"Search for Predictors of Exceptional Human Longevity: Using Computerized Genealogies and Internet Resources for Human Longevity Studies," Natalia S. Gavrilova and Leonid A. Gavrilov, January 2007*

Bert Kestenbaum†

The achievement of extreme old age seems to hold a great fascination for many of us (probably disproportionate to its actual importance for actuarial practice in terms of life table construction and the like). In their paper on attainments of age 100 in the United States, the authors lead those of us who are attracted to the subject through what no doubt are new territories for most: historical records over 100 years old, record linkage, conclusions based on relatively small numbers of observations, and the new frontier of genealogies on the World Wide Web.

The authors' main substantive conclusions are that (a) first-borns are almost twice as likely to achieve age 100 than persons of all other birth orders combined, a conclusion for which the authors offer an explanation, and that (b) persons born in the West are almost three times as likely to achieve centenarianship as people born in the Northeast, a surprising finding that they cannot explain. The main methodological contribution of the paper is the achievement of a 91% success rate in linkage to old census records, much higher than the runner-up figure of 69% achieved by Hill and his coauthors (Hill, Preston, and Rosenwaike 2000a).

The authors invited me to comment on their paper, presumably because I coauthored two papers in the subject area based on Social Security Administration (SSA) files that were published in this journal (Kestenbaum and Ferguson 2002, 2006).

Research Thesis

The authors' research thesis is that valid inferences about the achievement of extreme old age in the United States can be drawn from some small subset of Internet genealogies. They recognize the need to demonstrate the accuracy of the extreme ages recorded in these records, and endeavor to do so by checking the recorded dates of death against the SSA death file and the recorded months/years of birth against the 1900 Census of Population and later decennial censuses. They make a strong case that the records that they retain are now verified by independent sources; however, I find it difficult to set aside the suspicion that *those responsible for creating the genealogical data may have used these very same sources in their creation*.

My chief concern about the research thesis is with the representativeness of the cases obtained. For example, not a single black centenarian was found. Also, the age distribution is odd: more persons age 101 at death than age 100, and almost as many at age 102 as at age 100—even though $q(x)$ is about 0.4 at those ages. What assurances do we have that the authors' allegation that "There is no reason to believe that household characteristics are different for families covered by genealogies and the Caucasian population in general" is more than wishful thinking?

Another matter of uneasiness for the actuary typically used to large-scale experience studies is that the results here are based on a few hundred observations, and the results for males are based on fewer than that. While the study begins with 991 centenarians born in the United States since 1875, the study window is immediately narrowed to those born in the period 1890–99, and the number of records

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* Opinions expressed in this discussion are those of the author, and no official endorsement by the Social Security Administration should be inferred.

† Bert Kestenbaum, ASA, is Supervisory Actuary with the Office of the Chief Actuary, Social Security Administration, in Baltimore, MD, bert.m.kestenbaum@ssa.gov.

for persons born in that period who are successfully matched to both the SSA Death Master File and the early censuses is 485. Other results are based on the 358 cases found in the 1900 census and on just 198 families of centenarians born in the narrower window 1890–93.

While it is true that sampling error is the means to judge statistical significance and the authors are conscientious about presenting *p*-values and confidence intervals in the tables, the narrative uses the point estimates. For example, they note that the odds of achieving age 100 for persons born in the West are three times the odds for persons born in the Northeast, but, in fact, the 95% confidence interval begins at 1.04.

MEASUREMENT

The demonstrated region-of-birth differential in the probability of achieving centenarianship for persons born 1890–99 is one of several differentials measured relative to a control group consisting of all Caucasian households enumerated in the 1900 census with a child born 1890–99 present. What appears to be lacking, however, is a control for family size: clearly the chance that a family will have an ultimate centenarian increases with the number of children in the family. Thus, for example, the finding that families owning farms are the most likely to produce a centenarian may be attributable to larger-size families, rather than to the farm environment per se.

The authors used a different analysis to demonstrate a higher likelihood of reaching age 100 among first-born children, relative to children of higher birth orders. Selecting a sample of 198 families with a centenarian and a total of 950 children, they computed that among these families "the odds of becoming a centenarian are indeed 1.7 times higher for first-born children compared to their later-born siblings from exactly the same family."

There is a measurement issue here, and it is unclear to me whether it was accounted for. Because the 198 families are not the same size, the odds for first-borns will be greater than the odds for not first-borns even if in the absence of a true relationship, as demonstrated now.

We don't know the distribution by family size of the 198 families with 950 children in the study, but for illustration purposes let's assume a uniform distribution from one to nine. That is, 22 families have one child, 22 families have two, . . . , and 22 families have nine (for a total of 990 children—not far from 950). The value of the statistic of interest under the null hypothesis is computed as follows:

- The probability of centenarianship for the 198 first-born, given that there is a centenarian in their family, equals the average of $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, and $\frac{1}{9}\}$, or 0.314.
- The probability of centenarianship for the 792 not first-born, given that there is a centenarian in their family, equals the quotient by $(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$ of $\{1 \times \frac{1}{2} + 2 \times \frac{1}{3} + 3 \times \frac{1}{4} + 4 \times \frac{1}{5} + 5 \times \frac{1}{6} + 6 \times \frac{1}{7} + 7 \times \frac{1}{8} + 8 \times \frac{1}{9}\}$, which computes to 0.174.
- The ratio of the two probabilities is 0.314 to 0.174, or 1.80.

OTHER DATA

The authors point out the general consistency of their results with the literature, with the notable exception of their finding of a survival-to-100 advantage for those born in the West, which seems in conflict with the finding by Hill and his colleagues (2000b) of a survival-to-85 advantage for persons born in the Northeast and Midwest.

Survival advantages are measured differently by Doblhammer in her intriguing work on month-of-birth differentials, namely, by comparisons of the average age at death among persons who die after age 50. For the United States, for example, using a national file of death certificates for deaths occurring between 1989 and 1997, she calculated an advantage in the average lifetime of persons born in September of 0.45 years, relative to persons born in June (Doblhammer 2004).

To inform the discussion of regional differentials in survival, I have applied the Doblhammer method to a file of death certificates for U.S. deaths occurring between 1989 and 2003. I find that the average

age at death for white persons surviving to age 50 is 78.5 for persons born in the Midwest, 77.9 for persons born in the Northeast, 77.0 for persons born in the South, and 76.0 for persons born in the West.

(Incidentally, the month-of-birth differential that Doblhammer uncovered seems to be lessening with the passage of time. If the 15-year period 1989–2003 is divided into three five-year periods, the range in average age at death (conditional on survival to age 50) by month of birth is 0.52 years in 1989–93, 0.40 in 1994–98, and 0.29 in 1999–2003.)

The authors were eminently successful in locating records in early censuses for persons in their study. Their success rate of 91% is far better than the next-best 69% figure achieved in the aforementioned study by Hill and his colleagues. The authors attribute their success to a characteristic of genealogical data, namely, that the availability of places of birth for siblings of the study subjects allows for ''locating'' a mobile family at the time of a later census. It should be pointed out, however, that in the current study only records for which date of death information agreed with date-of-death information in Social Security records were linked to the early censuses, while in the study by Hill only records for which the date of death information *did not* agree with date-of-death information in Social Security records were linked to early censuses. One would expect that records of more dubious quality would be more difficult to link to census records.

The possibility that census records were used in the construction of the genealogies was mentioned earlier. If true, the linkage success rates are inflated.

SSA FILES

Three SSA files are mentioned in the paper: (1) the file of applications for a Social Security card, (2) the file of deaths recorded in SSA records that can be shared with the public, and (3) the Master Beneficiary Record of entitlements to Social Security benefits and/or enrollments in the Medicare program. Below are brief descriptions of the files.

