"The Discounted Joint Distribution of the Surplus Prior to Ruin and the Deficit at Ruin in a Sparre Andersen Model," Jiandong Ren, July 2007

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In this paper Professor Ren has obtained some very elegant matrix expressions for the Gerber-Shiu function and in particular for the discounted joint distribution of the surplus prior to ruin and the deficit at ruin in a Sparre Andersen model with phase-type interclaim times. I would like to comment that some techniques used in the paper can be used to analyze the first time when the surplus attains a certain level $b \geq u$, from the initial surplus u in the Sparre Andersen model with phase-type interclaim times.

For $b \geq u$, define

$$T_b = \min\{t \geq 0 : U(t) = b\} \tag{D.1}$$

to be the first time when the surplus reaches level b, and define for $\delta > 0$

$$R(u; b) = \mathbb{E}[e^{-\delta T_b} | U(0) = u]$$

to be the Laplace transform of T_b. Further define

$$R_{i,j}(u; b) = \mathbb{E}[e^{-\delta T_b} I(J_{T_b} = j) | U(0) = u, J_0 = i], \qquad i, j = 1, 2, \ldots, n,$$

to be the Laplace transform of T_b given that the initial state is i, the state when the process hits b is j, and the initial surplus is u, where $\{J_t\}_{t\geq 0}$ is the state process as defined in Cheung (2007). Then $R(u; b)$ can be computed by

$$R(u; b) = \boldsymbol{\alpha}\mathbf{R}(u; b)\mathbf{e}^{\top}, \tag{D.2}$$

where $\mathbf{R}(u; b) = (R_{i,j}(u; b))_{i,j=1}^{n}$. Note that the initial surplus u in $R(u; b)$ and $R_{i,j}(u; b)$ could be negative.

Using the same arguments as in Ko (2007), we can show that matrix $\mathbf{R}(u; b)$ satisfies the following equation for $u \leq b$:

$$c\mathbf{R}'(u; b) = (\delta \mathbf{I} - \mathbf{B})\mathbf{R}(u; b) - \mathbf{b}^{\top}\boldsymbol{\alpha} \int_0^{\infty} \mathbf{R}(u - x; b) p(x)\, dx. \tag{D.3}$$

Clearly, when $u = b$, we have

$$\mathbf{R}(b; b) = \mathbf{I}. \tag{D.4}$$

Since the solution to (D.3) with boundary condition (D.4) is unique, we assume that $\mathbf{R}(u; b)$is of the form

$$\mathbf{R}(u; b) = \mathbf{C}(b)e^{\mathbf{K}u},$$

where $\mathbf{C}(b)$ and $\mathbf{K}$ are two $n \times n$ matrices. The boundary condition $\mathbf{R}(b; b) = \mathbf{I}$ gives $\mathbf{C}(b) = e^{-\mathbf{K}b}$, so that

$$\mathbf{R}(u; b) = e^{-\mathbf{K}(b-u)}, \qquad u \leq b. \tag{D.5}$$

Note that all eigenvalues of $\mathbf{K}$ have positive real parts, because otherwise it would be a contradiction to the fact that $\lim_{b\to\infty} \mathbf{R}(u; b) = \mathbf{0}$.

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To determine matrix **K,** we substitute (D.5) into (D.3), yielding

$$c\mathbf{K}e^{-\mathbf{K}(b-u)} = (\delta\mathbf{I} - \mathbf{B})e^{-\mathbf{K}(b-u)} - \mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}x}p(x)\,dx\,e^{-\mathbf{K}(b-u)}.$$

Because $e^{-\mathbf{K}(b-u)}$ is nonsingular, **K** must satisfy the equation

$$c\mathbf{K} = (\delta\mathbf{I} - \mathbf{B}) - \mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} e^{-\mathbf{K}x}p(x)\,dx. \tag{D.6}$$

To solve the matrix equation above, let

$$\mathbf{L}_{\delta}(s) = \left(s - \frac{\delta}{c}\right)\mathbf{I} + \frac{1}{c}\mathbf{B} + \frac{1}{c}\mathbf{b}^{\top}\boldsymbol{\alpha}\hat{p}(s).$$

It follows from the paper under discussion that the solutions to the equation

$$\det[\mathbf{L}_{\delta}(s)] = 0 \tag{D.7}$$

and the solutions to Lundberg's fundamental equation

$$\hat{a}(\delta - cs)\hat{p}(s) = 1 \tag{D.8}$$

are identical. It can be proved by Rouché's theorem that equation (D.8) has exactly n solutions in the right half of the complex plane, denoted as $\rho_1, \rho_2, \ldots, \rho_n$. In the following we assume that these n solutions are distinct.

