Ruin Minimization for Insurers with Borrowing Constraints

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Abstract

We consider an optimal dynamic control problem for an insurance company with opportunities of proportional reinsurance and investment. The company can purchase proportional reinsurance to reduce its risk level and invest its surplus in a financial market that has a Black-Scholes risky asset and a risk-free asset. When investing in the risk-free asset, three practical borrowing constraints are studied individually: (B1) the borrowing rate is higher than lending (saving) rate, (B2) the dollar amount borrowed is no more than $K > 0$, and (B3) the proportion of the borrowed amount to the surplus level is no more than $k > 0$. Under each of the constraints, the objective is to minimize the probability of ruin. Classical stochastic control theory is applied to solve the problem. Specifically, the minimal ruin probability functions are obtained in closed form by solving Hamilton-Jacobi-Bellman (HJB) equations, and their associated optimal reinsurance-investment policies are found by verification techniques.

1. Introduction

In recent years there has been an upsurge in applications of stochastic control theory to optimization problems of insurance companies (see, e.g., Browne 1997; Emanuel, Harrison, and Taylor 1975; Hipp and Plum 2000; Höjgaard and Taksar 1998a, b; Taksar and Markussen 2003). Among the applications allowing proportional reinsurance and investment to minimize the ruin probability, Schmidli (2001, 2002) considers a model based on the Cramer-Lundberg risk process. Taksar and Markussen (2003) study a diffusion approximation model with all the surplus invested in a single Black-Scholes risky asset. Promislow and Young (2005) and Luo, Taksar, and Tsoi (2008) deal with the case when the insurance company can distribute its surplus into two assets subject to certain constraints on investment such as no borrowing and/or no shortselling. In most of the models involving investment, it is assumed that the interest rate on the risk-free asset is the same for borrowing (shortselling) and lending (buying). As the first of three scenarios of interest in this paper, we consider the following more realistic restriction: (B1) the interest rate of borrowing is higher than that of lending. This constraint (B1) has also been imposed earlier in the literature, for example, in Promislow and Young (2005) under a diffusion model and in Gerber and Yang (2007) under a jump-diffusion model. Promislow and Young (2005), when considering the constraint (B1), allow control policies to involve investment but not a reinsurance purchase. In the present paper the constraint (B1) is considered for the first time within a framework allowing both investment and proportional reinsurance.

It has been found (see, e.g., Browne 1997; Luo, Taksar, and Tsoi 2008; Promislow and Young 2005; Yang and Zhang 2005; Young 2004) that the policy minimizing ruin probability for an entity with a low wealth level is highly leveraged; that is, the entity tends to borrow a fixed amount of money and invest it in the risky asset together with all its wealth. Such a leveraging strategy creates nontrivial risks; thus it may not be allowed by insurance regulations or other practical restrictions. To accommodate certain insurance regulations, or to lower leveraging levels for practical purposes, we introduce

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and study two other borrowing constraints: (B2) The amount borrowed by the insurance company should be no more than a fixed limit $K > 0$, and (B3) the amount borrowed should be no more than a fixed proportion of surplus $k > 0$. If $K = 0$ or $k = 0$, borrowing is not allowed. This extreme case has been solved in Luo, Taksar, and Tsoi (2008).

The constraints (B1), (B2), and (B3) are financially more realistic than the assumptions often made in the literature (see, e.g., Gerber and Yang 2007; Promislow and Young 2005; Schmidli 2001, 2002; Taksar and Markussen 2003). The optimal solutions under these constraints exhibit rather significant changes from the earlier results. Some special classifications of optimal solutions and more demanding treatments on exogenous parameters are now needed to solve the HJB equations. With minimization of the ruin probability as the objective, we obtain solutions of the HJB equations in closed form and generate explicit optimal control policies under each of the three borrowing constraints. The results in this paper complement or generalize the earlier work in Browne (1995), Luo, Taksar, and Tsoi (2008), Promislow and Young (2005), Schmidli (2001, 2002), Taksar and Markussen (2003), and Young (2004).

We start with the *diffusion approximation* model (see, e.g., Emanuel, Harrison, and Taylor 1975; Garrido 1989; Hipp and Plum 2000; Höjgaard and Taksar 1998a, b; Promislow and Young 2005; Taksar and Markussen 2003) for the surplus process without taking reinsurance or investment into account, which is assumed to follow a drifted (arithmetic) Brownian motion:

$$dR_t = \mu dt + \sigma dw_t^{(0)},$$
$$R_0 = x,$$

where $w^{(0)} = \{w_t^{(0)}\}_{t\geq 0}$ is a standard Brownian motion, μ is the return (premium) rate of the insurance company, and σ in the diffusion term measures the risk level. We now introduce *proportional reinsurance* to the model. The proportional reinsurance level is associated with a value $(1 - a)$, where $0 \leq a \leq 1$ is called *risk exposure* of the insurance company. If the risk exposure a is fixed, then the insurance company pays $100a\%$ of each claim while the rest $100(1 - a)\%$ is paid by the reinsurer. In return, the insurance company diverts part of the premiums to the reinsurer at the rate of $\lambda(1 - a)$, where λ is called a *reinsurance rate*. If $\lambda = \mu$, the proportional reinsurance is called *cheap,* and *expensive* if $\lambda > \mu$. With proportional reinsurance, the consequent diffusion approximation dynamics are given by

$$dR_t = [\mu - (1 - a)\lambda]\, dt + a\sigma\, dw_t^{(0)},$$
$$R_0 = x.$$

Next, we impose investment and assume that a risky asset (stock) in the market follows the Black-Scholes dynamics:

$$dS_t = \mu_1 S_t\, dt + \sigma_1 S_t\, dw_t^{(1)},$$

where $w^{(1)} = \{w_t^{(1)}\}_{t\geq 0}$ is a standard Brownian motion independent of $w_t^{(0)}$, $\mu_1(> 0)$ is the stock return rate, and $\sigma_1(> 0)$ is the stock volatility. We further assume that a risk-free asset has the following dynamics:

$$dB_t = \phi(B_t)\, dt,$$

where ϕ is Lipschitz continuous. We denote by b the fraction of the surplus invested in the risky asset; the fraction $(1 - b)$ is then invested in the risk-free asset. We note that if $(1 - b) < 0$, it represents borrowing; if $(1 - b) > 0$, it represents lending (saving); and if $(1 - b) > 1$ or $b < 0$, it represents shortselling the risky asset. The risk exposure a and the fraction of the surplus invested in the risky asset b are the two controllable parameters. At any time $t \geq 0$, the fractions $a = a_t$ and $b = b_t$ are chosen by the insurance company. We denote by $\pi_t = (a_t, b_t)$ the *control policy* at time t. Once the policy $\pi =: \{\pi_t\}_{t\geq 0}$ is chosen, the dynamics of the surplus process become

$$dR_t^\pi = [\mu - (1 - a_t)\lambda + \phi((1 - b_t)R_t^\pi) + \mu_1 b_t R_t^\pi]\, dt + a_t\sigma\, dw_t^{(0)} + b_t\sigma_1 R_t^\pi\, dw_t^{(1)},$$
$$R_0 = x. \tag{1.1}$$

Ruin is defined as the event that the surplus becomes nonpositive. Our objective is to find the minimal ruin probability and the corresponding optimal reinsurance-investment control policy. We need only to study the case of expensive proportional reinsurance, that is, $\lambda > \mu$. The cheap reinsurance case is trivial under the criterion of ruin minimization. In fact, if $\lambda = \mu$, the insurance company can buy 100% reinsurance at the cheap rate and keep the surplus level positive forever; thus, the ruin probability is zero.

In Section 2 we formulate the problem and provide the HJB equation and the verification theorem. Closed-form solutions of the optimization problem under the three borrowing constraints are provided in detail in Sections 3–5. We conclude with some interesting observations in Section 6.

2. FORMULATION, HJB EQUATIONS, AND VERIFICATION THEOREM

We assume all the random variables are defined on a complete probability space denoted by the triple $(P, \Omega, \mathcal{F})$, endowed with a filtration $\mathcal{G} = \{\mathcal{G}_t\}_{t\geq 0}$. The two independent standard Brownian motions $w^{(0)}$ and $w^{(1)}$ are adapted to $\mathcal{G}$.

DEFINITION 2.1

A policy $\pi = \{\pi_t\}_{t\geq 0}$ is said to be admissible if it is predictable with respect to the filtration $\mathcal{G}$, and for all $t \geq 0$ it satisfies $0 \leq a_t \leq 1$, $\int_0^t b_s^2\, ds < \infty$, and

1. *$-\infty < b_t < \infty$ for borrowing constraint (B1) or*
2. *$(1 - b_t)R_t^\pi > -K$ for borrowing constraint (B2) or*
3. *$(1 - b_t) > -k$ for borrowing constraint (B3),*

almost surely. We denote the set of admissible controls by Π.

With an admissible policy π, the ruin time is given by

$$\tau^\pi = \inf\{t \geq 0 : R_t^\pi \leq 0\}. \tag{2.1}$$

Denote by $V_\pi(x)$ the corresponding ruin probability function under policy π with intial surplus of x:

$$V_\pi(x) = P_x(\tau_\pi < \infty) = P(\tau_\pi < \infty | R_0 = x). \tag{2.2}$$

The ultimate objective is to find the minimal ruin probability (*optimal value function*)

$$V(x) = \inf_{\pi\in\Pi} V_\pi(x) \tag{2.3}$$

and the optimal control policy π^* such that

$$V(x) = V_{\pi^*}(x). \tag{2.4}$$

If we assume that the optimal value function V is twice continuously differentiable, then it solves the following HJB equation:

$$0 = \inf_{0\leq a\leq 1, b\in\mathcal{B}} \{[\mu - (1 - a)\lambda + \phi((1 - b)x) + b\mu_1 x]V'(x) + \frac{1}{2}(a^2\sigma^2 + b^2\sigma_1^2 x^2)V''(x)\}, \tag{2.5}$$

where control region $\mathcal{B}$ is defined as the following:

1. $\mathcal{B} = (-\infty, \infty)$ for borrowing constraint (B1),
2. $\mathcal{B} = (-\infty, 1+ K/x)$ for borrowing constraint (B2), and
3. $\mathcal{B} = (-\infty, 1 + k)$ for borrowing constraint (B3).