1. In the first file, a record exists for each application for a Social Security card. Date of birth and place of birth information are collected on the application, as well as names of parents. The file was computerized during the 1970s; in the process, place of birth information and parental names were lost for persons who had previously filed a claim for benefits, which would include many of the persons born in the study window of 1890–99.
2. Attached to this file of applications for a Social Security card is a file of deaths that were reported to SSA from various sources. In order that a death report be placed on this death file, the name and date of birth of the decedent on the report must match within tolerances the name and date of birth on the applications file. Some death reports for women are accordingly rejected because of discrepancies in surname attributable to marriage and divorce. Furthermore, the public-use version, called the Death Master File, is missing certain deaths that may not be shared with the public. The article about the Death Master File written by Hill and Rosenwaike (2001) and referenced in the paper is a good one.
3. The Master Beneficiary Record file contains a record for each person who is or was entitled to Social Security benefits or enrolled in the Medicare program. The record contains the date of birth as ascertained at the time of entitlement or enrollment and the date of death for those deceased. Information about the dates of birth and death of the spouse may or may not be available, depending on certain circumstances.

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AUTHORS' REPLY

We would like to thank Mr. Kestenbaum for his time and interest given to our research and for his useful comments that provide us with additional suggestions for improvements of our future studies of centenarians. We agree with many of his comments, and we appreciate other comments that we do not agree with, and for which we provide here our response and clarifications.

It is true that our studies are not based on thousands of cases with extreme longevity (we worked with several hundred cases). Achievement of centenarianship is still a rare event, and most studies of centenarians use fewer than 300 individuals aged 100 years and more (Lee et al. 2004; Martin, da Rosa, and Siegler 2006). Using case-control design and within-family design in particular allowed us to achieve a good statistical power even with relatively small sample size. It is also true that the sample of centenarians found in computerized genealogies may not be representative for all American centenarians. We discussed this limitation in our article; again this is a problem with many other centenarian studies as well, which often are based on voluntary recruitment (Perls 2001; Martin, da Rosa, and Siegler 2006). These problems demonstrate that studies of extreme longevity, although fascinating, face many methodological challenges. It is well known that the lack of representativeness is not crucial for testing specific hypotheses (Woodward 2005) such as the birth-order effects, etc. In this case the problem can be resolved by selecting a proper control group of shorter-lived individuals drawn from the same genealogies. When we confirmed the validity and usefulness of genealogical data it is possible to conduct a large-scale study with automated search of computerized genealogies for thousands of centenarian cases.

Another very important comment of Mr. Kestenbaum's is a suggestion that the comparison of households with a future centenarian to a control group of all Caucasian households enumerated in the 1900 census requires a control for family size. Given that the chances of finding centenarians in a family increase with the number of children in the family, centenarian families may be larger than the families of individuals with average lifespan. In this case findings based on comparing centenarian households to a population-based sample may be attributed to the fact of being born in a larger family rather than to achievement of extreme longevity. This is an important comment, which can be applied to other studies using a similar design (Preston, Hill, and Drevenstedt 1998; Hill et al. 2000b; Stone 2003). Census data do not allow researchers to provide an adequate control for family size and provide only information about the number of children in a household at the time of enumeration. Number of children at the time of enumeration is a poor indicator of family size because this number would depend on the birth order of a centenarian. Fortunately, genealogical data provide information about family size, which can be verified using data from 1900 and 1910 censuses. Using a control group taken from the same population universe (genealogies) would give us an opportunity to rectify the study design and to resolve this methodological problem in future studies.

We also greatly appreciate the very useful description of three SSA files. This information would definitely be interesting to *NAAJ* readers. In our study we used the publicly available SSA Death Master File.

Below we provide clarifications for other questions raised by Mr. Kestenbaum, which probably resulted from a lack of detailed explanations in our article due to limited space.

Mr. Kestenbaum believes that we achieved a good linkage success rate to early censuses because "those responsible for creating the genealogical data may have used these very same sources in their creation.'' In our study we used only those genealogies in which the centenarian and his or her parents had information about complete (day, month, year) birth and death dates. Our 15-year experience working with genealogies convinced us that this requirement is essential for selecting genealogies of good accuracy and ensures that genealogy compilers used written sources (parish records, family Bible, birth certificates, etc.) in their work. Taking into account that census data do not have this detailed information, it is absolutely impossible that genealogies used in our study were compiled on the basis of early censuses alone. Genealogies that we used in our study were collected before 2002. At this time online censuses were not available to the public, and family researchers had to visit the National Archives for census data, so very few genealogies in our study referred to census data as an additional source of information.

Mr. Kestenbaum questions whether the odds of first-borns in our within-family study of centenarians and their siblings would always be greater than the odds for not first-borns even in the absence of true relationship. The method of conditional logistic regression used in our study accounts only for effects within each stratum (family in our case). Calculation of likelihood function using this method is conducted in such a way that the contribution of all terms constant within the stratum (including family size) are canceled out, leaving for analysis only effects of variables that vary within the stratum (Hosmer and Lemeshow 1989). To be certain of our results, we conducted a computer simulation using the design suggested by Mr. Kestenbaum: 176 families with 968 children assuming a uniform distribution from two to nine siblings in a family (as in our previous studies, we did not use noninformative cases with only one child in a family). There were 22 families for each sibship size (number of siblings). Assuming only one centenarian per family, we randomly assigned a birth order of the centenarian for each of 176 families. Running conditional logistic regression (clogit procedure in Stata; StataCorp 2005) on this simulated sample produced an odds ratio of centenarianship for the first-born child equal to 0.9 (0.6–1.3 95% C.I.) with $p = 0.543$, which is not significantly different from one and which should be expected if the method of conditional logistic regression produced correct results for within-family analysis.

Mr. Kestenbaum suggested using an approach based on cross-sectional comparisons of the average age at death, an approach that was initially proposed in the studies of month-of-birth differentials (Doblhammer 2002, 2004). However, the validity of this methodology is questioned in the scientific literature (Gavrilov and Gavrilova 2003; Gavrilova et al. 2003). Specifically, the following argument was published regarding this particular approach:

> This methodology is flawed and can produce both false positive and false negative findings. For example, if the seasonality of births and infant mortality were more expressed in the past, then the month-of-birth distribution of people would differ in different age groups of the population, thus producing a spurious month-of-birth effect on lifespan (if erroneously estimated through mean age at death). This mistake happens because the mean age at death depends on the age distribution of living people, which may differ depending on month of birth. Thus, even if the month of birth does not affect adult lifespan, nevertheless a false positive finding may occur, simply because the effects of population age structure are not taken into account. On the other hand, month-of-birth effects could be overlooked by [the] cross-sectional method if the seasonal effects on age-specific mortality are proportional. This false negative finding happens because proportional changes in death rates produce proportional changes in the numbers of deaths in all age groups, and such proportional changes in numbers have no effect on the mean age at death. Thus, a false negative finding may occur, because cross-sectional analysis of death records is blind to proportional changes in age-specific death rates. (Gavrilov and Gavrilova 2003, pp. 35–36)

These methodological problems of comparing the average age at death are not limited to the study of month-of-birth effects only. The same criticism is applicable to the studies of any other covariates, including educational differentials, as well as regional differentials, discussed by Mr. Kestenbaum. For example, ''life expectancy'' of individuals with basic education in the United States measured through mean age at death turned out to be higher than ''life expectancy'' of individuals with a high educational level (Doblhammer 2002). This result contradicts both the existing knowledge and common sense, which was acknowledged by the author (Doblhammer 2002). It is well known that older people tend to have less education compared to younger generations, which explains this paradox. So the average ages at death for persons born in the Midwest, Northeast, South and West, presented by Mr. Kestenbaum, may reflect compositional age differences in the sizes of particular region-of-birth cohorts rather than real differences in life expectancy. If birth rates in the Midwest and the Northeast were high in the past but then rapidly declined while the decline of birth rates in the West was slow, then the population of persons born in the West would be relatively younger, producing a low mean age at death. This suggestion is confirmed by the data on population aging in different regions of the United States. For example, in 1990 the percentage of persons aged 85 and over was 1.4 in the Midwest and the Northeast, 1.2 in the South, and only 1.0 in the West (Bean et al. 1994). Higher proportions of the oldest-old in the Midwest and the Northeast increase mean age at death in these regions. Assuming that many persons stay in the region of their birth, lower mean age at death for persons born in the West is most likely due to a relatively younger population in this region. It is most likely that the low age at death of Westerners is caused by compositional effects in the living population and is not related to longevity. According to the Bureau of Census data, the majority of Western states (North and South Dakota in particular) occupy top ranks according to their life expectancy, whereas many Southern states are among the worst according to this indicator.