Let $\mathbf{h}_i$ be an eigenvector of $\mathbf{L}_{\delta}(\rho_i)$ corresponding to the eigenvalue 0, for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} 0 &= \mathbf{L}_{\delta}(\rho_i)\mathbf{h}_i \\ &= \rho_i\mathbf{h}_i + \frac{1}{c}(\mathbf{B} - \delta\mathbf{I})\mathbf{h}_i + \frac{1}{c}\mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} e^{-\rho_i x}\mathbf{h}_i p(x)\,dx, \qquad i = 1, 2, \ldots, n. \end{aligned}$$

In matrix notation, we have

$$\mathbf{H}\boldsymbol{\Lambda}_{\rho} = \frac{1}{c}(\delta\mathbf{I} - \mathbf{B})\mathbf{H} - \frac{1}{c}\mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} p(x)\mathbf{H}e^{-\boldsymbol{\Lambda}_{\rho}x}\,dx, \tag{D.9}$$

where $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$ and $\boldsymbol{\Lambda}_{\rho} = \mathrm{diag}(\rho_1, \rho_2, \ldots, \rho_n)$. We assume that $\rho_1, \rho_2, \ldots, \rho_n$ are distinct, so $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n$ are linearly independent and **H** is invertible. Right-multiplying both sides of (D.9) by $\mathbf{H}^{-1}$, we have

$$\mathbf{H}\boldsymbol{\Lambda}_{\rho}\mathbf{H}^{-1} = \frac{1}{c}(\delta\mathbf{I} - \mathbf{B}) - \frac{1}{c}\mathbf{b}^{\top}\boldsymbol{\alpha}\int_0^{\infty} p(x)\mathbf{H}e^{-\boldsymbol{\Lambda}_{\rho}x}\mathbf{H}^{-1}\,dx. \tag{D.10}$$

Comparing (D.10) with (D.6) we obtain

$$\mathbf{K} = \mathbf{H}\boldsymbol{\Lambda}_{\rho}\mathbf{H}^{-1},$$

and, thus, we have

$$\mathbf{R}(u; b) = e^{-\mathbf{K}(b-u)} = \mathbf{H}e^{-\boldsymbol{\Lambda}_{\rho}(b-u)}\mathbf{H}^{-1}, \qquad u \le b, \tag{D.11}$$

and

$$R(u; b) = \boldsymbol{\alpha}\mathbf{H}e^{-\boldsymbol{\Lambda}_{\rho}(b-u)}\mathbf{H}^{-1}\mathbf{e}^{\top}, \qquad u \le b, \tag{D.12}$$

where $e^{-\boldsymbol{\Lambda}_{\rho}(b-u)} = \mathrm{diag}(e^{-\rho_1(b-u)}, e^{-\rho_2(b-u)}, \ldots, e^{-\rho_n(b-u)})$.

REMARKS

1. The matrix $\mathbf{K} = \mathbf{H}\boldsymbol{\Lambda}_{\rho}\mathbf{H}^{-1}$ is different from $-\mathbf{Q} = \mathbf{V}^{-1}\boldsymbol{\Lambda}_{\rho}\mathbf{V}$, because in $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$, the column vector $\mathbf{h}_i$ is a right eigenvector of $\mathbf{L}_{\delta}(\rho_i)$ corresponding to the eigenvalue of 0, whereas in

$\mathbf{V} = (\mathbf{v}_1^\top, \mathbf{v}_2^\top, \ldots, \mathbf{v}_n^\top)^\top$, the row vector $\mathbf{v}_i$ is a left eigenvector of $\mathbf{L}_\delta(\rho_i)$ corresponding to the eigenvalue of 0.

2. When $n = 1$, we have $\boldsymbol{\alpha} = 1$, matrix $\mathbf{B}$ simplifies to a negative number, denoted as $-\lambda$, $\mathbf{b}^\top = -\mathbf{B}\mathbf{e}^\top = \lambda$, and $\mathbf{K}$ simplifies to a number, denoted by ρ as in Gerber and Shiu (1998). Here ρ satisfies the equation

$$c\rho - (\lambda + \delta) + \lambda \int_0^\infty p(x)e^{-\rho x}\, dx = 0.$$

Equation (D.12) simplifies to

$$R(u; b) = \mathbb{E}[e^{-\delta T_b} | U(0) = u] = e^{-\rho(b-u)}. \tag{D.13}$$

Formula (D.13) was given in Gerber and Shiu (1998) by applying the optional sampling theorem to the martingale $\{e^{-\delta t + \rho U(t)}\}_{t\geq 0}$.