The procedure of deriving the HJB equation is standard (see, e.g., Fleming and Soner 1993; Taksar and Markussen 2003; Whittle 1983; Yong and Zhou 1999). We omit the proof and concentrate on seeking a convex C^2 solution to HJB equation (2.5) subject to the boundary conditions

$$V(0) = 1, \qquad V(\infty) = 0. \tag{2.6}$$

Such a solution is then the minimized ruin probability function V (optimal value function) by virtue of the *verification theorem* as given below.

Theorem 2.1

Let W be a decreasing C^2 solution (possibly with a finite number of discontinuities in W'' where left and right limits do not match) on $(0, \infty)$ to HJB equation (2.5) subject to the boundary conditions (2.6). Then the optimal value function V given by (2.3) coincides with W. Furthermore, let $(a^*(x), b^*(x))$ be such that

$$0 = [\mu - (1 - a^*(x))\lambda + \phi((1 - b^*(x))x) + b^*(x)\mu_1 x]W'(x) + \frac{1}{2}[(a^*(x))^2\sigma_0^2 + (b^*(x))^2\sigma_1^2 x^2]W''(x), \tag{2.7}$$

for all $0 \leq x < \infty$. Then the policy π^* of the following feedback form $\pi_s^* = (a^*(R_s^{\pi^*}), b^*(R_s^{\pi^*}))$, where $R_s^{\pi^*}$ is the solution to (1.1), is the optimal policy; that is, $W(x) = V(x) = V_{\pi^*}(x)$.

Notice that the Lipschitz conditions on both drift and diffusion terms in (1.1) are satisfied, and Itô's Lemma is applicable to such W whose second derivative W'' has finite points of discontinuity; thus, the proof of the verification will be similar to that in Fleming and Soner (1993, p. 172) or Taksar and Markussen (2003) and hence is omitted.

In the next three sections, our main objective is to solve the HJB equations under the three borrowing constraints (B1), (B2), and (B3), respectively, for the minimal ruin probability functions and find their associated optimal investment-reinsurance policies by verification techniques.

3. THE CASE WITH HIGHER BORROWING RATE

In this section we solve the case when borrowing is more expensive than lending. Thus the function ϕ describing the dynamics of risk-free asset is given by

$$\phi(B_t) = \begin{cases} \alpha B_t & \text{if } B_t \geq 0 \\ \beta B_t & \text{if } B_t < 0 \end{cases}, \tag{3.1}$$

where β represents the borrowing rate and α the lending rate with $\beta > \alpha$. In this case the HJB equation (2.5) is given by

$$0 = \inf_{0 \leq a \leq 1, -\infty < b < \infty} \{[\mu - (1 - a)\lambda + (\phi(1 - b) + b\mu_1)x]V'(x) + \frac{1}{2}(a^2\sigma^2 + b^2\sigma_1^2 x^2)V''(x)\}. \tag{3.2}$$

It is easy to see that if the initial surplus satisfies $x \geq x^*$, then $V(x) = 0$, where x^* is given by

$$x^* = \frac{\lambda - \mu}{\alpha}, \tag{3.3}$$

and we call it a *safe level*. In fact, when the surplus reaches the safe level x^*, the company can buy 100% reinsurance and invest all the surplus in the risk-free asset to earn interest rate α. Thus the interest earned can cover the shortfall between premiums and expensive reinsurance purchase, and the surplus of the insurance company will never vanish. Consequently, we need only to construct a solution on the interval

$$\mathbb{O} = (0, x^*). \tag{3.4}$$

To solve HJB equation (3.2), some expressions are needed to determine the minimizers of the left-hand side of the equation. For any decreasing, convex, and twice continuously differentiable function W with $W'(x) < 0$ and $W''(x) > 0$ on $\mathbb{O}$, we write

$$\begin{aligned} a_W(x) &= -\frac{\lambda W'(x)}{\sigma^2 W''(x)}, \\ b_W^{\alpha}(x) &= -\frac{(\mu_1 - \alpha)W'(x)}{x\sigma_1^2 W''(x)}, \\ b_W^{\beta}(x) &= -\frac{(\mu_1 - \beta)W'(x)}{x\sigma_1^2 W''(x)}. \end{aligned} \tag{3.5}$$

It immediately follows that on $\mathbb{O}$ we have

$$a_W(x) > 0 \tag{3.6}$$

and

$$b_W^{\alpha}(x) > b_W^{\beta}(x). \tag{3.7}$$

Next, we define sets

$$\begin{aligned} \mathbb{O}_{W_1} &= \{x \in \mathbb{O} : 0 < a_W(x) \le 1,\ b_W^{\alpha}(x) < 1\}, \\ \mathbb{O}_{W_2} &= \{x \in \mathbb{O} : 0 < a_W(x) \le 1,\ b_W^{\beta}(x) > 1\}, \\ \mathbb{O}_{W_3} &= \{x \in \mathbb{O} : a_W(x) > 1,\ b_W^{\alpha}(x) < 1\}, \\ \mathbb{O}_{W_4} &= \{x \in \mathbb{O} : a_W(x) > 1,\ b_W^{\beta}(x) > 1\}, \\ \mathbb{O}_{W_5} &= \{x \in \mathbb{O} : b_W^{\beta}(x) \le 1 \le b_W^{\alpha}(x)\}. \end{aligned} \tag{3.8}$$

It is easy to check that these five sets form a partition of $\mathbb{O}$:

$$\mathbb{O} = \mathbb{O}_{W_1} \cup \mathbb{O}_{W_2} \cup \mathbb{O}_{W_3} \cup \mathbb{O}_{W_4} \cup \mathbb{O}_{W_5}.$$

Write

$$\mathcal{L}^{(a,b)}W(x) = [\mu - (1 - a)\lambda + (\phi(1 - b) + b\mu_1)x]W'(x) + \frac{1}{2}(a^2\sigma^2 + b^2\sigma_1^2 x^2)W''(x). \tag{3.9}$$

We see that for any fixed $x \in \mathbb{O}$, $\mathcal{L}^{(a,b)}W(x)$ is a continuous function in variables a and b, and

$$\lim_{b \to \pm\infty} \mathcal{L}^{(a,b)}W(x) = \infty$$

uniformly in a. Thus, for any $x \in \mathbb{O}$, there always exist minimizers $a^*(x)$ and $b^*(x)$ with $0 \le a^*(x) \le 1$ and $b^*(x) \in (-\infty, \infty)$ such that

$$\mathcal{L}^{(a^*(x),b^*(x))}W(x) = \inf_{0 \le a \le 1, -\infty < b < \infty} \mathcal{L}^{(a,b)}W(x). \tag{3.10}$$

If minimizers $a^*(x)$ and $b^*(x)$ are determined, then from $\mathcal{L}^{(a^*(x),b^*(x))}V(x) = 0$, we can detect the structure of the slolution V. This idea immediately yields several lemmas in the following, which are applied to construct the solution of HJB equation (3.2).

Lemma 3.1

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = f_1^\alpha(x), \tag{3.11}$$

where

$$f_1^\alpha(x) = -\frac{\dfrac{\lambda^2}{\sigma^2} + \dfrac{(\mu_1 - \alpha)^2}{\sigma_1^2}}{2(\lambda - \mu - \alpha x)}, \tag{3.12}$$

then V solves HJB equation (3.2) *on* $\mathbb{O}_{V_1}$ *and vice versa.*

PROOF

For any $x \in \mathbb{O}_{V_1}$, the minimum on the right-hand side of HJB equation (3.2) is attained at the minimizers $a^*(x) = a_V^\alpha(x)$ and $b^*(x) = b_V^\alpha(x)$. Replace a and b by

$$a^*(x) = a_V^\alpha(x) = -\frac{\lambda}{\sigma^2 f_1^\alpha(x)},$$

$$b^*(x) = b_V^\alpha(x) = -\frac{(\mu_1 - \alpha)}{x\sigma_1^2 f_1^\alpha(x)}, \tag{3.13}$$

respectively, in (3.2) and simplify; then it results in (3.11). □

Similarly we have the following lemma:

Lemma 3.2

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = f_1^\beta(x), \tag{3.14}$$

where

$$f_1^\beta(x) = -\frac{\dfrac{\lambda^2}{\sigma^2} + \dfrac{(\mu_1 - \beta)^2}{\sigma_1^2}}{2(\lambda - \mu - \beta x)}, \tag{3.15}$$

then V solves HJB equation (3.2) *on* $\mathbb{O}_{V_2}$ *and vice versa.*

Lemma 3.3

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = f_2^\alpha(x), \tag{3.16}$$

where

$$f_2^\alpha(x) = \frac{\mu + \alpha x + \sqrt{(\mu + \alpha x)^2 + \sigma^2(\mu_1 - \alpha)^2/\sigma_1^2}}{\sigma^2}, \tag{3.17}$$

then V solves HJB equation (3.2) *on* $\mathbb{O}_{V_3}$ *and vice versa.*

PROOF

For any $x \in \mathbb{O}_{V_1}$, the minimizers are $a^*(x) = 1$ and $b^*(x) = b_V^\alpha(x)$ in HJB equation (3.2). Replace a and b by

$$a^*(x) = 1, \qquad (3.18)$$
$$b^*(x) = b_V^\alpha(x) = -\frac{(\mu_1 - \alpha)}{x\sigma_1^2 f_2^\alpha(x)},$$

respectively, in (3.2) and simplify; then it results in

$$\frac{(\mu_1 - \alpha)^2}{\sigma_1^2}\left(\frac{V'(x)}{V''(x)}\right)^2 - 2(\mu + \alpha x)\left(\frac{V'(x)}{V''(x)}\right) - \sigma^2 = 0, \qquad (3.19)$$

which immediately gives equation (3.16).