Similar arguments can be applied to the month-of-birth findings by Doblhammer's method. A correct approach to the studies of month-of-birth effects on life expectancy should include either studies of cohort mortality by month of birth or accounting for exposure (living population with information on month of birth). We applied a cohort approach to the month-of-birth study using the method of extinct generations with data from the Social Security Administration Death Master File (DMF) (Gavrilova and Gavrilov 2005). Availability of a large set of individual death records in the DMF allowed us to arrange data by single-year birth cohorts and to calculate life expectancy for each month of birth within each year of birth. We found that life expectancy at age 80 indeed depends on the month of birth, and this is replicated seasonally in successive birth cohorts (Gavrilova and Gavrilov 2005).

Mr. Kestenbaum also notes that in the study by Hill (with 69% linkage success rate) only records for which the date of death information *did not* agree with date of death information in Social Security records were linked to early censuses, suggesting that Hill's data were of more dubious quality than our genealogical data (Hill, Preston, and Rosenwaike 2000a). In our study we initially had no official records, but only genealogical records. For this reason we had first to conduct a linkage of genealogies to the Social Security records in order to verify death dates in genealogies. This was an absolutely necessary step because genealogical records (even in genealogies of good quality) are prone to error. On the other hand, Hill and colleagues started their study with a set of official death certificates that obviously have more reliable information about death dates than computerized genealogies, even if these dates do not agree with SSA information. So we believe that the quality of records in Hill's and our studies is about the same (in both cases death dates relied on one official data source).

Finally, we agree with Mr. Kestenbaum that achievement of extreme old age is fascinating to both researchers and a general public. We believe that this attention to centenarians is probably not a matter of pure curiosity because studies of extreme cases (cases of extreme lifespan in our case) may be important for understanding mechanisms of aging and longevity.

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"A Risk Model with Multilayer Dividend Strategy," Hansjörg Albrecher and Jürgen Hartinger, April 2007

RAMIN OKHRATI*

In this paper Hansjörg Albrecher and Jürgen Hartinger consider a risk model with multilayer dividend strategy. In the first example of the paper in particular, they derive a closed form for the ruin probability in the multilayer case when the dynamic of the dividend payments at time t is given by $dD(t) = a_i\, dt$ whenever $U(t)$ is in layer i, that is, $b_{i-1} \le U(t) < b_i$. In this discussion we will derive the ruin probability of that example in the continuous dividend strategy, that is, $dD(t) = h(U(t))\, dt$, where h satisfies some appropriate conditions that will be stated later. As in Example 1 of the paper, we consider exponential claim sizes.

EXTENSION OF EXAMPLE 1

We change the dynamic of the surplus process to

$$dU(t) = cdt - h(U(t))\, dt - dS(t),$$

where h is a differentiable function (except for a point α), and for $u > \alpha$, h is constant; we suppose $h(0) = 0$ and $c - h(x)$ is bigger than $\lambda\mu$. The latter assumption is a reasonable assumption so that in the discrete case the net profit condition is fulfilled in each layer, that is, $(c - a_i) > \lambda\mu$. In fact, h may not be differentiable, and it even may have some possible jumps, but here, for simplicity, we

* Ramin Okhrati is a PhD student in the Department of Mathematics and Statistics at Concordia University, Montreal, QC, Canada, raminokhrati@yahoo.com.

consider this case to show the general idea. Also, we assume that the functions involved are so nice that we can interchange the order of operators.

By these new assumptions we want to obtain a closed formula for the ruin probability

$$\psi(u) = \mathbb{P}(\inf_{t>0}\{t : U(t) < 0 | U(0) = u\} < \infty).$$

Consider the partition $\{b_0^k, b_1^k, b_2^k, \ldots, b_{k-1}^k\}$ of $[0, \alpha]$, where $b_i^k = i\alpha/(k-1)$ and k refers to k layers. Let $b_k^k = \infty$ and define $a^k(u) = \Sigma_{i=1}^k a_i^k 1_{\{b_{i-1}^k \le u < b_i^k\}}$, where $a_i^k = h(b_{i-1}^k) = h((i-1)\alpha/(k-1))$; then it is easy to see that $a^k(u) \to h(u)$ uniformly.

We approximate the new dynamic of surplus by $dU(t) = (c - a_i)dt - dS(t)$ where $a_i = a_i^k$. For a fixed k, by Example 1 of the paper we have

$$\psi_k^{(i)}(u) = \frac{1}{1 - L_k^k}\left(L_i^k - L_k^k + \frac{\beta - R_{1,i}^k}{\beta}\frac{R_{1,1}^k}{R_{1,i}^k}\, e^{\Sigma_{j=1}^{i-1}(R_{1,j+1}^k - R_{1,j}^k)b_j^k - R_{1,i}^k u}\right), \tag{D.1}$$

where

$$R_{1,i}^k = \beta - \frac{\lambda}{c - a_i^k},$$

and

$$L_i^k = R_{1,1}^k e^{-R_{1,1}^k b_1^k}\sum_{j=1}^{i-1}\left(\frac{1}{R_{1,j}^k} - \frac{1}{R_{1,j+1}^k}\right) e^{\Sigma_{l=2}^{j}(b_{l-1}^k - b_l^k)R_{1,l}^k}, \qquad \text{for } i = 1, \ldots, k.$$

(Index k in the above variables refers to k layers.)

Let $u \in [0, \alpha]$ be fixed. If $(i-1)\alpha/(k-1) \le u < i\alpha/(k-1)$, then $\psi_k^{([(k-1)u/\alpha]+1)}(u) = \psi_k^{(i)}(u)$. We have the ruin probability (D.1), and by increasing the number of the layers ($k \to \infty$), we want to find the ruin probability for our new example with the new assumptions. So for a fixed point u we want to find

$$\psi(u) = \lim_{k\to\infty} \psi_k^{[(k-1)u/\alpha]+1}.$$

For doing this, we first obtain the limit of $L_{[(k-1)u/\alpha]+1}^k$ as $k \to \infty$. Let $f(x) = \beta - \lambda/(c - h(x))$ and $g(x) = 1/f(x)$; then

$$R_{1,j}^k = f\left(\frac{(j-1)\alpha}{k-1}\right) \text{ and } \frac{1}{R_{1,j}^k} = g\left(\frac{(j-1)\alpha}{k-1}\right),$$

$$L_{[(k-1)u/\alpha]+1}^k = f(0)e^{-f(0)b_1^k}\sum_{j=1}^{[(k-1)u/\alpha]}\left(\left(g\left(\frac{(j-1)\alpha}{k-1}\right) - g\left(\frac{j\alpha}{k-1}\right)\right) e^{\Sigma_{l=2}^{j}(b_{l-1}^k - b_l^k)f((l-1)\alpha/(k-1))}\right). \tag{D.2}$$

Since g is differentiable for every j, there exists a $C_{\alpha,k}^j \in [(j-1)\alpha/(k-1), j\alpha/(k-1)]$ such that $g(j\alpha/k-1) - g((j-1)\alpha/(k-1)) = g'(C_{\alpha,k}^j)(\alpha/k-1)$; so for large k, we can approximate the above sum by $-\int_0^u g'(x)e^{-\int_0^x f(y)dy}\,dx$. In the following we show that in the limit, this is really the case.