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Professor Ren has obtained closed-form solutions for the discounted joint distribution of the surplus prior to ruin and the deficit at ruin under the phase-type renewal risk model. The present discussion shows that the work of Professor Ren can be further extended to a more general framework assuming a Markovian arrival process (MAP; see Neuts 1979). Several point processes such as phase-type renewal processes, Markov-modulated Poisson processes, as well as certain semi-Markov point processes are contained under the MAP umbrella. Moreover, under these premises, one can further relax the independence assumption between the interclaim times and the claim sizes assuming a more general dependent structure.

A MAP with representation $MAP\ (\boldsymbol{\alpha}, \boldsymbol{D}_0, \boldsymbol{D}_1)$ of order m is a two-dimensional Markov process on the state space $\mathbb{N}_0 \times \{1, \ldots, m\}$. For this process, an underlying continuous-time Markov chain (CTMC) on the state space $E = \{1, \ldots, m\}$ evolves such that the instantaneous rate of transition from state i to state $j \neq i$ in E without an accompanying claim is given by the (i, j)-th element of $\boldsymbol{D}_0$, namely, $D_0(i, j) \geq 0$. Similarly the instantaneous rate of transition from state i to state j (possibly $j = i$) in E with an accompanying claim is given by the quantity $D_1(i, j) \geq 0$. The diagonal elements of $\boldsymbol{D}_0$ are assumed to be negative such that the sum of the elements on each row of the matrix $\boldsymbol{D}_0 + \boldsymbol{D}_1$ are all zero. We denote by $\boldsymbol{\alpha}$ the probability of starting the CTMC in one of its underlying phases. We further denote by $B_{i,j}$ $(\tilde{b}_{i,j}(s))$ the cumulative distribution function (Laplace transform) of the claim amounts that occur at a transition of the underlying Markovian process from a state i to a state j. Under these assumptions the claim amounts are neither independent nor identically distributed random variables, the class of risk processes under consideration containing the phase-type renewal risk model studied by Professor Ren, the semi-Markovian risk model studied by Albrecher and Boxma (2005), as well as the Markov-modulated risk model studied by Ng and Yang (2006).

In this discussion, using steps similar to those in Ng and Yang (2006), we derive a system of integro-differential equations for the Gerber-Shiu discounted penalty function showing how the analysis can

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be pursued further. Denoting by $\phi_i(u)$, the conditional Gerber-Shiu discounted penalty function given the initial MAP phase i, three possible scenarios can happen: no change in the MAP phase, a change in the MAP phase accompanied by no claim arrival, and a change in the MAP phase at a claim instant. Mathematically, this can be written as

$$\phi_i(u) = [1 + D_0(i, i)\,dt]\,e^{-\delta\,dt}\,\phi_i(u + c\,dt) + \sum_{j=1, j\neq i}^{m} D_0(i, j)\,dt\,e^{-\delta\,dt}\,\phi_j(u + c\,dt)$$

$$+ \sum_{j=1}^{m} D_1(i, j)\,dt\,e^{-\delta\,dt}\left(\int_0^{u+c\,dt} \phi_j(u + c\,dt - x)dB_{i,j}(x) + \omega_{i,j}(u + c\,dt)\right) + o(dt), \quad \text{(D.1)}$$

where

$$\omega_{i,j}(u) = \int_u^{\infty} w(u, x - u)\,dB_{i,j}(x). \quad \text{(D.2)}$$

Further manipulations of (D.1) lead to

$$c\phi_i'(u) = \delta\phi_i(u) - \sum_{j=1}^{m} D_0(i, j)\phi_j(u) - \sum_{j=1}^{m} D_1(i, j)\left(\int_0^{u} \phi_j(u - x)\,dB_{i,j}(x) + \omega_{i,j}(u)\right). \quad \text{(D.3)}$$

Taking the Laplace transform on both sides of (D.3) we further obtain

$$cs\tilde{\phi}_i(s) = c\phi_i(0) + \delta\tilde{\phi}_i(s) - \sum_{j=1}^{m} D_0(i, j)\tilde{\phi}_j(s) - \sum_{j=1}^{m} D_1(i, j)\tilde{\phi}_j(s)\tilde{b}_{i,j}(s) - \sum_{j=1}^{m} D_1(i, j)\tilde{\omega}_{i,j}(s), \quad \text{(D.4)}$$

where $\tilde{\phi}_i(s) = \int_0^\infty e^{-su}\phi_i(u)\,du$ and $\tilde{\omega}_{i,j}(s) = \int_0^\infty e^{-su}\omega_{i,j}(u)\,du$, for $i = 1, \ldots, m$. Under matrix notation equation (D.4) can be written as

$$\left[\left(s - \frac{\delta}{c}\right)\boldsymbol{I} + \frac{1}{c}\,(\boldsymbol{D_0} + \boldsymbol{\Lambda}(s))\right]\tilde{\phi}(s) = \phi(0) - \frac{1}{c}\,\tilde{\omega}(s), \quad \text{(D.5)}$$