□

The next lemma can be obtained similarly:

Lemma 3.4

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = f_2^\beta(x), \qquad (3.20)$$

where

$$f_2^\beta(x) = -\frac{\mu + \beta x + \sqrt{(\mu + \beta x)^2 + \sigma^2(\mu_1 - \beta)^2/\sigma_1^2}}{\sigma^2}, \qquad (3.21)$$

then V solves HJB equation (3.2) *on $\mathbb{O}_{V_4}$ and vice versa.*

Lemma 3.5

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$0 = \inf_{0\le a\le 1}\{[\mu - (1-a)\lambda + \mu_1 x]V'(x) + \frac{1}{2}(a^2\sigma^2 + \sigma_1^2 x^2)V''(x)\}, \qquad (3.22)$$

then V solves HJB equation (3.2) *on $\mathbb{O}_{V_5}$ and vice versa.*

PROOF

Write

$$\mathcal{L}^b V(x) = \inf_{0\le a\le 1}\{[\mu - (1-a)\lambda]V'(x) + \frac{1}{2}a^2\sigma^2 V''(x)\}$$
$$+ (\phi(1-b) + b\mu_1)xV'(x) + b^2\sigma_1^2 x^2 V''(x). \qquad (3.23)$$

Thus, the HJB equation becomes

$$\inf_{-\infty<b<\infty} \mathcal{L}^b V(x) = 0. \qquad (3.24)$$

Now we prove the lemma by contradiction. Suppose the minimizer on the left-hand side of (3.24) is $b^*(x) > 1$ for some $x \in \mathbb{O}_{V_5}$. Thus, by differentiation we have

$$\frac{d\mathcal{L}^b V(x)}{db}\Big|_{b=b^*(x)} = 0,$$

and it leads to $b^*(x) = b_V^\beta(x)$ and thus $b_V^\beta(x) > 1$, which contradicts the definition of $\mathbb{O}_{V_5}$. Similarly it cannot hold that $b^*(x) < 1$; hence the minimizer must be $b^*(x) = 1$, wherefrom the HJB equation simplifies to equation (3.22). □

In the following subsections, we solve HJB equation (3.2) for the minimal ruin probability function V and find its optimal control policy of reinsurance and investment as well. First, we focus on constructing the solution V to the HJB equation under the assumption

$$\mu_1 \geq \beta (> \alpha).$$

Because of the significant interplay between the solution and exogenous parameters, we need to consider the following four cases individually in the next four subsections:

1. $\mu < \lambda \leq \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$
2. $\mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2} < \lambda < \mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$
3. $\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2} \leq \lambda < \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$
4. $\lambda \geq \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$.

As the reinsurance becomes more and more expensive, we can see the gradual quantitative changes in the solution and its associated optimal reinsurance-investment control policy. The other case $\beta > \mu_1$ can be reduced to the known results in the literature; a brief discussion on this case is given in the last subsection.

3.1 The Case of $\mu < \lambda \leq \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$

Suppose V is a decreasing convex C^2 function on $\mathbb{O}$ that solves HJB equation (3.2). To construct the solution, first we need to locate the five sets in (3.8) on which the corresponding expressions of the solution are given in Lemmas 3.1–3.5 and then piece them together smoothly. We first identify the set $\mathbb{O}_{V_1} = \{x \in \mathbb{O} : 0 < a_V(x) \leq 1, b_V^{\alpha}(x) < 1\}$. From Lemma 3.1, the solution V solves equation (3.11) on $\mathbb{O}_{V_1}$; thus, for $x \in \mathbb{O}_{V_1}$ we have

$$a_V(x) = \frac{2\lambda(\lambda - \mu - \alpha x)}{\lambda^2 + \sigma^2(\mu_1 - \alpha)^2/\sigma_1^2}. \tag{3.25}$$

Hence, $a_V(x) \leq 1$ implies $x \geq x_1^{\alpha}$, where

$$x_1^{\alpha} = \frac{\lambda^2 - 2\mu\lambda - \sigma^2(\mu_1 - \alpha)^2/\sigma_1^2}{2\lambda\alpha}. \tag{3.26}$$

Further,

$$b_V^{\alpha}(x) = \frac{2(\mu_1 - \alpha)(\lambda - \mu - \alpha x)}{x[\sigma_1^2\lambda^2/\sigma^2 + (\mu_1 - \alpha)^2]}. \tag{3.27}$$

Thus, $b_V^{\alpha}(x) < 1$ gives $x > x_2^{\alpha}$, where

$$x_2^{\alpha} = \frac{2(\mu_1 - \alpha)(\lambda - \mu)}{\mu_1^2 - \alpha^2 + \lambda^2\sigma_1^2/\sigma^2}. \tag{3.28}$$

We note that $0 < x_1^{\alpha} < x^*$ and $0 < x_2^{\alpha} < x^*$. Now we conclude

$$\mathbb{O}_{V_1} \subseteq [x_1^{\alpha}, x^*) \cap (x_2^{\alpha}, x^*). \tag{3.29}$$

Lemma 3.6

If V is a convex C^2 function on $\mathbb{O}$ and it solves HJB equation (3.2) with boundary conditions $V(0) = 1$ and $V(x^*) = 0$, then

$$\mathbb{O}_{V_1} = [x_1^{\alpha}, x^*) \cap (x_2^{\alpha}, x^*) \cap \mathbb{O}.$$

PROOF

Based on the previous discussion in this section, we need only to show $[x_1^\alpha, x^*) \cap (x_2^\alpha, x^*) \cap \mathbb{O} \subseteq \mathbb{O}_{V_1}$. Suppose it does not hold; then because of the fact that V is a C^2 function, there must exist x^l and x^u with $x^l < x^u$, such that $(x^l, x^u) \subseteq [x_1^\alpha, x^*) \cap (x_2^\alpha, x^*) \cap \mathbb{O}$ and $(x^l, x^u) \cap \mathbb{O}_{V_1} = \emptyset$. Now we define a new function

$$\tilde{V}(x) = \begin{cases} 1 - C_1(1 - V(x)) & 0 < x \le x^l \\ C_2 - e^{C_3} \int_{x^l}^{x} \exp\{\int_0^u f_1^\alpha(v)\, dv\}\, du & x^l < x < x^u \\ C_4 V(x) & x^u \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.30}$$

where constants C_1 (> 0), C_2, C_3, and C_4 (> 0) solve the smooth-fit conditions, which equate the left and right limits of $\tilde{V}$ and $\tilde{V}'$ at threshold points x^l and x^u:

$$\tilde{V}(x^l-) = \tilde{V}(x^l+),\ \tilde{V}'(x^l-) = \tilde{V}'(x^l+),$$
$$\tilde{V}(x^u-) = \tilde{V}(x^u+),\ \tilde{V}'(x^u-) = \tilde{V}'(x^u+).$$

Such constants exist by direct computation. Therefore $\tilde{V}$ is a C^2 function on $(0, \infty)$ with possibly a finite number of discontinuities in its second derivative. Because function V solves HJB equation (3.2) on $\mathbb{O}$, it is easy to check that $\tilde{V}$ solves the equation on $(0, x^l] \cap [x^u, x^*)$. Further, equation $\tilde{V}''(x)/\tilde{V}'(x) = f_1^\alpha(x)$ holds on (x^l, x^u), thus from Lemma 3.1, $\tilde{V}$ solves the HJB equation on (x^l, x^u), and hence on $\mathbb{O}$. In addition, the boundary conditions are also satisfied: $\tilde{V}(0) = 1$ and $\tilde{V}(x^*) = 0$. Thus, from the verification theorem, V and $\tilde{V}$ must both coincide with the minimal ruin probability function, which is unique because of its definition. Therefore it holds that $V = \tilde{V}$, and we have $(x^l, x^u) \subseteq \mathbb{O}_{\tilde{V}_1} = \mathbb{O}_{V_1}$, which is a contradiction. □

Lemma 3.7

If $\mu_1 > \alpha$ *and* $\lambda < (>) \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$, *then* $x_1^\alpha < (>) x_2^\alpha$.

From Lemma 3.6 and Lemma 3.7, noticing $x_2^\alpha > 0$, we conclude $\mathbb{O}_{V_1} = (x_2^\alpha, x^*)$. Next we locate $\mathbb{O}_{V_2}$. For any $x \in \mathbb{O}_{V_2}$, from Lemma 3.2, we have

$$a_V(x) = \frac{2\lambda(\lambda - \mu - \beta x)}{\lambda^2 + \sigma^2(\mu_1 - \beta)^2/\sigma_1^2}. \tag{3.31}$$

Hence, $a_V(x) \le 1$ implies $x \ge x_1^\beta$, where

$$x_1^\beta = \frac{\lambda^2 - 2\mu\lambda - \sigma^2(\mu_1 - \beta)^2/\sigma_1^2}{2\lambda\beta}. \tag{3.32}$$

Notice that

$$b_V^\beta(x) = \frac{2(\mu_1 - \beta)(\lambda - \mu - \beta x)}{x(\sigma_1^2\lambda^2/\sigma^2 + (\mu_1 - \beta)^2)}; \tag{3.33}$$

thus, $b_V^\beta(x) > 1$ gives $x < x_2^\beta$, where

$$x_2^\beta = \frac{2(\mu_1 - \beta)(\lambda - \mu)}{\mu_1^2 - \beta^2 + \lambda^2\sigma_1^2/\sigma^2}. \tag{3.34}$$

As in Lemma 3.6, we conclude

$$\mathbb{O}_{V_2} = [x_1^\beta, x_2^\beta) \cap \mathbb{O}. \tag{3.35}$$

Thus, $\mathbb{O}_{V_2} = (0, x_2^\beta)$ because $x_1^\beta \le 0$ when $\lambda < \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$.