For a fixed $u \in [0, \alpha]$, let

$$\xi(u) = \lim_{k\to\infty}\sum_{j=1}^{[(k-1)u/\alpha]}\left(\left(g\,\frac{(j-1)\alpha}{k-1}\right) - g\left(\frac{j\alpha}{k-1}\right)\right) e^{\Sigma_{l=2}^{j}(b_{l-1}^k - b_l^k)f((l-1)\alpha/(k-1))}\right),$$

and $\varphi_j^k(\alpha) = \Sigma_{l=2}^j(b_{l-1}^k - b_l^k)f((l-1)\alpha/(k-1))$. Now consider the Taylor expansion of $e^{\varphi_j^k(\alpha)}$ in $\xi(u)$. So we will have

$$\xi(u) = \lim_{k\to\infty}\sum_{j=1}^{[(k-1)u/\alpha]}\left(\left(g\left(\frac{(j-1)\alpha}{k-1}\right) - g\left(\frac{j\alpha}{k-1}\right)\right)\sum_{n=0}^{\infty}\frac{1}{n!}[\varphi_j^k(\alpha)]^n\right),$$

and after interchanging the order of operations we will get

$$\xi(u) = \sum_{n=0}^{\infty} \left(\frac{1}{n!} \lim_{k\to\infty} \sum_{j=1}^{[(k-1)u/\alpha]} \left(\left(g\left(\frac{(j-1)\alpha}{k-1}\right) - g\left(\frac{j\alpha}{k-1}\right)\right) [\varphi_j^k(\alpha)]^n \right)\right).$$

Let

$$\zeta^n(u) = \lim_{k\to\infty} \sum_{j=1}^{[(k-1)u/\alpha]} \left(\left(g\left(\frac{(j-1)\alpha}{k-1}\right) - g\left(\frac{j\alpha}{k-1}\right)\right) [\varphi_j^k(\alpha)]^n \right).$$

Now

$$\begin{aligned}\zeta^n(u) &= -\lim_{k\to\infty} \sum_{j=1}^{[(k-1)u/\alpha]} \left(g'(C^j_{\alpha,k}) \left(\frac{\alpha}{k-1}\right) [\varphi_j^k(\alpha)] \cdots [\varphi_j^k(\alpha)] \right)\\ &= (-1)^{n+1} \int_0^u dx \int_0^x dx_1 \int_0^x dx_2 \cdots \int_0^x g'(x)\cdot f(x_1)f(x_2)\cdots f(x_n)\, dx_n\\ &= (-1)^{n+1} \int_0^u g'(x)(F(x))^n\, dx,\end{aligned}$$

where in the above, $C^j_{\alpha,k} \in [(j-1)\alpha/(k-1), j\alpha/(k-1)]$ (as already mentioned), and $F(x) = \int_0^x f(y)\, dy$.

Therefore,

$$\xi(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^n(u) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \left(\int_0^u g'(x)(F(x))^n\, dx \right).$$

By switching the order of summation and integration and applying the integration by part,

$$\xi(u) = -\int_0^u g'(x)e^{-F(x)}\, dx = -e^{-F(u)}g(u) - \int_0^u e^{-F(x)}\, dx + e^{-F(0)}g(0),$$

and so from (D.2) we get

$$\lim_{k\to\infty} L^k_{[(k-1)u/\alpha]+1} = f(0)[-e^{-F(u)}g(u) - \textstyle\int_0^u e^{-F(x)}\, dx + g(0)]. \tag{D.3}$$

Let

$$R(u) = f(0)\left[-\frac{e^{-F(u)}}{f(u)} - \textstyle\int_0^u e^{-F(x)}\, dx \right] + 1. \tag{D.4}$$

By taking $u = \alpha$, then

$$\lim_{k\to\infty} L^k_k = R(\alpha). \tag{D.5}$$

Also, it is easy to see that

$$\lim_{k\to\infty} \sum_{j=1}^{[(k-1)u/\alpha]} (R^k_{1,j+1} - R^k_{1,j})b^k_j = \int_0^u xf'(x)\, dx. \tag{D.6}$$

Finally by (D.3), (D.4), (D.5), and (D.6) we have

$$\lim_{k\to\infty} \psi_k^{[(k-1)u/\alpha]+1} = \frac{1}{1-R(\alpha)} \left(R(u) - R(\alpha) + \frac{\beta - f(u)}{\beta}\frac{f(0)}{f(u)}\, e^{\int_0^u xf'(x)dx} e^{-f(u)\cdot u} \right).$$

Therefore for a fixed $u \in [0, \alpha]$ we have a closed form of the ruin probability for the extended example:

$$\psi(u) = \frac{1}{1-R(\alpha)} \left(R(u) - R(\alpha) + \frac{\beta - f(u)}{\beta}\frac{f(0)}{f(u)}\, e^{-\int_0^u f(x)dx} \right),$$

where $R(u)$ is given by (D.4).

If $u \geq \alpha$, then we are in the kth layer, and the ruin probability is given by (D.1) for $i = k$. In this case by (D.5) we will have

$$\psi(u) = \frac{1}{1 - R(\alpha)} \left(\frac{\beta - f(\alpha)}{\beta} \frac{f(0)}{f(\alpha)} e^{\int_0^\alpha x f'(x) dx} \right) e^{-f(\alpha).u}.$$

Example

Let $\lambda = 1$, $\beta = 1$, $c = 1.4$, $\alpha = 15$, and $h(x) = 0.02x$ (for $x = 5, 10, 15$ we have the same dividend strategy in the four-layer case as in the numerical result of Example 1 of the paper). Then the ruin probabilities are the following:

If $u < 15$, then

$$\psi(u) = (-4.472396478 \times 10^6) \cdot \left(\frac{e^{-u}}{(1.4 - 0.02u)^{50}} + \int_0^u \frac{e^{-x}}{(1.4 - 0.02x)^{50}} dx \right) + 1.$$

If $u \geq 15$, then

$$\psi(u) = 0.4557309860 e^{-0.0909090909u}.$$

ACKNOWLEDGMENTS

I am very grateful to Professor Xiaowen Zhou for suggesting the topic of this discussion and helping me with the mathematics of it. Also special thanks to Professor Hansjörg Albrecher for many helpful comments and suggestions.

AUTHORS' REPLY

We are grateful for receiving two interesting discussions. In the first one, which has appeared in the previous issue of the *NAAJ,* Mr. Cheung shows how the differential approach for the multilayer model can be extended to derive higher moments of the discounted dividend payments recursively.

In the second discussion here, Mr. Okhrati illustrates how the formula for the ruin probability with exponential claim size distribution of Example 2.1 in the paper can be used to obtain an exact formula for a continuous surplus-dependent premium intensity for exponential claims, by letting the number of layers go to infinity.

We would like to point out that there is also an alternative direct way to obtain the exact formula for the ruin probability for continuous surplus-dependent premium intensity $c(u)$ in the exponential claim case, which can be found in Tichy (1984): The survival probability $\phi(u) = 1 - \psi(u)$ for arbitrary claim size distribution function $F(z)$ is the solution of the integro-differential equation

$$c(u)\phi'(u) - \lambda\phi(u) + \lambda \int_0^u \phi(u - z)\, dF(z) = 0.$$

For Exp(1)-claims, it follows that $\phi(u)$ is the solution of

$$c(u)\phi''(u) + (c'(u) - \lambda + c(u))\phi'(u) = 0.$$

Together with the initial condition $\lim_{u\to\infty} \phi(u) = 1$, this leads to the explicit formula

$$\phi(u) = \frac{1 + \lambda \int_0^u \frac{1}{c(v)} e^{\lambda m(v) - v} dv}{1 + \lambda \int_0^\infty \frac{1}{c(v)} e^{\lambda m(v) - v} dv}, \tag{D.1}$$

where $m(v) = \int_0^v dt/c(t)$.