where $\tilde{\phi}(s) = (\tilde{\phi}_1(s), \ldots, \tilde{\phi}_m(s))^T$, $\boldsymbol{\Lambda}(s)$ is a matrix with the (i, j)-th element given by $D_1(i, j)\tilde{b}_{i,j}(s)$ and $\tilde{\omega}(s)$ is a column vector with the ith element given by $\tilde{\omega}_i(s) = \sum_{j=1}^m D_1(i, j)\tilde{\omega}_{i,j}(s)$, $i = 1, \ldots, m$. Equation (D.5) generalizes equation (2.5) obtained by Professor Ren under the phase-type Sparre Andersen insurance risk model. If one keeps the author's notation and assumes that the interclaim times are independent and identically distributed $PH(\boldsymbol{\alpha}, \boldsymbol{B})$ random variables, independent with respect to the claim sizes, then $\boldsymbol{D_0} = \boldsymbol{B}$, $\boldsymbol{D_1} = \boldsymbol{b^T\alpha}$, and $\tilde{b}(s) = \tilde{p}(s)$. The formula given in equation (D.5) also generalizes the results obtained in Theorem 3 in Ng and Yang (2006) for the Markov-modulated risk model, as well as equation (7) in Albrecher and Boxma (2005) for their semi-Markov dependent risk model. Furthermore, using the same line of logic as in the present paper we can prove that the roots of

$$\text{Det}\left[\left(s - \frac{\delta}{c}\right)\boldsymbol{I} + \frac{1}{c}\,(\boldsymbol{D_0} + \Lambda(s))\right] = 0$$

coincide with the roots of the generalized Lundberg equation, m of those being located in the right half complex plane (using a matrix generalization of Rouché's theorem; see, e.g., De Smit 1983). To find the solution for $\tilde{\phi}(0)$, for each root, say, ρ_i, with $\text{Re}(\rho_i) > 0$, $i = 1, \ldots, m$, one has to find the left linearly independent row vectors, say, $\boldsymbol{v}_i$, such that

$$\boldsymbol{V}\left[\left(s - \frac{\delta}{c}\right)\boldsymbol{I} + \frac{1}{c}\,(\boldsymbol{D_0} + \boldsymbol{\Lambda}(s))\right] = \boldsymbol{0}, \quad \text{(D.6)}$$

where $\boldsymbol{V} = (\boldsymbol{v}_1^T, \boldsymbol{v}_2^T, \ldots, \boldsymbol{v}_m^T)^T$. Using (D.5) and (D.6) the Laplace transform of the Gerber-Shiu discounted penalty function at 0 can be further written as

$$\phi(0) = \frac{1}{c}\sum_{i=1}^{m} V^{-1}\boldsymbol{R}_i\boldsymbol{V}\boldsymbol{e}_i, \tag{D.7}$$

where

$$\boldsymbol{R}_i = \text{Diag}[\tilde{\omega}_i(\rho_1), \tilde{\omega}_i(\rho_2), \ldots, \tilde{\omega}_i(\rho_m)],$$

and e_i corresponds to the ith column vector of the identity matrix. Denoting

$$\boldsymbol{Q} = V^{-1}\text{Diag}[-\rho_1, -\rho_2, \ldots, -\rho_m]\boldsymbol{V},$$

one can finally obtain

$$\phi(0) = \frac{1}{c}\sum_{i=1}^{m} \tilde{\omega}_i(-\boldsymbol{Q})\boldsymbol{e}_i, \tag{D.8}$$

which generalizes equation (3.6) of Professor Ren's paper. Moreover, because of the more general model assumptions, equation (D.8) represents an elegant generalization of formula (2.7) in Li and Lu (2008) for the Markov-modulated risk process, as well as Proposition (3.1) in Albrecher and Boxma (2005). The diagonalization form involved in equation (D.8) will provide us with a probabilistic interpretation for the roots of the generalized Lundberg equation. This together with further developments for the more general Gerber-Shiu discounted penalty function represent the subject of a forthcoming paper.

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"Asset Allocation with Hedge Funds on the Menu," Phelim Boyle and Sun Siang Liew, October 2007

TAK KUEN SIU*

It is a great pleasure to congratulate Professor Phelim Boyle and Sun Siang Liew on this stimulating and timely paper, which provides a valuable account of the asset allocation problem of an investor when hedge funds are included as investment vehicles. The results presented in this paper are of much interest to both the actuarial and the finance profession. Recently hedge funds have come under the spotlight in global financial markets with rapid growth in the hedge fund industry over the past decade in terms of both number and assets under management. This phenomenal growth has also stimulated the development of the industry of funds of hedge funds, also called funds of funds. According to Tremont Capital Management (2006), assets under management increased from around \$100 billion in 1995 to close to \$1 trillion by mid-2006. Hedge funds are also becoming an important investment vehicle for pension funds and insurance companies. Many pension funds and insurance companies have increased their investments in hedge funds, which offer returns not easily generated from traditional or conventional assets and provide diversification into a portfolio of conventional assets. So it is important for actuarial practitioners to gain a better and deeper understanding of the risk and return characteristics of hedge funds, and to know how to make better or rational investment decisions on hedge funds so as to enhance their risk and return profiles.