For $x \in \mathbb{O}_{V_3}$, from Lemma 3.3, we have

$$a_V(x) = \frac{\lambda}{\sigma^2} \frac{\sqrt{(\mu + \alpha x)^2 + (\mu_1 + \alpha)^2\sigma^2/\sigma_1^2} - (\mu + \alpha x)}{(\mu_1 - \alpha)^2/\sigma_1^2}; \tag{3.36}$$

hence, $a_V(x) > 1$ implies $x < x_1^\alpha$. Further, we notice that

$$b_V^\alpha(x) = \frac{\sqrt{(\mu + \alpha x)^2 + (\mu_1 - \alpha)^2\sigma^2/\sigma_1^2} - (\mu + \alpha x)}{(\mu_1 - \alpha)x}; \tag{3.37}$$

thus, $b_V^\alpha(x) < 1$ gives $x > x_3^\alpha$, where

$$x_3^\alpha = \frac{-\mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}}{\mu_1 + \alpha}. \tag{3.38}$$

Therefore we conclude

$$\mathbb{O}_{V_3} = (x_3^\alpha, x_1^\alpha), \tag{3.39}$$

which leads to $\mathbb{O}_{V_3} = \emptyset$ by the following lemma.

Lemma 3.8

If $\mu_1 > \alpha$ and $\lambda < (>)\mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$, then $x_1^\alpha < (>)x_3^\alpha$.

PROOF

Notice that $x_1^\alpha = \lambda^2 - 2\mu\lambda - (\mu_1 - \alpha)^2\sigma^2/\sigma_1^2/2\lambda\alpha := G(\lambda)$, which is an increasing function in λ. Thus,

$$x_1^\alpha = G(\lambda) < G(\mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}) = x_3^\alpha. \qquad \square$$

For $x \in \mathbb{O}_{V_4}$, from Lemma 3.4, we have

$$a_V(x) = \frac{\lambda}{\sigma^2} \frac{\sqrt{(\mu + \beta x)^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2} - (\mu + \beta x)}{(\mu_1 - \beta)^2/\sigma_1^2}; \tag{3.40}$$

hence, $a_V(x) > 1$ implies $x < x_1^\beta$. Further, we have

$$b_V^\beta(x) = \frac{\sqrt{(\mu + \beta x)^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2} - (\mu + \beta x)}{(\mu_1 - \beta)x}, \tag{3.41}$$

and $b_V^\beta(x) > 1$ implies $x < x_3^\beta$, where

$$x_3^\beta = \frac{-\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}}{\mu_1 + \beta}. \tag{3.42}$$

Therefore

$$\mathbb{O}_{V_4} = (0, x_1^\beta) \cap (0, x_3^\beta). \tag{3.43}$$

Noting that $x_1^\beta \le 0$ when $\mu < \lambda \le \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$, we have $\mathbb{O}_{V_4} = \emptyset$. Further, x_2^β decreases as a function of β, thus $x_2^\beta < x_2^\alpha$. Hence, from the discussions above, to form a partition of $\mathbb{O}$, we must have

$$\mathbb{O}_{V_5} = [x_2^\beta, x_2^\alpha]. \tag{3.44}$$

From Lemma 3.5, the solution V solves equation (3.22) on $\mathbb{O}_{V_5} = [x_2^\beta, x_2^\alpha]$. For those values of x in $\mathbb{O}_{V_5}$ such that $a_V(x) \le 1$, the minimizer of the left-hand side of equation (3.22) is $a^*(x) = a_V(x)$. Plug in $a_V(x)$ for a in the equation and simplify; thus, V solves

$$\frac{V''(x)}{V'(x)} = f_1(x), \tag{3.45}$$

where

$$f_1(x) = \frac{\lambda - \mu - \mu_1 x - \sqrt{(\lambda - \mu - \mu_1 x)^2 + \lambda^2\sigma_1^2 x^2/\sigma^2}}{\sigma_1^2 x^2}. \tag{3.46}$$

Thus,

$$a_V(x) = \frac{\sqrt{(\lambda - \mu - \mu_1 x)^2 + \lambda^2\sigma_1^2 x^2/\sigma^2} - (\lambda - \mu - \mu_1 x)}{\lambda}. \tag{3.47}$$

Hence, $a_V(x) \leq 1$ gives

$$x_1^- \leq x \leq x_1^+, \tag{3.48}$$

where

$$x_1^{\pm} = \frac{\mu_1 \pm \sqrt{\mu_1^2 - \lambda(\lambda - 2\mu)\sigma_1^2/\sigma^2}}{\lambda\sigma_1^2/\sigma^2}. \tag{3.49}$$

Lemma 3.9

If $\lambda \leq \mu + \sqrt{\mu^2 + \mu_1^2\sigma^2/\sigma_1^2}$, $\mu_1 > \beta$ *and* $\lambda < (>)\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$ *hold, then* $x_1^- < (>) x_2^\beta$.

Lemma 3.10

If $\lambda \leq \mu + \sqrt{\mu^2 + \mu_1^2\sigma^2/\sigma_1^2}$, *then* $x_1^+ > 2\mu_1(\lambda - \mu)/\mu_1^2 + \lambda^2\sigma_1^2/\sigma^2 > x_2^\alpha$.

Hence, from Lemma 3.9 and Lemma 3.10 we have $x_1^- < x_2^\beta < x_2^\alpha < x_1^+$, wherefrom we conclude for all $x \in \mathbb{O}_{V_5} \subset (x_1^-, x_1^+)$, the minimizer in equation (3.22) is $a^*(x) = a_V(x)$, and the function V solves equation (3.45). To this end we can construct the solution satisfying the following:

$$\frac{V''(x)}{V'(x)} = \begin{cases} f_1^\beta(x) & \text{if } 0 < x < x_2^\beta \\ f_1(x) & \text{if } x_2^\beta \leq x \leq x_2^\alpha \\ f_1^\alpha(x) & \text{if } x_2^\alpha < x < x^* \end{cases}, \tag{3.50}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$. The solution V can be found by applying smooth fits at the threshold points x_2^β and x_2^α and equating the following left and right limits:

$$V(x_2^\beta-) = V(x_2^\beta+),\; V'(x_2^\beta-) = V'(x_2^\beta+),$$
$$V(x_2^\alpha-) = V(x_2^\alpha+),\; V'(x_2^\alpha-) = V'(x_2^\alpha+). \tag{3.51}$$

The discussion in this section can be summarized as the following theorem.

Theorem 3.1

If $\mu_1 > \beta > \alpha$ *and* $\mu < \lambda < \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$, *then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at* x^**) function given by*

$$V(x) = \begin{cases} 1 - e^{C_1}\int_0^x \exp\{\int_0^u f_2^\beta(v)\,dv\}\,du & 0 < x < x_2^\beta \\ C_3 - e^{C_2}\int_{x_2^\beta}^x \exp\{\int_0^u f_1(v)\,dv\}\,du & x_2^\beta \leq x \leq x_2^\alpha \\ e^{C_4}\int_x^{x^*} \exp\{\int_{x_2^\alpha}^u f_1^\alpha(v)\,dv\}\,du & x_2^\alpha < x < x^* \\ 0 & x \geq x^* \end{cases}, \tag{3.52}$$

where f_2^β *is given by* (3.21), f_1 *by* (3.46), f_1^α *by* (3.12), x_2^β *by* (3.34), x_2^α *by* (3.28), x^* *by* (3.3), *and*

$$C_1 = -\ln\left[\int_0^{x_2^\beta} \exp\left\{\int_0^u f_1^\beta(v)\,dv\right\} du + \exp\left\{\int_0^{x_2^\beta} f_1^\beta(v)\,dv\right\} \int_{x_2^\beta}^{x_2^\alpha} \exp\left\{\int_0^u f_1(v)\,dv\right\} du \right.$$
$$\left. + \exp\left\{\int_0^{x_2^\beta} f_1^\beta(v)\,dv + \int_{x_2^\beta}^{x_2^\alpha} f_1(v)\,dv\right\} \int_{x_2^\alpha}^{x^*} \exp\left\{\int_{x_2^\alpha}^u f_1^\alpha(v)\,dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{x_2^\beta} f_1^\beta(v)\,dv,$$

$$C_3 = 1 - e^{C_1}\int_0^{x_2^\beta} \exp\left\{\int_0^u f_1^\beta(v)\,dv\right\} du,$$

$$C_4 = C_1 + \int_0^{x_2^\beta} f_1^\beta(v)\,dv + \int_{x_2^\beta}^{x_2^\alpha} f_1(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} -\dfrac{\lambda}{\sigma^2 f_1^\beta(x)} & 0 < x < x_2^\beta \\ -\dfrac{\lambda}{\sigma^2 f_1(x)} & x_2^\beta \le x \le x_2^\alpha \\ -\dfrac{\lambda}{\sigma^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.53}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - \beta}{x\sigma_1^2 f_1^\beta(x)} & 0 < x < x_2^\beta \\ 1 & x_2^\beta \le x \le x_2^\alpha \\ -\dfrac{\mu_1 - \alpha}{x\sigma_1^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{3.54}$$

PROOF

It is easy to check that V is a convex decreasing C^2 function on $\mathbb{O}$. From the discussion in this subsection, we see that V solves HJB equation (3.2). Thus, from the verification theorem (Theorem 2.1), the results in this theorem immediately follow. □

3.2 The Case of $\mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2} < \lambda \le \mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$

As in the previous section, we have $\mathbb{O}_{V_1} = (x_2^\alpha, x^*)$ and $\mathbb{O}_{V_3} = \emptyset$. Replacing α by β in Lemma 3.7, we see that $x_1^\beta < x_2^\beta$; further, from $\lambda > \mu + \sqrt{\mu^2 + (\mu_1 - \beta)^2\sigma^2/\sigma_1^2}$ we notice $x_1^\beta > 0$; thus, we have $\mathbb{O}_{V_2} = [x_1^\beta, x_2^\beta)$. Similarly, from Lemma 3.8 by replacing α by β, we have $x_1^\beta < x_3^\beta$; hence, $\mathbb{O}_{V_4} = (0, x_1^\beta)$. At last, we have $\mathbb{O}_{V_5} = [x_2^\beta, x_2^\alpha]$, on which the minimizer is given by (3.47). Therefore, the solution V solves the following:

$$\frac{V''(x)}{V'(x)} = \begin{cases} f_2^\beta(x) & \text{if } 0 < x < x_1^\beta \\ f_1^\beta(x) & \text{if } x_1^\beta \le x < x_2^\beta \\ f_1(x) & \text{if } x_2^\beta \le x \le x_2^\alpha \\ f_1^\alpha(x) & \text{if } x_2^\alpha < x < x^* \end{cases}, \tag{3.55}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$. The solution is found by applying smooth fits at the threshold points x_1^β, x_2^β, and x_2^α:

$$\begin{aligned} V(x_1^\beta-) &= V(x_1^\beta+), V'(x_1^\beta-) = V'(x_1^\beta+), \\ V(x_2^\beta-) &= V(x_2^\beta+), V'(x_2^\beta-) = V'(x_2^\beta+), \\ V(x_2^\alpha-) &= V(x_2^\alpha+), V'(x_2^\alpha-) = V'(x_2^\alpha+). \end{aligned} \tag{3.56}$$

The discussion in this section can be summarized in the following theorem.