The function $c(u)$ corresponds to Okhrati's expression $c - h(u)$, and hence the numerical example in the discussion above is retrieved by choosing

$$c(u) = \begin{cases} 1.4 - u/50, & u < 15 \\ 1.1, & u \geq 15. \end{cases}$$

With $\lambda = 1$ and some algebra, one indeed obtains from (D.1)

$$\psi(u) = \frac{\int_u^\infty \frac{1}{c(v)} e^{\lambda m(v) - v}\, dv}{1 + \int_0^\infty \frac{1}{c(v)} e^{\lambda m(v) - v}\, dv} = \frac{1}{4.53} \cdot \begin{cases} \frac{10e^{-15}}{(1 - 0.3/1.4)^{50}} + \frac{1}{1.4} \int_u^{15} \frac{e^{-v}\, dv}{(1 - 0.02v/1.4)^{51}}, & u < 15 \\ \frac{10e^{-15/1.1}}{(1 - 0.3/1.4)^{50}} e^{-u/11}, & u \geq 15 \end{cases},$$

which is another way of writing the last two formulas in Okhrati's example.

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"On the Class of Erlang Mixtures with Risk Theoretic Applications," Gordon E. Willmot and Jae-Kyung Woo, April 2007

SARALEES NADARAJAH*

Professor Willmot and Ms. Woo should be congratulated for the excellent treatment in their paper of the class of Erlang mixtures and its application to risk theory. My comment here relates to Section 2 of the paper. In this section Willmot and Woo show that many well-known distributions can be expressed as Erlang mixtures. The first distribution considered is that of a sum of exponentials. If X_m, m = 1, 2, . . . , M, are independent exponential random variables with expected values $1/\lambda_m$, $m = 1, 2, \ldots, M$, it is pointed out that the probability density function (pdf) of $Z = X_1 + X_2 + \cdots + X_M$ is an Erlang mixture. However, in this case, Willmot and Woo assume that λ_m's are distinct. The result that the pdf of Z is an Erlang mixture also holds if not all of the λ_m's are distinct. Suppose K of the X_m's have the same expected value, say, $1/\lambda_e$, and the remaining $N = M - K$ have distinct expected values. Then the pdf of Z is given by

$$f_Z(z) = \left(\sum_{n=1}^{N} E_n \lambda_n \exp(-\lambda_n z) + \sum_{k=1}^{K} A_k \frac{z^{k-1} \lambda_e^k \exp(-\lambda_e z)}{\Gamma(k)} \right), \tag{D.1}$$

where

$$E_n = \left(\frac{1}{1 - \lambda_n/\lambda_e} \right)^K \prod_{u=1, n \neq n}^{N} \frac{1}{1 - \lambda_n/\lambda_u},$$

$$C_{uv} = \frac{1}{(1 - B_u/\lambda_e)^v},$$

$$D_u = \left(\frac{1}{1 - B_u/\lambda_e} \right)^K \prod_{n=1}^{N} \frac{1}{1 - B_u/\lambda_n} - \sum_{n=1}^{N} \frac{E_n}{1 - B_u/\lambda_n},$$

* Saralees Nadarajah is Senior Lecturer in Statistics in the School of Mathematics, University of Manchester, Manchester M60 1QD, United Kingdom, saralees.nadarajah@manchester.ac.uk.

C is a $K \times K$ matrix with the elements $\{C_{uv}\}$, D is a $K \times 1$ vector with the elements $\{D_u\}$, $A = C^{-1}D$, and $\{B_p\}$ satisfy the equations

$$\left(1 - \frac{B_p}{\lambda_e}\right)^{-K} \prod_{n=1}^{N} \left(1 - \frac{B_p}{\lambda_n}\right)^{-1} = \sum_{n=1}^{N} E_n \left(1 - \frac{B_p}{\lambda_n}\right)^{-1} + \sum_{k=1}^{K} A_k \left(1 - \frac{B_p}{\lambda_e}\right)^{-k}$$

for $p = 1, 2, \ldots, K$. The result in (D.1) is due to Khuong and Kong (2006).

Another distribution considered in Section 2 is that of a sum of gammas. Let Y_i, $i = 1, 2, \ldots, N$, be independent gamma random variables with parameters $(\alpha_i, 1/\beta_i)$, $i = 1, 2, \ldots, N$, and let $Z = Y_1 + Y_2 + \cdots + Y_N$. Willmot and Woo argue that the pdf of Z is an Erlang mixture without actually giving an explicit expression for it. Various representations are available in the literature for the pdf of Z. For example, Moschopoulos (1985) showed that

$$f_Z(z) = \prod_{n=1}^{N} \left(\frac{\beta_1}{\beta_n}\right)^{\alpha_n} \sum_{k=0}^{\infty} \frac{\delta_k z^{\sum_{n=1}^{N} \alpha_n + k - 1} \exp(-z/\beta_1)}{\beta_1^{\sum_{n=1}^{N} \alpha_n + k} \Gamma\left(\sum_{n=1}^{N} \alpha_n + k\right)} \tag{D.2}$$

for $z > 0$, where $\beta_1 = \min \beta_n$ and the coefficients δ_k satisfy the recurrence relations

$$\delta_0 = 1$$

and

$$\delta_{k+1} = \frac{1}{k+1} \sum_{i=1}^{k+1} \left[\sum_{j=1}^{N} \alpha_j \left(1 - \frac{\beta_1}{\beta_j}\right)^i\right] \delta_{k+1-i}.$$

It follows from (D.2) that the pdf of Z is an Erlang mixture if $\sum_{n=1}^{N} \alpha_n$ is an integer. For other equivalent representations for the pdf of Z, see Mathai and Saxena (1978), Springer (1979), Provost (1988, 1989), Mathai (1993), Efthymoglou and Aalo (1995), and Aalo, Piboongungon, and Efthymoglou (2005).

A distribution not considered in Section 2 is that of a sum of correlated gammas. Suppose Y_i, $i = 1, 2, \ldots, N$, are gamma random variables with parameters $(\alpha, 1/\beta_i)$, $i = 1, 2, \ldots, N$, and $\operatorname{corr}(Y_i, Y_j) = \rho_{ij}$. In this case Alouini, Abdi, and Kaveh (2001) showed that the pdf of $Z = Y_1 + Y_2 + \cdots + Y_N$ can be expressed as

$$f_Z(z) = \prod_{n=1}^{N} \left(\frac{\lambda_1}{\lambda_n}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{\delta_k z^{N\alpha + k - 1} \exp(-z/\lambda_1)}{\lambda_1^{N\alpha + k} \Gamma(N\alpha + k)} \tag{D.3}$$

for $z > 0$, where $\lambda_1 = \min \lambda_n$, $\{\lambda_n\}$, are the eigenvalues of the matrix $A = DC$, where D is the $N \times N$ diagonal matrix with the entries $\{\beta_n\}$, and C is the $N \times N$ positive definite matrix defined by

$$C = \begin{bmatrix} 1 & \sqrt{\rho_{12}} & \cdots & \sqrt{\rho_{1N}} \\ \sqrt{\rho_{21}} & 1 & \cdots & \sqrt{\rho_{2N}} \\ \vdots & \vdots & \cdots & \vdots \\ \sqrt{\rho_{N_1}} & \cdots & \cdots & 1 \end{bmatrix}$$

and the coefficients δ_k satisfy the recurrence relations

$$\delta_0 = 1$$

and

$$\delta_{k+1} = \frac{\alpha}{k+1} \sum_{i=1}^{k+1} \left[\sum_{j=1}^{N} \left(1 - \frac{\lambda_1}{\lambda_j}\right)^i\right] \delta_{k+1-i}.$$

It follows from (D.3) that the pdf of Z is an Erlang mixture if $N\alpha$ is an integer.