It is commonly known that returns from hedge funds exhibit nonlinear behavior, which may be attributed to highly leveraged and unconventional trading strategies adopted by hedge fund managers. It has been documented empirically that hedge fund returns are not normally distributed, and they are skewed and heavy-tailed with time-varying mean and variance. Professors Boyle and Liew introduce the use of regime-switching models to capture these important empirical features and to model the joint returns of the hedge fund and equity markets. Under the regime-switching models, those important empirical features of hedge fund returns can be generated by the presence of a hidden Markov chain. The regime-switching models are nonlinear and nonstationary. They provide important insights into understanding the empirical behavior of hedge fund returns and the impact of the nonlinear risk and return characteristics of hedge fund returns on the asset allocation decision. Here I would like to highlight the potential use of two alternative models to describe and explain the nonlinear behavior of hedge fund returns in studying the asset allocation problem when hedge funds are included.

The first model is the self-exciting threshold autoregressive (SETAR) model first proposed by Howell Tong in the late 1970s (see Tong 1977, 1978, 1983, 1990). For illustration, consider a special case of the first-order, two-regime SETAR model. Let $\{R_t\}$ denote the return process of a hedge fund. Then $\{R_t\}$ can be assumed to be governed by the following special case of the first-order, two-regime SETAR model:

$$R_t = \begin{cases} \mu_1 + \sigma_1\xi_t & \text{if } R_{t-1} \geq r; \\ \mu_2 + \sigma_2\xi_t & \text{if } R_{t-1} < r. \end{cases}$$

Here $\{\xi_t\}$ is a sequence of independent and identically distributed standard normal random variables. One may consider other distributions for $\{\xi_t\}$ in general; μ_i and σ_i represent the expected return and the volatility of the hedge fund in the ith regime, respectively, where $i = 1, 2$; r is called the threshold parameter, which is the key parameter dividing the state space of $\{R_t\}$ into two regimes; the time delay parameter is set to 1; and the autoregressive parameters in both regimes are assumed to be 0.

The key idea of the SETAR model is to provide a piecewise linear approximation to an nonlinear autoregressive model by introducing regimes via the threshold principle (see Tong 1990, Chap. 3). It is one of the oldest nonlinear time series models in the literature. Instead of governing the switching regimes by a hidden Markov chain, the switching regimes in the SETAR model are governed by the past values of the process itself. In particular, it depends on whether the past return

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R_{t-1} is above or below the threshold parameter r; this is called self-exciting. The SETAR model can generate some empirical features of hedge fund returns, such as nonnormality, skewed, heavy-tailed, and time-varying mean and variance. So it presents some potential applications to modeling hedge fund returns and studying the asset allocation problem when hedge funds are included. The optimization procedure for the maximization of expected utility under the SETAR model may not be very different from that under the regime-switching model. It is perhaps worth mentioning that the SETAR model can also describe some cyclical phenomena, such as the limit cycle, and can be related to some more general nonlinear dynamical systems, such as stochastic chaos. Tong (1990) and Chan and Tong (2001) provide an excellent account of these topics.

Both the regime-switching and SETAR models focus on describing the statistical or empirical behavior of hedge fund returns. They may be considered a reduced-form-style model in a certain sense. It may be beneficial to consider a more structural approach to model the behavior of hedge fund returns and investigate the corresponding asset allocation problem. Fung and Hsieh (2002) pioneered the asset-based style model for hedge fund returns. The asset-based style model is described as

$$R_t = \alpha + \sum_{k=1}^{n} \beta_k SF_{k,t} + \xi_t,$$

where $SF_{k,t}$ is the value of the kth style factor at time t; β_k is the factor loading of the kth factor; and ξ_t is the residual at time t.

The asset-based style model seems to be in the spirit of the style model for mutual funds introduced in Sharpe (1992). However, the Sharpe style model is a linear factor model and cannot capture the nonlinear behavior of hedge fund returns (see Fung and Hsieh 2002). The rationale of the asset-based style model is to provide a simple model, which can capture the nonlinear behavior of hedge fund returns, and the model's key idea is to explain hedge fund returns by asset-based style factors. More specifically, hedge fund returns are expressed as a linear regression on some style factors, which are themselves nonlinear in nature. The nonlinear feature of hedge fund returns can be captured by these nonlinear style factors. These factors represent the returns of trading strategies on traditional asset classes that can explain the returns of a group of hedge funds. They link hedge fund returns to observed market prices.