Theorem 3.2

If $\mu_1 > \beta > \alpha$ and $\mu < \lambda < \mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u f_2^\beta(v)\, dv\}\, du & 0 < x < x_1^\beta \\ C_3 - e^{C_2} \int_{x_1^\beta}^x \exp\int_0^u f_1^\beta(v)\, dv\}\, du & x_1^\beta \le x < x_2^\beta \\ C_5 - e^{C_4} \int_{x_2^\beta}^x \exp\{\int_0^u f_1(v)\, dv\}\, du & x_2^\beta \le x \le x_2^\alpha \\ e^{C_6} \int_x^{x^*} \exp\{\int_{x_2^\alpha}^u f_1^\alpha(v)\, dv\}\, du & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.57}$$

where f_2^β is given by (3.21), f_1^β *by* (3.15), f_1 *by* (3.46), f_1^α *by* (3.12), x_1^β *by* (3.32), x_2^β *by* (3.34), x_2^α *by* (3.28), x^* *by* (3.3), *and*

$$\begin{aligned} C_1 = -\ln \Bigg[& \int_0^{x_1^\beta} \exp\left\{\int_0^u f_2^\beta(v)\, dv\right\} du + \exp\left\{\int_0^{x_1^\beta} f_2^\beta(v)\, dv\right\} \int_{x_1^\beta}^{x_2^\beta} \exp\left\{\int_0^u f_1^\beta(v)\, dv\right\} du \\ & + \exp\left\{\int_0^{x_1^\beta} f_2^\beta(v)\, dv + \int_{x_1^\beta}^{x_2^\beta} f_1^\beta(v)\, dv\right\} \int_{x_2^\beta}^{x_2^\alpha} \exp\left\{\int_0^u f_1(v)\, dv\right\} du \\ & + \exp\left\{\int_0^{x_1^\beta} f_2^\beta(v)\, dv + \int_{x_1^\beta}^{x_2^\beta} f_1^\beta(v)\, dv + \int_{x_2^\beta}^{x_2^\alpha} f_1(v)\, dv\right\} \int_{x_2^\alpha}^{x^*} \exp\left\{\int_{x_2^\alpha}^u f_1^\alpha(v)\, dv\right\} du \Bigg], \end{aligned}$$

$$C_2 = C_1 + \int_0^{x_1^\beta} f_2^\beta(v)\, dv,$$

$$C_3 = 1 - e^{C_1} \int_0^{x_1^\beta} \exp\left\{\int_0^u f_2^\beta(v)\, dv\right\} du,$$

$$C_4 = C_2 + \int_{x_1^\beta}^{x_2^\beta} f_1^\beta(v)\, dv,$$

$$C_5 = C_3 - e^{C_2} \int_{x_1^\beta}^{x_2^\beta} \exp\left\{\int_0^u f_1^\beta(v)\, dv\right\} du,$$

$$C_6 = C_4 + \int_{x_2^\beta}^{x_2^\alpha} f_1(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1^\beta \\ -\dfrac{\lambda}{\sigma^2 f_1^\beta(x)} & x_1^\beta \le x < x_2^\beta \\ -\dfrac{\lambda}{\sigma^2 f_1(x)} & x_2^\beta \le x \le x_2^\alpha \\ -\dfrac{\lambda}{\sigma^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.58}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - \beta}{x\sigma_1^2 f_2^\beta(x)} & 0 < x < x_1^\beta \\ -\dfrac{\mu_1 - \beta}{x\sigma_1^2 f_1^\beta(x)} & x_1^\beta \le x < x_2^\beta \\ 1 & x_2^\beta \le x \le x_2^\alpha \\ -\dfrac{\mu_1 - \alpha}{x\sigma_1^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{3.59}$$

3.3 The Case of $\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2} \le \lambda \le \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$

We assume V solves the HJB equation. From Lemma 3.7 and equation (3.29), we see that $\mathbb{C}_{V_1} = (x_2^\alpha, x^*)$. Replacing α by β in Lemma 3.7, we have $x_1^\beta \ge x_2^\beta$; thus $\mathbb{C}_{V_2} = \emptyset$ by (3.35). As in the previous subsection, from (3.39) and Lemma 3.8, we have $\mathbb{C}_{V_3} = \emptyset$; from (3.43) and Lemma 3.8, we have $\mathbb{C}_{V_4} = (0, x_1^\beta)$; consequently, $\mathbb{C}_{V_5} = [x_3^\beta, x_2^\alpha]$.

Lemma 3.11

If $\mu_1 > \beta$ *and* $\lambda \ge (<)\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$, *then* $x_1^- \ge (<)x_3^\beta$.

PROOF

Treat x_1^- as a function of λ: $x_1^- = f(\lambda)$, which is strictly increasing. Thus

$$x_1^- = f(\lambda) \ge (<)f(\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}) = x_3^\beta,$$

when $\lambda \ge (<)\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2}$. □

From Lemma 3.9 replacing β by α, we have $x_1^- < x_2^\alpha$. From Lemma 3.11 it holds that $x_1^- \ge x_3^\beta$; further from Lemma 3.10, we have $x_1^+ > x_2^\alpha$; thus, it holds that $x_3^\beta \le x_1^- < x_2^\alpha < x_1^+$. As in the previous subsection, for $x \in (x_1^-, x_2^\alpha]$, we see that the minimizer in equation (3.22) is

$$a^*(x) = a_V(x) = \frac{\sqrt{(\lambda - \mu - \mu_1 x)^2 + \lambda^2\sigma_1^2 x^2/\sigma^2} - (\lambda - \mu - \mu_1 x)}{\lambda}.$$

For $x \in [x_3^\beta, x_1^-]$, we have $a^*(x) = 1$; plug in $a^*(x) = 1$ into HJB equation (3.22) and simplify, and then the solution V solves

$$\frac{V''(x)}{V'(x)} = f_2(x), \tag{3.60}$$

where

$$f_2(x) = -\frac{2(\mu + \mu_1 x)}{\sigma^2 + \sigma_1^2 x}. \tag{3.61}$$

Now we can construct the solution V as the following:

$$\frac{V''(x)}{V'(x)} = \begin{cases} f_2^\beta(x) & \text{if } 0 < x < x_3^\beta \\ f_2(x) & \text{if } x_3^\beta \le x \le x_1^- \\ f_1(x) & \text{if } x_1^- < x \le x_2^\alpha \\ f_1^\alpha(x) & \text{if } x_2^\alpha < x < x^* \end{cases}, \tag{3.62}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$. Applying smooth fits at the threshold points x_3^β, x_1^-, and x_2^α:

$$\begin{aligned} &V(x_3^\beta -) = V(x_3^\beta +),\ V'(x_3^\beta -) = V'(x_3^\beta +), \\ &V(x_1^- -) = V(x_1^- +),\ V'(x_1^- -) = V'(x_1^- +), \\ &V(x_2^\alpha -) = V(x_2^\alpha +),\ V'(x_2^\alpha -) = V'(x_2^\alpha +), \end{aligned} \tag{3.63}$$

we obtain the following theorem.

Theorem 3.3

If $\mu_1 > \beta > \alpha$ and $\mu + \sqrt{\mu^2 + (\mu_1^2 - \beta^2)\sigma^2/\sigma_1^2} \le \lambda \le \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except point x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1}\int_0^x \exp\{\int_0^u f_2^\beta(v)\,dv\}\,du & 0 < x < x_3^\beta \\ C_3 - e^{C_2}\int_{x_3^\beta}^x \exp\{\int_0^u f_2(v)\,dv\}\,du & x_3^\beta \le x \le x_1^- \\ C_5 - e^{C_4}\int_{x_1}^x \exp\{\int_0^u f_1(v)\,dv\}\,du & x_1^1 < x \le x_2^\alpha \\ e^{C_6}\int_x^{x^*} \exp\{\int_{x_2^\alpha}^u f_1^\alpha(v)\,dv\}\,du & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.64}$$

where f_2^β is given by (3.21), f_1 *by* (3.46), f_2 *by* (3.61), f_1^α *by* (3.12), x_3^β *by* (3.42), x_1^- *by* (3.49), x_2^α *by* (3.28), x^* *by* (3.3), *and*

$$
\begin{aligned}
C_1 = -\ln\Bigg[& \int_0^{x_3^\beta} \exp\left\{\int_0^u f_2^\beta(v)\,dv\right\} du + \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\,dv\right\} \int_{x_3^\beta}^{x_1} \exp\left\{\int_0^u f_2(v)\,dv\right\} du \\
& + \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\,dv + \int_{x_3^\beta}^{x_1} f_2(v)\,dv\right\} \int_{x_1}^{x_2^\alpha} \exp\left\{\int_0^u f_1(v)\,dv\right\} du \\
& + \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\,dv + \int_{x_3^\beta}^{x_1} f_2(v)\,dv + \int_{x_1}^{x_2^\alpha} f_1(v)\,dv\right\} \int_{x_2^\alpha}^{x^*} \exp\left\{\int_{x_2^\alpha}^u f_1^\alpha(v)\,dv\right\} du\Bigg],
\end{aligned}
$$

