{4.3}$$

As a function of  $x$ ,  $f(x|u)$  has a discontinuity of

$$\frac{1}{c} \boldsymbol{\alpha} \mathbf{b}^\top (1 - P(x)) \tag{4.4}$$

at  $x = u$ . Similar to equation (4.3) in Gerber and Shiu (1997) for classical risk model, it does not depend on  $\delta$ .

When the distribution of the interclaim time is a convolution of  $n$  independent exponential distributions, because of the special form of vectors  $\boldsymbol{\alpha}$  and  $\mathbf{b}$  shown in (3.11), the size of the discontinuity is zero for  $n \geq 2$ . This was pointed out by Dickson and Drekić (2004), Tsai (2005), and Gerber and Shiu (2005b).

## 4.2 Dickson's Formula

This section presents a formula for  $f(x, y|u)$  that generalizes equations (5.1) of Gerber and Shiu (1997) and (3.1) and (3.2) of Dickson (1992). In derivation, we apply the methodology in Section 5 of Gerber and Shiu (1997).

Let  $\mathbf{M}(x, y|0) = 1/c p(x + y)e^{Qx}$ . Then  $f(x, y|0) = \boldsymbol{\alpha}\mathbf{M}(x, y|0)\mathbf{b}^\top$ . For  $u > 0$ , let  $\mathbf{M}(x, y|u)$  be a  $n \times n$  matrix solving the renewal equation

$$\begin{aligned} \mathbf{M}(x, y|u) &= \int_0^u g(z)\mathbf{M}(x, y|u - z) dz + I(u < x)\mathbf{M}(x - u, y + u|0) \\ &= \int_0^u g(z)\mathbf{M}(x, y|u - z) dz + \frac{1}{c} I(u < x)e^{Q(x-u)}p(x + y) \\ &= \int_0^u g(z)\mathbf{M}(x, y|u - z) dz + I(u < x)\mathbf{M}(x, y|0)e^{-Qu}. \end{aligned} \quad (4.5)$$

Comparing (4.2) and (4.5) shows that

$$f(x, y|u) = \boldsymbol{\alpha}\mathbf{M}(x, y|u)\mathbf{b}^\top. \quad (4.6)$$

Let  $\mathbf{Y}(u)$  be a matrix solving the renewal equation

$$\mathbf{Y}(u) = \int_0^u g(z)\mathbf{Y}(u - z) dz + I(u < x)e^{-Qu}. \quad (4.7)$$

Then  $\mathbf{M}(x, y|0)\mathbf{Y}(u)$  satisfies (4.5), and so

$$\mathbf{M}(x, y|u) = \mathbf{M}(x, y|0)\mathbf{Y}(u). \quad (4.8)$$

Taking Laplace transform of (4.7), we have

$$\begin{aligned} \hat{\mathbf{Y}}(\xi) &= \hat{g}(\xi)\hat{\mathbf{Y}}(\xi) + \int_0^x e^{-\xi u}e^{-Qu} du \\ &= \hat{g}(\xi)\hat{\mathbf{Y}}(\xi) + (\mathbf{I} - e^{-(\xi+\mathbf{Q})x})(\xi\mathbf{I} + \mathbf{Q})^{-1}, \end{aligned} \quad (4.9)$$

and so

$$\hat{\mathbf{Y}}(\xi) = \frac{1}{1 - \hat{g}(\xi)} (\mathbf{I} - e^{-(\xi+\mathbf{Q})x})(\xi\mathbf{I} + \mathbf{Q})^{-1}. \quad (4.10)$$

Next define the matrix of discounted penalty functions

$$\boldsymbol{\Psi}(u) = \mathbb{E}[e^{-\delta T}I(T < \infty)e^{-Qu(T)}|U(0) = u]. \quad (4.11)$$

It satisfies the matrix renewal equation

$$\boldsymbol{\Psi}(u) = \int_0^u g(z)\boldsymbol{\Psi}(u - z) dz + \int_u^\infty g(z)e^{-Q(u-z)} dz. \quad (4.12)$$

In particular,

$$\boldsymbol{\Psi}(0) = \int_0^\infty g(z)e^{Qz} dz = \hat{g}(-\mathbf{Q}). \quad (4.13)$$