One might think of the possibility of modeling hedge fund returns with an nonlinear regression and/or a nonparametric regression on some linear factors, such as the market prices or returns of some linear or traditional asset classes. However, it may not be easy to get some important economic insights and interpretations if such a regressogram-based approach is used. The asset-based style model provides an explicit, transparent, and intuitive link between hedge fund returns and hedge fund trading strategies via the linear relationship between hedge fund returns and the style factors. Important economic insights can be gained from this linear relationship. For example, under the asset-based style model, one can express the manager's α relative to a set of style factors. The β_k of the style factors provide information about the manager's capital allocation to the style factors and measure the degree of leverage the hedge fund manager adopts. The asset-based style model also provides a flexible way to describe the nature and quantity of risk. It allows the use of scenario-based analysis to provide a qualitative assessment of a hedge fund's risk exposure. This cannot be achieved using statistical models only.

The asset-based style model can provide a convenient way to deal with the asset allocation problem with hedge funds, especially when one wishes to consider the asset allocation problem of a large group of hedge funds, which can be explained by the same set of style factors. Suppose one considers the use of the mean-variance approach to make asset allocation decisions with hedge funds. This may be controversial because higher moments need to be taken into account due to the nonnormality of hedge fund returns. However, it may work well in an approximate sense. In fact, Fung and Hsieh (1999) show that the mean-variance criterion to rank hedge funds approximately preserves the ranking of preferences in some standard utility functions. Under the key assumption that the idiosyncratic risks described by the residuals of returns are independent over different hedge funds, the asset-based factor model provides a parsimonious and practical way for the mean-variance analysis of the asset allocation problem with hedge funds. The asset-based style model establishes the link between returns of different hedge funds through their link to the style factors. The idea is in the spirit of the original idea of introducing Sharpe's Capital Asset Pricing Model to reduce the number of parameter estimates required for Markowitz's mean-variance portfolio selection model.

Fung and Hsieh (2001) employ traded options to explicitly describe the nonlinear return characteristics of trend-following hedge funds. In particular, they show how to use a dynamically managed option-based strategy, namely, a ''lookback option,'' to replicate the returns from trend-following hedge fund strategies. By considering the lookback option as an asset-based style factor for trend-following hedge funds, one can get some insights into understanding the asset allocation problem with complex options, say, lookback options, through studying the asset allocation problem with hedge funds using the asset-based style model.

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AUTHORS' REPLY TO DISCUSSION BY BANGWON KO, JULY 2006

"On Optimal Dividend Strategies in the Compound Poisson Model," Hans U. Gerber and Elias S. W. Shiu, April 2006

This reply is to address an observation made by Ko (2006, p. 71). In his elegant discussion, Ko pointed out a set of interesting identities that arises from *Lundberg's fundamental equation* when the individual claim distribution is a mixture of exponential distributions. We (Gerber and Shiu 1998, p. 51) coined the term "Lundberg's fundamental equation" for the equation

$$1 + \frac{\delta}{\lambda} - \frac{c}{\lambda}\xi = \hat{p}(\xi) \tag{R.1}$$

because Lundberg (1932, p. 144) wrote that this "relation is fundamental to the whole collective risk theory."

Let

$$p(x) = \sum_{j=1}^{m} A_j \beta_j e^{-\beta_j x}, \quad x > 0, \tag{R.2}$$

where $0 < \beta_1 < \beta_2 < \cdots < \beta_m$ and A_j's are m positive numbers that add up to 1. Its Laplace transform is

$$\hat{p}(\xi) = \sum_{j=1}^{m} \frac{A_j \beta_j}{\beta_j + \xi}, \quad \text{Re } \xi > -\beta_1. \tag{R.3}$$

Although $\hat{p}(\xi)$ does not exist for Re $\xi \leq -\beta_1$, Lundberg's fundamental equation (R.1) can be analytically extended as

$$1 + \frac{\delta}{\lambda} - \frac{c}{\lambda}\xi = \sum_{j=1}^{m} \frac{A_j \beta_j}{\beta_j + \xi} \tag{R.4}$$

for ξ in the complex plane with the points $-\beta_1, -\beta_2, \ldots, -\beta_m$ removed. Because equation (R.4) is equivalent to a polynomial equation of degree $m + 1$, it has $m + 1$ roots. These roots depend on δ, and we denote them as $\{\rho_i(\delta); i = 1, \ldots, m + 1\}$. If $\delta > 0$, then the roots are real and satisfy the interweaving condition:

$$-\beta_m < \rho_{m+1}(\delta) < -\beta_{m-1} < \rho_m(\delta) < \cdots < -\beta_2 < \rho_3(\delta) < -\beta_1 < \rho_2(\delta) < 0 < \rho_1(\delta). \tag{R.5}$$

Figure R.1 illustrates the case for $m = 2$.