$$C_2 = C_1 + \int_0^{x_3^\beta} f_2^\beta(v)\,dv,$$

$$C_3 = 1 - e^{C_1}\int_0^{x_3^\beta} \exp\left\{\int_0^u f_2(v)\,dv\right\} du,$$

$$C_4 = C_2 + \int_{x_3^\beta}^{x_1} f_2^\beta(v)\,dv,$$

$$C_5 = C_3 - e^{C_2}\int_{x_3^\beta}^{x_1} \exp\left\{\int_0^u f_2(v)\,dv\right\} du,$$

$$C_6 = C_4 + \int_{x_1}^{x_2^\alpha} f_1(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$
a^*(x) = \begin{cases} 1 & 0 < x \le x_1^- \\ -\dfrac{\lambda}{\sigma^2 f_1(x)} & x_1^- < x \le x_2^\alpha \\ -\dfrac{\lambda}{\sigma^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.65}
$$

and the optimal investment feedback function is

$$
b^*(x) = \begin{cases} -\dfrac{\mu_1 - \beta}{x\sigma_1^2 f_2^\beta(x)} & 0 < x < x_3^\beta \\ 1 & x_3^\beta \le x \le x_2^\alpha \\ -\dfrac{\mu_1 - \alpha}{x\sigma_1^2 f_1^\alpha(x)} & x_2^\alpha < x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{3.66}
$$

3.4 The Case of $\lambda - \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$

In this case from Lemma 3.7 and equation (3.29), we have $\mathbb{O}_{V_1} = [x_1^\alpha, x^*)$. As in the previous subsection, it holds that $\mathbb{O}_{V_2} = \emptyset$. From (3.39) and Lemma 3.8, we conclude $\mathbb{O}_{V_3} = (x_3^\alpha, x_1^\alpha)$. Replacing α by β in Lemma 3.8, and from (3.43), we see that $\mathbb{O}_{V_4} = (0, x_3^\beta)$. At last we have $\mathbb{O}_{V_5} = [x_3^\beta, x_3^\alpha]$, and $a^*(x) = 1$ on $\mathbb{O}_{V_5}$. In fact, replacing β by α in Lemma 3.11, we have $x_1^- > x_3^\alpha$ when x_1^- exists, and $a_V(x) \ge 1$ for $x \in \mathbb{O}_{V_5} = [x_3^\beta, x_3^\alpha]$ when x_1^- does not exist; thus, the minimizer in equation (3.22) must be $a^*(x) = 1$, wherefrom the solution V solves equation (3.60) on $\mathbb{O}_{V_5}$. Therefore, we can construct the solution V satisfying the following:

$$\frac{V''(x)}{V'(x)} = \begin{cases} f_2^\beta(x) & \text{if } 0 < x < x_3^\beta \\ f_2(x) & \text{if } x_3^\beta \le x \le x_3^\alpha \\ f_2^\alpha(x) & \text{if } x_3^\alpha < x < x_1^\alpha \\ f_1^\alpha(x) & \text{if } x_1^\alpha \le x < x^* \end{cases}, \tag{3.67}$$

with boundary conditions $V(0) = 1$ and $V(x^*) = 0$. Applying smooth fits at the threshold points x_3^β, x_3^α, and x_1^α and setting

$$\begin{aligned} V(x_3^\beta-) &= V(x_3^\beta+),\ V'(x_3^\beta-) = V'(x_3^\beta+), \\ V(x_3^\alpha-) &= V(x_3^\alpha+),\ V'(x_3^\alpha-) = V'(x_3^\alpha+), \\ V(x_1^\alpha-) &= V(x_1^\alpha+),\ V'(x_1^\alpha-) = V'(x_1^\alpha+). \end{aligned} \tag{3.68}$$

we obtain the following theorem.

Theorem 3.4

If $\mu_1 > \beta > \alpha$ and $\lambda > \mu + \sqrt{\mu^2 + (\mu_1^2 - \alpha^2)\sigma^2/\sigma_1^2}$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u f_2^\beta(v)\, dv\}\, du & 0 < x < x_3^\beta \\ C_3 - e^{C_2} \int_{x_3^\beta}^x \exp\{\int_0^u f_2(v)\, dv\}\, du & x_3^\beta \le x < x_3^\alpha \\ C_5 - e^{C_4} \int_{x_3^\alpha}^x \exp\{\int_0^u f_2^\alpha(v)\, dv\}\, du & x_3^\alpha < x < x_1^\alpha \\ e^{C_6} \int_x^{x^*} \exp\{\int_{x_1^\alpha}^u (v)\, dv\}\, du & x_1^\alpha \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{3.69}$$

where f_2^β is given by (3.21), f_2 *by* (3.61), f_2^α *by* (3.17), f_1^α *by* (3.12), x_3^β *by* (3.42), x_3^α *by* (3.38), x_1^α *by* (3.26), x^* *by* (3.3), *and*

$$\begin{aligned} C_1 = -\ln \Bigg[&\int_0^{x_3^\beta} \exp\left\{\int_0^u f_2^\beta(v)\, dv\right\} du + \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\, dv\right\} \int_{x_3^\beta}^{x_3^\alpha} \exp\left\{\int_0^u f_2(v)\, dv\right\} du \\ &+ \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\, dv + \int_{x_3^\beta}^{x_3^\alpha} f_2(v)\, dv\right\} \int_{x_3^\alpha}^{x_1^\alpha} \exp\left\{\int_0^u f_2^\alpha(v)\, dv\right\} du \\ &+ \exp\left\{\int_0^{x_3^\beta} f_2^\beta(v)\, dv + \int_{x_3^\beta}^{x_3^\alpha} f_2(v)\, dv + \int_{x_3^\alpha}^{x_1^\alpha} f_2^\alpha(v)\, dv\right\} \int_{x_1^\alpha}^{x^*} \exp\left\{\int_{x_1^\alpha}^u f_1^\alpha(v)\, dv\right\} du \Bigg], \end{aligned}$$

$$C_2 = C_1 + \int_0^{x_3^\beta} f_2^\beta(v)\, dv,$$

$$C_3 = 1 - e^{C_1} \int_0^{x_3^\beta} \exp\left\{\int_0^u f_2^\beta(v)\, dv\right\} du,$$

$$C_4 = C_2 + \int_{x_3^\beta}^{x_3^\alpha} f_2(v)\, dv,$$

$$C_5 = C_3 - e^{C_2} \int_{x_3^\beta}^{x_3^\alpha} \exp\left\{\int_0^u f_2(v)\, dv\right\} du,$$

$$C_6 = C_4 + \int_{x_3^\alpha}^{x_1^\alpha} f_2^\alpha(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1^\alpha \\ -\dfrac{\lambda}{\sigma^2 f_1^\alpha(x)} & x_1^\alpha < x < x^*, \\ 0 & x \geq x^* \end{cases} \tag{3.70}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - \beta}{x\sigma_1^2 f_2^\beta(x)} & 0 < x < x_3^\beta \\ 1 & x_3^\beta \leq x \leq x_3^\alpha \\ -\dfrac{\mu_1 - \alpha}{x\sigma_1^2 f_2^\alpha(x)} & x_3^\alpha < x < x_1^\alpha . \\ -\dfrac{\mu_1 - \alpha}{x\sigma_1^2 f_1^\alpha(x)} & x_1^\alpha \leq x < x^* \\ 0 & x \geq x^* \end{cases} \tag{3.71}$$

REMARK 3.1

The optimal reinsurance and investment feedback functions $a^*(x)$ (*risk exposure*), $b^*(x)$ (*proportion of surplus invested in the risky asset*), and $xb^*(x)$ (*amount of surplus invested in the risky asset*) *in Theorems 3.1–3.4 decrease in x.*

Although it is neither obvious to detect directly from the HJB equation nor financially intuitive, the monotonicity result in Remark 3.1 is quite noticeable from the explicit expressions of the feedback controls. In fact, it is easy to check that $a^*(x)$ and $b^*(x)$ are continuous at the threshold points, functions $f_1^\alpha(x), f_1^\beta(x), f_2^\alpha(x), f_2^\beta(x)$ decrease, and $f_1(x)$ decreases when $x < \bar{x}$ where $\bar{x} = 2\mu_1(\lambda - \mu)/\lambda^2\sigma_1^2/(\sigma^2 + \mu_1^2)$ and $x_2^\alpha < \bar{x}$. Thus, the feedback functions $a^*(x)$ and $b^*(x)$ decrease. This result tells us that, as the surplus level increases, the optimal control policy becomes more conservative.

3.5 The Other Cases

When $\beta \geq \mu_1$, for any $x \in \mathbb{O}$, the minimizer of the HJB equation for b is never $b_V^\beta(x)$. In fact, under the assumption $\beta \geq \mu_1$, we always have $b_V^\beta(x) < 0$. Suppose the minimizer $b^*(x) = b_V^\beta(x)(< 0)$, then by differentiation we must have $b^*(x) = b_V^\alpha(x) > b_V^\beta(x) = b^*(x)$, which is a contradiction. Hence, the minimizer is either 1 or $b_V^\alpha(x) < 1$, and it indicates that borrowing never occurs. Thus, we conclude that when the borrowing rate is higher than the stock return rate, the optimal control policy never involves borrowing although borrowing is allowed. Consequently, the solution of the problem is identical to that in the case with one interest rate. To this end, all the possible cases on exogenous parameters are considered.

4. THE CASE WITH LIMITED DOLLAR AMOUNT OF BORROWING

In this section we study the case when there is a limit $K > 0$ for the borrowed dollar amount by the insurance company. For simplicity, we assume the same rate for borrowing and lending and focus on the effect of the borrowing limit K. Introducing this new exogenous parameter complicates the optimization problem, and the interplay between optimal solutions and exogenous parameters is not so clearly shown as in the previous section. However, the optimal solutions can be found by enumerating all the possible forms.