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AUTHORS' REPLY

We thank the discussant for her comments, which undoubtedly add to the paper.

We remark that the sum of independent exponential random variables with some of the parameters possibly being equal is actually a special case of the model considered in Subsection 2.3 (''Erlangian Sums of Gammas''). In particular, all of the α_i's being nonnegative integers in equation (2.22) corresponds to the Laplace transform of this special case (in fact, countable mixtures of pdf's of these distributions are still within the same mixed Erlang class, as may easily be seen by examination of the associated Laplace transform). It is interesting to have an explicit formula for the mixing weights, and we thank the discussant for this contribution. For computational purposes, however, the recursive scheme for these weights given by equation (2.25) is straightforward to use and is numerically stable because it is in fact a compound Poisson recursion.

Similarly, the discussant gives an explicit formula for the pdf with Laplace transform (2.22) when the sum of the α's is a positive integer. This is the same assumption we used in the paper, and so this explicit formula is equivalent to obtaining the coefficient q_j of z^j in $Q(z)$ as given in equation (2.23). That is, an explicit formula for the probabilities of the convolution of Erlang (negative binomial with integer shape parameters) distributions is given.

We thank the discussant for the example involving the sum of dependent gamma random variables as well.

It is worth mentioning that the approach that we employed in the paper is essentially to express the distribution of interest in all these cases as a compound distribution with exponential secondary or ''claim size'' distribution. This may be viewed as ''actuarially equivalent'' to the ''method of stages'' approach originally used in Markovian queuing theory. Following this compound viewpoint, the primary distribution is normally characterized via its pgf. One could argue that this is in fact equivalent to an explicit expression for the distribution by the uniqueness of the pgf. Thus, it is not really the case that we do not give ''an explicit expression for it,'' as is stated in the paragraph before equation (D.2) of the discussion. In fact, the reason that we do not attempt to explicitly obtain the coefficient of z^j in this pgf is because this is normally not the best way to numerically evaluate these probabilities.

Recursive or Fast Fourier transform (FFT) methods are often well suited to these situations. More complicated Erlang mixtures may actually be such that it is extremely difficult to explicitly identify the mixing weights, but numerical evaluation via the pgf is not difficult.

"Moments of the Dividend Payments and Related Problems in a Markov-Modulated Risk Model," Shaunming Li and Yi Lu, April 2007

ERIC C. K. CHEUNG*

Professors Li and Lu have written a stimulating paper that provides many insights for future research. Many techniques used in the paper can be applied to a barrier strategy in an ordinary Sparre Andersen model (or ordinary renewal risk model) with a phase-type interclaim time distribution. This discussion aims at showing how Professors Li and Lu's results can be easily modified to find the nth moment of the discounted total dividends in the above model.

THE MODEL

The surplus process without dividend payments is given by (1.1) of the paper, but now $\{N(t); t \geq 0\}$ is a counting process with $N(t) = \sup\{k \in \mathbb{N}; \sum_{i=1}^{k} W_i \leq t\}$, where $\{W_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) positive continuous random variables representing the interclaim times (a claim is assumed to occur at time 0), with common probability density function (p.d.f.)

$$f_W(y) = \mathbf{a}e^{\mathbf{Q}y}\mathbf{q}, \qquad y > 0. \tag{D.1}$$

Here $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ is the initial probability row vector, $\mathbf{Q} = (\lambda_{i,j})_{i,j=1}^{m}$ is the generator matrix, and $\mathbf{q} = -\mathbf{Q}\mathbf{1} = (\lambda_{1,m+1}, \lambda_{2,m+1}, \ldots, \lambda_{m,m+1})^T$ ($\mathbf{1}$ is an m-dimensional column vector of ones as in the paper). For convenience, we define $\lambda_i = -\lambda_{i,i}$ for $i \in E = \{1, 2, \ldots, m\}$. Furthermore, $\{X_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables, independent of $\{W_i\}_{i=1}^{\infty}$, with common p.d.f. f_X, tail probability $\bar{F}_X$, Laplace transform (of f_X) $\tilde{f}_X$, and finite mean μ_X.

At time $t \geq 0$ before ruin occurs, the surplus process $\{U(t); t \geq 0\}$ is at one of the states $i \in E$, where

1. With rate $\lambda_{i,j}$ the process goes to state j for $j \in E/\{i\}$ without a claim occurring, and
2. With rate $\lambda_{i,m+1}$ a claim governed by f_X occurs, and the process immediately enters state j for $j \in E$ with probability a_j if the claim does not cause ruin.

With the above interpretation, it will be natural to define $\{J(t); t \geq 0\}$ to be the underlying state process (instead of the external environment process in the paper).

Under identical dividend modifications as described in the paper, when the surplus reaches a constant barrier b, the entire premium income is paid out as dividends until the next claim occurs.

MOMENT-GENERATING FUNCTION AND HIGHER MOMENTS OF DIVIDENDS

If $D_{u,b}$ represents the present value of all dividends paid until ruin, the moment-generating function given the initial state is denoted by $M_i(u, y; b) = \mathbb{E}[e^{yD_{u,b}} | J(0) = i]$ for $0 \leq u \leq b$, $i \in E$, as defined in the paper. Then, using similar arguments as in Section 3, we have that for sufficiently small $h > 0$,

$$\begin{aligned} M_i(u, y; b) &= (1 - \lambda_i h)M_i(u + ch, e^{-\delta h}y; b) + \sum_{k=1,k\neq i}^{m} \lambda_{i,k} h M_k(u + ch, e^{-\delta h}y; b) \\ &\quad + \lambda_{i,m+1} h \left\{ \sum_{k=1}^{m} a_k \int_0^{u+ch} M_k(u + ch - x, e^{-\delta h}y; b) f_X(x)\, dx + \bar{F}_X(u + ch) \right\} \\ &\quad + o(h), \qquad 0 \leq u < b, i \in E, \end{aligned}$$

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* Eric C. K. Cheung is a PhD student in the Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1, ericbergkamp@gmail.com.

which gives

$$c\,\frac{\partial M_i(u, y; b)}{\partial u} - \delta y\,\frac{\partial M_i(u, y; b)}{\partial y} + \sum_{k=1}^{m} \lambda_{i,k} M_k(u, y; b)$$
$$+ \lambda_{i,m+1} \left\{ \sum_{k=1}^{m} a_k \int_0^u M_k(u - x, y; b) f_X(x)\, dx + \bar{F}_X(u) \right\} = 0, \qquad 0 \le u < b,\ i \in E. \quad \text{(D.2)}$$

For $u = b$, we have

$$M_i(b, y; b) = (1 - \lambda_i h)e^{ych} M_i(b, e^{-\delta h} y; b) + \sum_{k=1,k\neq i}^{m} \lambda_{i,k} h e^{ych} M_k(b, e^{-\delta h} y; b)$$
$$+ \lambda_{i,m+1} h e^{ych} \left\{ \sum_{k=1}^{m} a_k \int_0^b M_k(b - x, e^{-\delta h} y; b) f_X(x)\, dx + \bar{F}_X(b) \right\} + o(h), \qquad i \in E,$$

which yields

$$cyM_i(b, y; b) - \delta y\,\frac{\partial M_i(b, y; b)}{\partial y} + \sum_{k=1}^{m} \lambda_{i,k} M_k(b, y; b)$$
$$+ \lambda_{i,m+1} \left\{ \sum_{k=1}^{m} a_k \int_0^b M_k(b - x, y; b) f_X(x)\, dx + \bar{F}_X(b) \right\} = 0, \qquad i \in E. \quad \text{(D.3)}$$

Comparing (D.2) and (D.3), we arrive at the same boundary conditions as (3.3).