Taking Laplace transform of (4.12) yields

$$\begin{aligned} \hat{\Psi}(\xi) &= \hat{g}(\xi)\hat{\Psi}(\xi) + \int_0^\infty e^{-\xi u} \int_u^\infty g(z)e^{-Q(u-z)} dz du \\ &= \hat{g}(\xi)\hat{\Psi}(\xi) + \int_0^\infty g(z)e^{Qz} \int_0^z e^{-(\xi I+Q)u} du dz \\ &= \hat{g}(\xi)\hat{\Psi}(\xi) + (\hat{g}(-Q) - \hat{g}(\xi I))(\xi I + Q)^{-1}. \end{aligned} \tag{4.14}$$

Thus

$$\begin{aligned} \hat{\Psi}(\xi) &= \frac{1}{1 - \hat{g}(\xi)} (\hat{g}(-Q) - \hat{g}(\xi I))(\xi I + Q)^{-1} \\ &= \frac{1}{1 - \hat{g}(\xi)} (\Psi(0) - \hat{g}(\xi I))(\xi I + Q)^{-1}. \end{aligned} \tag{4.15}$$

Define

$$\Phi(u) = \begin{cases} e^{-Qu}, & x > u \geq 0 \\ e^{-Qx}\Psi(u - x), & 0 < x \leq u. \end{cases} \tag{4.16}$$

It has Laplace transform

$$\begin{aligned} \hat{\Phi}(\xi) &= \int_0^x e^{-(\xi I+Q)u} du + \int_x^\infty e^{-\xi u} e^{-Qx}\Psi(u - x) du \\ &= (I - e^{-(\xi I+Q)x})(\xi I + Q)^{-1} + e^{-(\xi I+Q)x}\hat{\Psi}(\xi), \end{aligned} \tag{4.17}$$

and so

$$\hat{\Phi}(\xi) - \hat{\Psi}(\xi) = (I - e^{-(\xi I+Q)x})(\xi I + Q)^{-1} - \hat{\Psi}(\xi). \tag{4.18}$$

Applying (4.15) to the right-hand side of (4.18) yields

$$\hat{\Phi}(\xi) - \hat{\Psi}(\xi) = \frac{1}{1 - \hat{g}(\xi)} (I - e^{-(\xi I+Q)x})(\xi I + Q)^{-1}(I - \Psi(0)). \tag{4.19}$$

Comparing (4.19) with (4.10), we have

$$\hat{Y}(\xi) = (\hat{\Phi}(\xi) - \hat{\Psi}(\xi))(I - \Psi(0))^{-1}, \tag{4.20}$$

and so

$$Y(u) = (\Phi(u) - \Psi(u))(I - \Psi(0))^{-1}. \tag{4.21}$$

Plugging (4.21) into (4.8) yields

$$M(x, y|u) = M(x, y|0)(\Phi(u) - \Psi(u))(I - \Psi(0))^{-1}. \tag{4.22}$$

In view of (4.6),

$$\begin{aligned} f(x, y|u) &= \alpha M(x, y|0)(\Phi(u) - \Psi(u))(I - \Psi(0))^{-1}b^\top \\ &= \begin{cases} \alpha M(x, y|0)(e^{-Qu} - \Psi(u))(I - \Psi(0))^{-1}b^\top, & x > u \geq 0 \\ \alpha M(x, y|0)(e^{-Qx}\Psi(u - x) - \Psi(u))(I - \Psi(0))^{-1}b^\top, & 0 < x \leq u. \end{cases} \end{aligned} \tag{4.23}$$

This generalizes equations (5.1) of Gerber and Shiu (1997) and (3.1) and (3.2) of Dickson (1992).

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## DISCUSSIONS

### BANGWON KO\*

Professor Ren has obtained many compact matrix expressions in a Sparre Andersen model where the interclaim time random variables are phase-type. In this discussion, we investigate two results used in the paper. First, we derive the system of integro-differential equations (9), which has first appeared in Schmidli (2005) but without a detailed explanation. Second, we provide a proof of the matrix-determinant identity, given below equation (2.7).