We have $\phi(B_t) = rB_t$, where r is the common interest rate. Then the dynamics of the risk-free asset are given by $dB_t = rB_t dt$, and HJB equation (2.5) becomes

$$0 = \inf_{0 \le a \le 1, -\infty < b < 1+K/x} \{[\mu - (1-a)\lambda + (r(1-b) + b\mu_1)x]V'(x) + \frac{1}{2}(a^2\sigma^2 + b^2\sigma_1^2 x^2)V''(x)\}. \tag{4.1}$$

We need only to consider the case $\mu_1 > r$. In fact, if $\mu_1 \le r$, we see that the optimal policy does not involve borrowing; thus, the limit on borrowed amount is of no effect, and the resulting optimal solution is the same as that in the case without the borrowing constraint.

Similar to the previous section, we have the following result: if the surplus level is high, that is, $x \ge x^*$, then $V(x) = 0$, where

$$x^* = \frac{\lambda - \mu}{r}. \tag{4.2}$$

Thus, we need only to construct a solution on the interval

$$\mathbb{O} = (0, x^*). \tag{4.3}$$

For any decreasing, convex, and twice continuously differentiable function W with $W'(x) < 0$ and $W''(x) > 0$ on $\mathbb{O}$, we write $a_W(x) = -\lambda W'(x)/\sigma^2 W''(x)$ as in the previous section, and define

$$b_W(x) = -\frac{(\mu_1 - r)W'(x)}{x\sigma_1^2 W''(x)}. \tag{4.4}$$

Now we redefine sets

$$\mathbb{O}_{W_1} = \{x \in \mathbb{O} : 0 < a_W(x) \le 1, b_W(x) \le 1 + K/x\},$$

$$\mathbb{O}_{W_2} = \{x \in \mathbb{O} : 0 < a_W(x) \le 1, b_W(x) > 1 + K/x\},$$

$$\mathbb{O}_{W_3} = \{x \in \mathbb{O} : a_W(x) > 1, b_W(x) \le 1 + K/x\},$$

$$\mathbb{O}_{W_4} = \{x \in \mathbb{O} : a_W(x) > 1, b_W(x) > 1 + K/x\}. \tag{4.5}$$

Obviously these four sets form a partition of $\mathbb{O}$. Next we give several lemmas that are applied to construct the minimal ruin probability function.

Lemma 4.1

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = g_1(x), \tag{4.6}$$

where

$$g_1(x) = -\frac{\frac{\lambda^2}{\sigma^2} + \frac{(\mu_1 - r)^2}{\sigma_1^2}}{2(\lambda - \mu - rx)}, \tag{4.7}$$

then V solves HJB equation (4.1) *on $\mathbb{O}_{V_1}$ and vice versa.*

Lemma 4.2

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = g_2(x), \tag{4.8}$$

where

$$g_2(x) = \frac{(\mu_1 - r)K + \mu - \lambda + \mu_1 x + \sqrt{[(\mu_1 - r)K + \mu - \lambda + \mu_1 x]^2 + \frac{\lambda^2\sigma_1^2}{\sigma^2}(x + K)^2}}{-\sigma_1^2(x + K)^2}, \quad (4.9)$$

then V solves HJB equation (4.1) *on* $\mathbb{O}_{V_2}$ *and vice versa.*

Lemma 4.3

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. *If V solves*

$$\frac{V''(x)}{V'(x)} = g_3(x), \quad (4.10)$$

where

$$g_3(x) = -\frac{\mu + rx + \sqrt{(\mu + rx)^2 + (\mu_1 - r)^2\sigma^2/\sigma_1^2}}{\sigma^2}, \quad (4.11)$$

then V solves HJB equation (4.1) *on* $\mathbb{O}_{V_3}$ *and vice versa.*

Lemma 4.4

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. *If V solves*

$$\frac{V''(x)}{V'(x)} = g_4(x), \quad (4.12)$$

where

$$g_4(x) = -\frac{2[\mu + (\mu_1 - r)K + \mu_1 x]}{\sigma^2 + \sigma_1^2(x + K)^2}, \quad (4.13)$$

then V solves HJB equation (4.1) *on* $\mathbb{O}_{V_4}$ *and vice versa.*

Next we assume that V solves HJB equation (4.1) and proceed as in the previous section to identify sets $\mathbb{O}_{V_1}$, $\mathbb{O}_{V_2}$, $\mathbb{O}_{V_3}$, $\mathbb{O}_{V_4}$, and the expression of V.

From Lemma 4.1, V solves equation (4.6) on $\mathbb{O}_{V_1}$; thus, for any $x \in \mathbb{O}_{V_1}$ we have $a_V(x) = -\lambda/(\sigma^2 g_1(x))$. Hence, $a_V(x) \leq 1$ implies $x \geq x_1$, where

$$x_1 = \frac{\lambda^2 - 2\mu\lambda - (\mu_1 - r)^2\sigma^2/\sigma_1^2}{2\lambda r}. \quad (4.14)$$

Further, $b_V(x) = (-\mu_1 - r)/(x\sigma_1^2 g_1(x))$; thus, $b_V(x) \leq 1 + K/x$ gives $x \geq x_2$, where

$$x_2 = \frac{2(\mu_1 - r)(\lambda - \mu)}{\mu_1^2 - r^2 + \lambda^2\sigma_1^2/\sigma^2}. \quad (4.15)$$

Now we conclude

$$\mathbb{O}_{V_1} \subseteq [x_1, x^*) \cap [x_2, x^*). \quad (4.16)$$

From Lemma 4.2, for $x \in \mathbb{O}_{V_2}$, it holds that $a_V(x) = -\lambda/(\sigma^2 g_2(x))$, and $b_V(x) = (-\mu_1 - r)/(x\sigma^2 g_2(x))$. Hence, $a_V(x) \leq 1$ implies $x \in [x_2^-, x_2^+]$, where

$$x_2^{\pm} = \frac{\mu_1 - K\lambda\sigma_1^2/\sigma^2}{\lambda\sigma_1^2/\sigma^2} \pm \frac{\sqrt{(\mu_1 - K\lambda\sigma_1^2/\sigma^2)^2 - \lambda\sigma_1^2/\sigma^2[\lambda - 2\mu - 2(\mu_1 - r)K + \lambda\sigma_1^2/\sigma^2K^2]}}{\lambda\sigma_1^2/\sigma^2}, \tag{4.17}$$

and $b_V(x) > 1 + K/x$ gives $x < x_2$, where x_2 is in (4.15). Now we conclude

$$\mathbb{O}_{V_2} \subseteq [x_2^-, x_2^+) \cap [0, x_2). \tag{4.18}$$

Similarly, we have

$$\mathbb{O}_{V_3} \subseteq [x_3, x_1), \tag{4.19}$$

where x_1 is in (4.14) and x_3 is given by

$$x_3 = \frac{\sqrt{(\mu - rK)^2 + (\mu_1^2 - r^2)\sigma^2/\sigma_1^2} - (\mu + \mu_1 K)}{\mu_1 + r}, \tag{4.20}$$

and

$$\mathbb{O}_{V_4} \subseteq \{(0, x_2^-) \cup (x_2^+, x^*)\} \cap (0, x_3), \tag{4.21}$$

where $x_2^{\pm}$ are in (4.17) and x_3 is in (4.20). Following a similar proof to that of Lemma 3.6, we have the following.

Lemma 4.5

Suppose V is a convex decreasing C^2 function that solves HJB equation (4.1) *on $\mathbb{O}$ with boundary conditions $V(0) = 1$ and $V(x^*) = 0$; then*

$$\begin{aligned}
\mathbb{O}_{V_1} &= [x_1, x^*) \cap [x_2, x^*) \cap \mathbb{O}, \\
\mathbb{O}_{V_2} &= [x_2^-, x_2^+] \cap (0, x_2) \cap \mathbb{O}, \\
\mathbb{O}_{V_3} &= [x_3, x_1) \cap \mathbb{O}, \\
\mathbb{O}_{V_4} &= \{(0, x_2^-) \cup (x_2^+, x^*)\} \cap (0, x_3) \cap \mathbb{O},
\end{aligned} \tag{4.22}$$

where $x_1, x_2, x_2^{\pm}, x_3, x^*$ are given by (4.14), (4.15), (4.17), (4.20), *and* (4.2), *respectively*.

We can check that inequalities $x_1 < x^*$, $x_2 < x^*$ always hold, and inequality $x_1 < x_2^+$ holds when x_2^+ exits. Thus, from Lemma 4.5, we come to the following lemma.

Lemma 4.6

Suppose V is a convex decreasing C^2 function that solves HJB equation (4.1) *on $\mathbb{O}$ with boundary conditions $V(0) = 1$ and $V(x^*) = 0$; then the partition of $\mathbb{O}$ must have one of the following five forms*:

$$\begin{array}{ccccc}
 & \mathbb{O}_{V_1} & \mathbb{O}_{V_2} & \mathbb{O}_{V_3} & \mathbb{O}_{V_4} \\
Case1 & (0, x^*) & \emptyset & \emptyset & \emptyset \\
Case2 & [x_2, x^*) & (0, x_2) & \emptyset & \emptyset \\
Case3 & [x_2, x^*) & [x_2^-, x_2) & \emptyset & (0, x_2^-) \\
Case4 & [x_1, x^*) & \emptyset & (0, x_1) & \emptyset \\
Case5 & [x_1, x^*) & \emptyset & [x_3, x_1) & (0, x_3)
\end{array}. \tag{4.23}$$

Given the exogenous parameters, we can identify the order of the threshold points x_1, x_2, x_3, $x_2^{\pm}$, etc. Thus, from Lemma 4.6, we can observe which form the solution follows. Consequently, the minimal ruin probability function and its associated optimal reinsurance-investment policy can be derived. Next we summarize and state the theorems corresponding to these five cases.