Now, the interesting identities pointed out in Ko's discussion (2006) are

$$\prod_{j=1}^{m+1} [\beta_i + \rho_j(\delta)] = \frac{\lambda}{c} A_i \beta_i \prod_{k=1, k\neq i}^{m} (\beta_i - \beta_k), \quad i = 1, 2, \ldots m, \tag{R.6}$$

where the right-hand sides are independent of δ. As we can see from Figure R.1, changing δ means shifting the slanted straight line in a parallel fashion, resulting in moving the roots $\{\rho_i(\delta)\}$. Surprisingly, for $i = 1, 2, \ldots, m$, the product

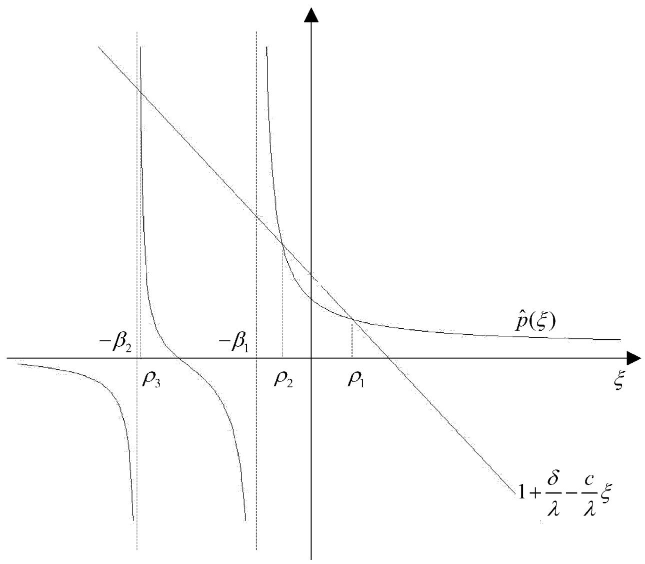
$$\prod_{j=1}^{m+1} [\beta_i + \rho_j(\delta)]$$

is *invariant* with respect to changes in δ, in accordance to (R.6).

Having been told of these identities, we examined our paper Gerber and Shiu (2005), which is a generalization of Gerber and Shiu (1998). The time between successive claims is generalized from an

Figure R.1

Roots of Lundberg's Fundamental Equation for $m = 2$



exponential random variable with mean $1/\lambda$ to a sum of n independent exponential random variables with respective means $1/\lambda_1, \ldots, 1/\lambda_n$. It turned out that (R.6) can be generalized, as (9.13) in Gerber and Shiu (2005) can be rewritten as

$$\prod_{j=1}^{m+n} [\beta_i + \rho_j(\delta)] = \frac{\lambda_1 \cdots \lambda_n}{c^n} A_i\beta_i \prod_{k=1,k\neq i}^{m} (\beta_i - \beta_k), \quad i = 1, 2, \ldots m. \tag{R.7}$$

Again, the right-hand sides of these m identities do not depend on δ, that is, each of the m products on the left-hand side of (R.6) is invariant with respect to changes in δ. In this more general case, Lundberg's equation (R.4) is generalized as

$$\prod_{i=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_i}\right) - \frac{c}{\lambda_i}\xi\right] = \sum_{j=1}^{m} \frac{A_j\beta_j}{\beta_j + \xi}. \tag{R.8}$$

As (R.8) is equivalent to a polynomial equation of degree $n + m$, it has $n + m$ roots $\{\rho_i(\delta)\}$; n of these roots, $\{\rho_i(\delta); i = 1, \ldots, n\}$, are in the right half of the complex plane, and m roots, $\{\rho_i(\delta); i = n+ 1, \ldots, n + m\}$, are negative and interwoven with the exponential parameters, generalizing (R.5):

$$-\beta_m < \rho_{n+m}(\delta) < -\beta_{m-1} < \rho_{n+m-1}(\delta) < \cdots < -\beta_2 < \rho_{n+2}(\delta) < -\beta_1 < \rho_{n+1}(\delta) < 0. \tag{R.9}$$