To obtain equation (2.4), we begin with the equation

$$\phi_j(u)(\delta dt) = E(d\phi_j(U)(t)|U(t) = u). \quad (D.1)$$

A consequence of an equation analogous to (D.1) is the celebrated Black-Scholes differential equation (McDonald 2006, eq. 21.31).

The right-hand side of equation (D.1) is

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$$\begin{aligned}
 E[d\phi_j(U(t))|U(t) = u] &= (1 + b_j dt)\phi_j(u + cdt) + \sum_{k=1, k \neq j}^m (b_{jk} dt)\phi_k(u + cdt) \\
 &\quad + (b_j dt)E\left[\sum_{k=1}^n \alpha_k \phi_k(u + cdt - X)\right] - \phi_j(u).
 \end{aligned}
 \tag{D.2}$$

This is obtained by conditioning on whether there is a transition or a claim between time  $t$  and time  $t + dt$ . If there is no claim during the time interval, we obtain the first two terms because the transition matrix from time  $t$  to time  $t + dt$  is given by  $\mathbf{I} + dt\mathbf{B}$ . In this case we have

$$\phi_k(u + cdt) = \phi_k(u) + \phi'_k(u)(cdt),
 \tag{D.3}$$

for  $k \neq 0$ . If there is a claim during the interval, we obtain the third term on the right-hand side of (D.2) because the Markov process  $\{J_t : t \geq 0\}$  restarts with the initial distribution  $\alpha$ . Here the random variable  $X$  represents the claim size; its density is denoted by  $p(x)$  in the paper. Then, ignoring terms of order  $(dt)^2$ , we have

$$(b_j dt)E[\phi_k(u + cdt - X)] = (b_j dt)E[\phi_k(u - X)].$$

where the expectation is

$$E[\phi_k(u - X)] = \int_0^u \phi_k(u - x)p(x)dx + \int_u^\infty \varpi(u, x - u)p(x)dx.
 \tag{D.4}$$

It follows from equations (D.1), (D.2), (D.3), and (D.4) and ignoring terms of order  $(dt)^2$  that

$$\phi_j(u)\delta = \phi'_j(u)c + \sum_{k=1}^n b_{jk}\phi_k(u) + b_j \left( \sum_{k=1}^n \alpha_k \int_0^u \phi_k(u - x)p(x)dx + \int_u^\infty \varpi(u, x - u)p(x)dx \right).$$

Thus, we obtain the system of integro-differential equations (9). For an analogous derivation, see Ko (2007).

Next, we prove a matrix-determinant identity. Let  $\mathbf{X}$  be an  $m \times n$  matrix and  $\mathbf{Y}$  an  $n \times m$  matrix. Observe that

$$\begin{pmatrix} \mathbf{I}_{m \times m} & \mathbf{X} \\ \mathbf{0}_{n \times m} & \mathbf{I}_{n \times n} \end{pmatrix} \begin{pmatrix} \lambda \mathbf{I}_{m \times m} - \mathbf{X}\mathbf{Y} & \mathbf{0}_{m \times n} \\ \mathbf{Y} & \lambda \mathbf{I}_{n \times n} \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{I}_{m \times m} & \lambda \mathbf{X} \\ \mathbf{Y} & \lambda \mathbf{I}_{n \times n} \end{pmatrix} = \begin{pmatrix} \lambda \mathbf{I}_{m \times m} & \mathbf{0}_{m \times n} \\ \mathbf{Y} & \lambda \mathbf{I}_{n \times n} - \mathbf{Y}\mathbf{X} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{m \times m} & \mathbf{X} \\ \mathbf{0}_{n \times m} & \mathbf{I}_{n \times n} \end{pmatrix},$$

where  $\mathbf{0}_{m \times n}$ 's denote the  $m \times n$  matrix of 0's. Taking determinant on each side, we see that the characteristic polynomials of  $\mathbf{X}\mathbf{Y}$  and  $\mathbf{Y}\mathbf{X}$  are identical up to a factor of  $\lambda^{n-m}$ . The identity,  $\det(\mathbf{I} + \mathbf{X}\mathbf{Y}) = \det(\mathbf{I} + \mathbf{Y}\mathbf{X})$ , is an immediate consequence.

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