Using the representation $M_i(u, y; b) = 1 + \sum_{n=1}^{\infty} (y^n/n!) V_{i,n}(u; b)$ given by the paper (with the same definition for $V_{i,n}(u; b)$ as stated in the equation following (3.3)), (D.2) and the boundary conditions (3.3) become

$$cV'_{i,n}(u; b) - n\delta V_{i,n}(u; b) + \sum_{k=1}^{m} \lambda_{i,k} V_{k,n}(u; b)$$
$$+ \lambda_{i,m+1} \sum_{k=1}^{m} a_k \int_0^u V_{k,n}(u - x; b) f_X(x)\, dx = 0, \qquad 0 \le u < b, \qquad i \in E. \quad \text{(D.4)}$$

and (3.5), respectively.

Analogous to (2.6), for each fixed $n \in \mathbb{N}$, let $v_i(u; n)$, $0 \le u < \infty$, $i \in E$, be the solutions to (D.4). Therefore, we have

$$cv'_i(u; n) - n\delta v_i(u; n) + \sum_{k=1}^{m} \lambda_{i,k} v_k(u; n) + \lambda_{i,m+1} \sum_{k=1}^{m} a_k \int_0^u v_k(u - x; n) f_X(x)\, dx = 0, \qquad i \in E. \quad \text{(D.5)}$$

The solution techniques given by Professors Li and Lu are also applicable here (see, e.g., Lakshmikantham and Rao 1995, p. 50), that is, for $j \in E$, we define $v_{1,j}(u; n), v_{2,j}(u; n), \ldots, -v_{m,j}(u; n)$, with initial conditions $v_{i,j}(0; n) = I(i = j)$, to be the m particular solutions of the integro-differential equations (D.5). Then the results (together with the definitions) in the last paragraph of Section 3 all hold true. Therefore, to evaluate $V_{i,n}(u; b)$ for $0 \le u \le b$, $i \in E$, we only have to find the particular solutions $v_{i,j}(u; n)$ of the system (D.5). After all, the unconditional nth moment of $D_{u,b}$ is given by

$$\mathbb{E}[D_{u,b}^n] = \mathbf{a}\mathbf{V}_n(u; b), \qquad 0 \le u \le b.$$

LAPLACE TRANSFORMS

To find the particular solutions $v_{i,j}(u; n)$ of the system (D.5), we take Laplace transforms on both sides of (D.5) (with $v_{i,j}(u; n)$ in place of $v_i(u; n)$) and arrive at

$$\left(s - \frac{n\delta}{c}\right)\tilde{v}_{i,j}(s;\, n) + \sum_{k=1}^{m}\frac{\lambda_{i,k}}{c}\tilde{v}_{k,j}(s;\, n) + \frac{\lambda_{i,m+1}}{c}\sum_{k=1}^{m} a_k\tilde{v}_{k,j}(s;\, n)\tilde{f}_X(s) = v_{i,j}(0;\, n), \qquad i, j \in E,$$

where $\tilde{v}_{i,j}(s;\, n) = \int_0^\infty e^{-su}v_{i,j}(u;\, n)\, du$ for i, j $\in E$. In matrix form, the above system can be written as

$$\mathbf{A}_n(s)\hat{\mathbf{v}}_n(s) = \mathbf{I}, \tag{D.6}$$

where $\hat{\mathrm{v}}_n(\mathrm{s}) = (\tilde{v}_{i,j}(s;\, n))_{i,j=1}^m$, and

$$\begin{aligned}\mathbf{A}_n(s) &= \left(s - \frac{n\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{Q} + \frac{1}{c}\mathbf{qa}\tilde{f}_X(s)\\ &= \frac{1}{c}\{(cs - n\delta)\mathbf{I} + \mathbf{Q}\}\{\mathbf{I} + [(cs - n\delta)\mathbf{I} + \mathbf{Q}]^{-1}\mathbf{qa}\tilde{f}_X(s)\}.\end{aligned}$$

This implies

$$\begin{aligned}\det \mathbf{A}_n(s) &= \frac{1}{c^n}\det\{(cs - n\delta)\mathbf{I} + \mathbf{Q}\}\det\{\mathbf{I} + [(cs - n\delta)\mathbf{I} + \mathbf{Q}]^{-1}\mathbf{qa}\tilde{f}_X(s)\}\\ &= \frac{1}{c^n}\det\{(cs - n\delta)\mathbf{I} + \mathbf{Q}\}\{1 - \mathbf{a}[(n\delta - cs)\mathbf{I} - \mathbf{Q}]^{-1}\mathbf{q}\tilde{f}_X(s)\}\\ &= \frac{1}{c^n}\det\{(cs - n\delta)\mathbf{I} + \mathbf{Q}\}\{1 - \tilde{f}_W(n\delta - cs)\tilde{f}_X(s)\},\end{aligned}$$

where $\tilde{f}_W$ is the Laplace transform of f_W.

Therefore, solving det $\mathbf{A}_n(\xi) = 0$ for ξ is equivalent to solving $\tilde{f}_W\,(n\delta - c\xi)\tilde{f}_X(\xi) = 1$, which is exactly the generalized Lundberg's fundamental equation with force of interest $n\delta$. See, for example, Gerber and Shiu (2005, eq. (1.8)).

Turning back to (D.6), it is apparent that

$$\hat{\mathbf{v}}_n(s) = [\mathbf{A}_n(s)]^{-1}.$$

If $\tilde{f}_X(s)$ is a rational function, each element of $\hat{\mathbf{v}}_n(s)$ is also rational and can be resolved into partial fractions using the roots of the generalized Lundberg's fundamental equation. In such cases (each element of) $\hat{\mathbf{v}}_n(s)$ can be analytically inverted to give (the corresponding element of) $\mathbf{v}_n(u) = (v_{i,j}(u;\, n))_{i,j=1}^m$.

EXTENSIONS TO OTHER MODELS

If ${}_eD_{u,b}$ denotes the present value of dividends in a stationary Sparre Andersen model where the interclaim times are governed by (D.1), it is apparent that

$$\mathbb{E}[{}_eD_{u,b}^n] = \mathbf{a}_e\mathbf{V}_n(u;\, b), \qquad 0 \le u \le b,$$

and $\mathbf{a}_e = \mathbf{aQ}^{-1}/(\mathbf{aQ}^{-1}\mathbf{1})$ is the initial probability vector of the equilibrium distribution of f_W.

Another extension is the case where a claim does not occur at time 0, but the time elapsed from the last claim is known to be x. Since the residual lifetime p.d.f. of f_W (with definition $g_x(y) = f_W\,(x + y)/\bar{F}_W(x)$ and $\bar{F}_W$ is the tail of f_W) is given by

$$g_x(y) = \mathbf{a}_x e^{\mathbf{Q}y}\mathbf{q}, \qquad y > 0$$

where $\mathbf{a}_x = \mathbf{a}e^{\mathbf{Q}x}/(\mathbf{a}e^{\mathbf{Q}x}\mathbf{1})$, it is clear that

$$\mathbb{E}[{}_xD_{u,b}^n] = \mathbf{a}_x\mathbf{V}_n(u;\, b), \qquad 0 \le u \le b,$$

where ${}_xD_{u,b}$ denotes the present value of dividends.

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''Pension Plan Valuation and Mortality Projection: A Case Study with Mortality Data,'' Hélène Cossette, Antoine Delwarde, Michel Denuit, Frédérick Guillot, and Étienne Marceau, April 2007

STEVEN HABERMAN* AND ARTHUR RENSHAW†

The authors are to be commended on an extremely interesting paper. We would like to comment on two particular aspects of the paper: the presentation and use of the Cox-type proportional hazards Poisson regression model, and aspects of the binomial bilinear methodology.