In Gerber and Shiu (2005), equation (9.13) is a remark. Because the expectation

$$\mathrm{E}[e^{-\delta T}w(-U(T))I(T < \infty)|U(0) = 0] \tag{R.10}$$

can be expressed as

$$\sum_{i=1}^{m} \hat{w}(\beta_i)[\beta_i + \rho_{n+i}(\delta)] \prod_{j=1,j\neq i}^{m} \frac{\beta_i + \rho_{n+j}(\delta)}{\beta_i - \beta_j} \tag{R.11}$$

and as

$$\frac{\lambda_1 \cdots \lambda_n}{c^n} \sum_{i=1}^{m} \frac{A_i\beta_i\hat{w}(\beta_i)}{\prod_{j=1}^{n} [\beta_i + \rho_j(\delta)]}, \tag{R.12}$$

equating the coefficient of $\hat{w}(\beta_i)$ in (R.11) with that in (R.12) yields

$$[\beta_i + \rho_{n+i}(\delta)] \prod_{j=1, j\neq i}^{m} \frac{\beta_i + \rho_{n+j}(\delta)}{\beta_i - \beta_j} = \frac{\lambda_1 \cdots \lambda_n}{c^n} \frac{A_i\beta_i}{\Pi_{j=1}^{n} [\beta_i + \rho_j(\delta)]},$$

from which (R.7) follows.

The derivations of (R.11) and (R.12) are rather intricate. Thus, an independent proof of (R.7) is of interest. To this end, let

$$\gamma(\xi) = \prod_{i=1}^{n} \left[\left(1 + \frac{\delta}{\lambda_i}\right) - \frac{c}{\lambda_i} \xi\right], \tag{R.13}$$

and, with a slight abuse of notation, let $\hat{p}(\xi)$ denote the rational function on the right-hand side of (R.3). Then $\rho_1(\delta), \rho_2(\delta), \ldots, \rho_{n+m}(\delta)$ are the $n + m$ roots of the equation

$$\gamma(\xi) = \hat{p}(\xi).$$

Consequently, for each constant β, the $n + m$ numbers, $[\beta + \rho_1(\delta)], [\beta + \rho_2(\delta)], \ldots, [\beta + \rho_{n+m}(\delta)]$, are the roots of

$$\gamma(\xi - \beta) = \hat{p}(\xi - \beta).$$

Thus, the left-hand side of (R.7) is the product of the $n + m$ roots of

$$\gamma(\xi - \beta_i) = \hat{p}(\xi - \beta_i). \tag{R.14}$$

The theorem of François Viète (1540–1603) shows how certain symmetric expressions of the zeros of a polynomial are related to the coefficients of the polynomial. In particular, the product of the negative of the zeros of a polynomial is its constant term divided by its leading coefficient. (Viète, the greatest French mathematician of the sixteenth century, wrote under the Latinized name Franciscus Vieta.) We now rewrite (R.14) as a polynomial equation (of degree $n + m$) by multiplying both sides of (R.14) with the product

$$\xi \prod_{k=1, k\neq i}^{m} (\beta_k + \xi - \beta_i).$$

Then the ratio of the constant term to the leading coefficient of the resulting polynomial is

$$-\frac{A_i\beta_i \prod_{k=1, k\neq i}^{m} (\beta_k + 0 - \beta_i)}{\prod_{j=1}^{n} \left(-\frac{c}{\lambda_j}\right)}, \tag{R.15}$$

which does not involve δ. Rewriting expression (R.15) as

$$(-1)^{n+m} \frac{\lambda_1 \cdots \lambda_n}{c^n} A_i\beta_i \prod_{k=1, k\neq i}^{m} (\beta_i - \beta_k)$$

and comparing it with the right-hand side of (R.7), we now have an independent proof of (R.7).

The key properties needed for the proof above are that $\gamma(\xi)$ is a polynomial with a leading coefficient independent of δ and that $\hat{p}(\xi)$ is a proper rational function with simple poles and they are at $-\beta_1, -\beta_2, \ldots, -\beta_m$. We have assumed that $p(x)$ is a mixture of distinct exponential probability density functions; some authors call it a *hyperexponential* probability density function. Let us now assume that $p(x)$ is a *hypoexponential* probability density function, that is, $p(x)$ is the probability density function for a sum of m independent exponential random variables with means $1/\beta_1, \ldots, 1/\beta_m$. Then

$$\hat{p}(\xi) = \prod_{k=1}^{m} \frac{\beta_k}{\beta_k + \xi},$$

and (R.7) becomes

$$\prod_{j=1}^{m+n} [\beta_i + \rho_j(\delta)] = (-1)^{m+1} \frac{\lambda_1 \cdots \lambda_n}{c^n} \prod_{k=1}^{m} \beta_k, \quad i = 1, 2, \ldots m. \tag{R.16}$$

In this case the value of the product $\Pi_{j=1}^{m+n} [\beta_i + \rho_j(\delta)]$ is independent of both δ and i. We note that (R.16) remains valid if some of the β's are the same, for example, if $p(x)$ is an Erlang density.

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