In relation to the Poisson regression model (Section 4.4), by denoting the three explanatory variables, for each sex, as follows:

Maximal pension amounts:	classes ($i = 1, 2, \ldots, 6$, or 7)
Retirement status:	age < 65 ($j = 1$)
	age ≥ 65 and not disabled ($j = 2$)
	age ≥ 65 and disabled ($j = 3$)
Method of payment	RRQ ($k = 1$)
	RRQ + CPP ($k = 2$)

an alternative parameterization of the Poisson regression model is possible by writing

$$\beta_0 + \sum_{j=1}^{s} \beta_j z_{xtj} = \alpha + \alpha_i + \beta_j + \gamma_k, \text{ subject to } \alpha_1 = \beta_1 = \gamma_1 = 0.$$

By this means the parameters are readily identified with the respective rows of Tables 2 and 3 and give a clearer indication of how the multiplicative factors in the final columns of Tables A.1 and A.2 are compiled. (We are unsure of the authors' motives for including suffices in age x and year t when defining the binary z variables and the parametric structure; we note that the explanatory variables are independent of both x and t.)

We note that the authors stop short of investigating the significance and likely effect of any interactions between the three explanatory variables. This is possible by additionally introducing one or more of the second-order interaction terms $(\alpha\beta)_{ij}$, $(\alpha\gamma)_{ik}$, $(\beta\gamma)_{jk}$, culminating with the fully interactive model by further introducing the third-order interaction term $(\alpha\beta\gamma)_{ijk}$ (subject to the convention that all indexed parameters are preset to zero when one or more of the suffices $i, j, k = 1$). If statistically significant, the inclusion of such terms can have a material effect on the values of the multiplicative factors reported in the final columns of Tables A.1 and A.2. See, for example, Renshaw (1988) for a retrospective study of this type based on the impaired lives experience from a U.K. life insurer.

The choice between binomial and Poisson modeling in this context would appear to be something of a gray area and is discussed by the Continuous Mortality Investigation (CMI) Bureau in relation to the construction of static life tables for the U.K. life insurance industry; see Forfar, McCutcheon, and

* Steven Haberman, ASA, FIA, PhD, is a Professor of Actuarial Science and Deputy Dean, Cass Business School, City University, London, UK, s.haberman@city.ac.uk.

† Arthur Renshaw, PhD, is Senior Research fellow, Cass Business School, City University, London, UK.

Wilkie (1988). The conversion of central exposed to risk to initial exposed to risk by adding half the number of relevant deaths (Forfar et al. 1988, Section 2) provides a possible alternative source of exposure to L_{xt} under binomial modeling.

At the risk of stating the obvious, $q_x(t)$ is targeted directly under binomial modeling, while typically the values of $\mu_x(t)$ $(= -\log p_x(t))$ are approximated (or more accurately computed using numerical methods), while the situation is reversed under Poisson modeling. The two approaches lead to marginally different values for the statistics of interest, including the expected residual lifetimes (Section 5.1) and life annuity prices (Section 5.2). Given the much greater uncertainty associated with both the extreme age ''closing-out'' and period extrapolation procedures, the choice between binomial and Poisson modeling would appear to be of rather minor significance, provided that there is no advantage to be gained in risk measurement procedures (Section 6). We further note that the joint modeling and residual bootstrapping approaches to risk measurement (Renshaw and Haberman 2007) readily extend to binomial modeling.

Under the GLM approach to modeling, the structure of the binomial-Gumbel model (Section 3.4) is identified with the complementary log-log link function, with inverse

$$q_{xt} = 1 - \exp(-\exp \eta_{xt}),$$

in combination with the characteristic LC nonlinear parametric predictor

$$\eta_{xt} = \alpha_x + \beta_x \kappa_t.$$

Also, under the GLM binomial approach, the application of the canonical log-odds link, with inverse

$$q_{xt} = \frac{\exp \eta_{xt}}{1 + \exp \eta_{xt}}$$

is possible, where the iterative fitting formulas read as follows:

$$\text{updated}(\hat{\alpha}_x) = \hat{\alpha}_x + \frac{\sum_t \omega_{xt}(y_{xt} - \hat{q}_{xt})}{\sum_t \omega_{xt}\hat{p}_{xt}\hat{q}_{xt}},$$

$$\text{updated}(\hat{\beta}_x) = \hat{\beta}_x + \frac{\sum_t \omega_{xt}(y_{xt} - \hat{q}_{xt})\hat{\kappa}_t}{\sum_t \omega_{xt}\hat{p}_{xt}\hat{q}_{xt}\hat{\kappa}_t^2},$$

$$\text{updated}(\hat{\kappa}_t) = \hat{\kappa}_t + \frac{\sum_x \omega_{xt}(y_{xt} - \hat{q}_{xt})\hat{\beta}_x}{\sum_x \omega_{xt}\hat{p}_{xt}\hat{q}_{xt}\hat{\beta}_x^2},$$

subject to binomial responses $y_{xt} = D_{xt}/L_{xt} \sim \text{binomial}(1, q_{xt})$ and weights $\omega_{xt} = L_{xt}$.

With reference to the authors' application of the Binomial-Gumbel model, we note that the term in braces in the denominator of each updating expression (page 9), subject to the omission of the iterative index, should read

$$\left(D_{xt} \frac{q_{xt} - \mu_{xt} p_{xt}}{(q_{xt})^2} - L_{xt} \right)$$

and that the correct expression for binomial deviance residuals (page 11) is

$$\sqrt{2} \times \text{sign}(D_{xt} - \hat{D}_{xt}) \sqrt{(L_{xt} - D_{xt}) \ln \frac{1 - \dfrac{D_{xt}}{L_{xt}}}{1 - \hat{q}_{xt}} + D_{xt} \ln \frac{D_{xt}}{L_{xt}\hat{q}_{xt}}};$$

see, for example, McCullagh and Nelder (1989).

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AUTHOR REPLY TO DISCUSSION BY EDWARD FURMAN AND RICARDAS ZITIKIS, JULY 2007

"An Actuarial Premium Pricing Model for Nonnormal Insurance and Financial Risks in Incomplete Markets," Zinoviy Landsman and Michael Sherris, January 2007

We thank Dr. Edward Furman and Professor Ricardas Zitikis for pointing out the important relation between the proposed premium pricing measures in our paper and the weighted premium concept. We note that in our paper we gave the economic interpretation of our pricing model using the stochastic discount factor concept (see Section 3 of the paper), which can be considered as a generalization of weighted premium principles. This concept is found in economic asset pricing theory and is well documented in the economic literature (see, e.g., Ross 1976; Cochrane 2001, chs. 1, 4). In accordance with this theory the basic pricing equation is given by the formula

$$P = \mathrm{E}[ZX], \tag{D.1}$$

where X is the payoff and $Z > 0$ is a random variable with finite expectation such that

$$\mathrm{E}[Z] = \frac{1}{r_f},$$

with r_f the risk-free rate. The variable Z is called the stochastic discount factor.

The pricing equation (D.1) not only provides for the price to be a linear functional with respect to the space of payoffs, but also implies no arbitrage when $Z > 0$, which is a fundamental result in asset pricing theory. When the risk-free rate $r_f = 1$, which is the implied assumption in our paper, we have

$$\mathrm{E}[Z] = 1,$$

and the discount factor principle coincides with the weighted premium principle.

We wish to also make clearer a definition in our paper. In formulas (2.3) and (4.3) on pages 121 and 124, respectively, P_M is the market price of the market portfolio and $\mu_M = \mathrm{E}[M]$ is the expected value of the market portfolio M excluding any risk premium. In the case of returns, the return excluding any risk premium would be the risk-free rate. Our focus is on prices and risk premiums in the case $r_f = 1$, and so our results should be considered for prices or values rather than returns. We are grateful to David T. Kausch, who kindly pointed out this inaccuracy in the paper.

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