Theorem 4.1

If $x_1 \le 0$ and $x_2 \le 0$ (x_1 and x_2 are given by (4.14) and (4.15), respectively), then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} \left(\dfrac{\lambda - \mu - rx}{\lambda - \mu}\right)^{1+\lambda^2/2r\sigma^2 + (\mu_1 - r)^2/2r\sigma_1^2} & 0 < x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.24}$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} -\dfrac{\lambda}{\sigma^2 g_1(x)} & 0 < x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.25}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & 0 < x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.26}$$

According to Theorem 3.1 in Luo, Taksar, and Tsoi (2008), when $x_1 \le 0$, that is, $\lambda \le \mu + \sqrt{\mu^2 + \sigma^2/\sigma_1^2(\mu_1 - r)^2}$, the borrowed amount under the optimal policy with no borrowing constraint is bounded; thus, setting a borrowing limit higher than the bound should be of no effect on the optimal policy, and the optimal solution should be the same as in the case without borrowing limit constraint. This is indeed validated by Theorem 4.1. As we see, the solution in Theorem 4.1 is the same as the one in Theorem 3.1 of Luo et al. If $x_2 \le 0$ holds, that is, $K \ge K^*$, where

$$K^* = \frac{2(\mu_1 - r)(\lambda - \mu)}{\lambda^2\sigma_1^2/\sigma^2 + (\mu_1 - r)^2}, \tag{4.27}$$

it means that the borrowing limit K is sufficiently high; in fact, K^* is exactly the upper bound (attained as $x \to 0$) for the borrowed amount under the optimal policy with no borrowing constraint.

Theorem 4.2

If $x_2 > 0$, $x_2 \ge x_1$, and $x_2^- \le 0$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except point x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u g_2(v)\, dv\}\, du & 0 < x < x_2 \\ e^{C_2} \int_x^{x^*} \exp\{\int_{x_2}^u g_1(v)\, dv\}\, du & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.28}$$

where g_1, g_2 are given by (4.7) and (4.9), respectively, $x_2^{\pm}$, x_1, x_2, and x^* are given by (4.17), (4.14), (4.15), and (4.2), respectively, and constants C_1, C_2 are given by

$$C_1 = -\ln\left[\int_0^{x_2} \exp\left\{\int_0^u g_2(v)\, dv\right\} du + \exp\left\{\int_0^{x_2} g_2(v)\, dv\right\} \int_{x_2}^{x^*} \exp\left\{\int_{x_2}^u g_1(v)\, dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{x_2} g_2(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} -\dfrac{\lambda}{\sigma^2 g_2(x)} & 0 < x < x_2 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.29}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + K/x & 0 < x < x_2 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.30}$$

Theorem 4.3

If $x_2 > 0$, $x_2 \ge x_1$, and $x_2^- > 0$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u g_4(v)\, dv\}\, du & 0 < x < x_2^- \\ C_3 - e^{C_2} \int_{x_2^-}^x \exp\{\int_0^u g_2(v)\, dv\}\, du & x_2^- \le x < x_2 \\ e^{C_4} \int_x^{x^*} \exp\{\int_{x_2}^u g_1(v)\, dv\}\, du & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.31}$$

where g_1, g_2, and g_4 are given by (4.7), (4.9), *and* (4.13), *respectively; $x_2^{\pm}$, x_1, x_2, and x^* are given by* (4.17), (4.14), (4.15), *and* (4.2), *respectively; and constants C_1, C_2, C_3, and C_4 are given by*

$$C_1 = -\ln\left[\int_0^{x_2^-} \exp\left\{\int_0^u g_4(v)\, dv\right\} du + \exp\left\{\int_0^{x_2^-} g_4(v)\, dv\right\} \int_{x_2^-}^{x_2} \exp\left\{\int_0^u g_2(v)\, dv\right\} du \right.$$
$$\left. + \exp\left\{\int_0^{x_2^-} g_4(v)\, dv + \int_{x_2^-}^{x_2} g_2(v)\, dv\right\} \int_{x_2}^{x^*} \exp\left\{\int_{x_2}^u g_1(v)\, dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{x_2^-} g_4(v)\, dv,$$

$$C_3 = 1 - e^{C_1} \int_0^{x_2^-} \exp\left\{\int_0^u g_4(v)\, dv\right\} du,$$

$$C_4 = C_1 + \int_0^{x_2^-} g_4(v)\, dv + \int_{x_2^-}^{x_2} g_2(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_2^- \\ -\dfrac{\lambda}{\sigma^2 g_2(x)} & x_2^- \le x < x_2 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.32}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + K/x & 0 < x < x_2 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.33}$$

Theorem 4.4

If $x_1 > 0$, $x_1 > x_2$, *and* $x_3 \le 0$, *then the minimal ruin probability function* V *is a convex, decreasing, and twice continuously differentiable (except at* x^**) function given by*

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u g_3(v)\,dv\}\,du & 0 < x < x_1 \\ e^{C_2} \int_x^{x^*} \exp\{\int_{x_1}^u g_1(v)\,dv\}\,du & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.34}$$

where g_1 *and* g_3 *are given by* (4.7) *and* (4.11), *respectively;* x_1, x_2, x_3, *and* x^* *are given by* (4.14), (4.15), (4.20), *and* (4.2), *respectively; and constants* C_1, C_2 *are given by*

$$C_1 = -\ln\left[\int_0^{x_1} \exp\left\{\int_0^u g_3(v)\,dv\right\} du + \exp\left\{\int_0^{x_1} g_3(v)\,dv\right\} \int_{x_1}^{x^*} \exp\left\{\int_{x_1}^u g_1(v)\,dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{x_1} g_3(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.35}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - r}{x\sigma_1^2 g_3(x)} & 0 < x < x_1 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.36}$$

Theorem 4.5

If $x_1 > 0$, $x_1 > x_2$, *and* $x_3 > 0$, *then the minimal ruin probability function* V *is a convex, decreasing, and twice continuously differentiable (except point* x^**) function given by*

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u g_4(v)\,dv\}\,du & 0 < x < x_3 \\ C_3 - e^{C_2} \int_{x_2^-}^{x} \exp\{\int_0^u g_3(v)\,dv\}\,du & x_3 \le x < x_1 \\ e^{C_4} \int_x^{x^*} \exp\{\int_{x_1}^u g_1(v)\,dv\}\,du & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.37}$$

where g_1, g_3, *and* g_4 *are given by* (4.7), (4.11), *and* (4.13), *respectively;* x_1, x_2, x_3, *and* x^* *are given by* (4.14), (4.15), (4.20), *and* (4.2), *respectively; and constants* C_1, C_2, C_3, *and* C_4 *are given by*

$$C_1 = -\ln\left[\int_0^{x_3} \exp\left\{\int_0^u g_4(v)\,dv\right\} du + \exp\left\{\int_0^{x_3} g_4(v)\,dv\right\}\int_{x_3}^{x_1} \exp\left\{\int_0^u g_3(v)\,dv\right\} du\right.$$
$$\left. + \exp\left\{\int_0^{x_3} g_4(v)\,dv + \int_{x_3}^{x_1} g_3(v)\,dv\right\}\int_{x_1}^{x^*} \exp\left\{\int_{x_1}^u g_1(v)\,dv\right\} du\right],$$
$$C_2 = C_1 + \int_0^{x_3} g_4(v)\,dv,$$
$$C_3 = 1 - e^{C_1}\int_0^{x_3} \exp\left\{\int_0^u g_4(v)\,dv\right\} du,$$
$$C_4 = C_1 + \int_0^{x_3} g_4(v)\,dv + \int_{x_3}^{x_1} g_3(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{4.38}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + K/x & 0 < x < x_3 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_3(x)} & x_3 \le x < x_1 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{4.39}$$

Theorems 4.1–4.5 cover all the possible cases on the threshold points, and the solution of the optimal control problem is complete.

REMARK 4.1

The optimal reinsurance and investment feedback functions $a^*(x)$ and $b^*(x)$ in Theorems 4.1–4.5 decrease in x.

Notice that $g_1(x)$, $g_3(x)$ are decreasing functions on $\mathbb{O}$, and $g_2(x)$ decreases when $x < \tilde{x}$, where $\tilde{x} = (2\mu_1(\lambda - \mu) + K[\mu_1^2 - \lambda^2\sigma_1^2/\sigma^2 - 2\mu_1(\mu_1 - r)])/(\mu_1^2 + \lambda^2\sigma_1^2/\sigma^2)$ and $x_2 < \tilde{x}$; further, $a^*(x)$, $b^*(x)$ are continuous. Hence, the monotonicity follows. Note that in this case the amount of surplus in risky asset $xb^*(x)$ is not necessarily decreasing.

5. THE CASE WITH LIMITED PROPORTION OF BORROWING

In the previous section the maximum amount on borrowing K is fixed, and it does not depend on the insurer's surplus level. However, lenders may assess borrowers' borrowing capacity by considering their surplus level. In this section we consider the constraint that the borrowed proportion (borrowed amount to total surplus level) is no more than a fixed level $k > 0$; thus, the maximum amount allowed to borrow depends on the surplus level. As in the previous section, we also assume a common rate r for borrowing and lending. The HJB equation is given by

$$0 = \inf_{0\le a\le 1, -\infty<b<1+k} \{[\mu - (1-a)\lambda + (r(1-b) + b\mu_1)x]V'(x) + \frac{1}{2}(a^2\sigma^2 + b^2\sigma_1^2x^2)V''(x)\}. \tag{5.1}$$

We need only to consider the case $\mu_1 > r$, and it still holds that $V(x) = 0$ for $x \ge x^*$, where the safe level x^* is given in (4.2). Thus, we seek a solution on the interval $\mathbb{O} = (0, x^*)$. Note that functions $a_W(x)$ and $b_W(x)$ are defined in (3.5) and (4.4), respectively, and we redefine four sets that form a partition of $\mathbb{O}$ as follows:

$$\begin{aligned}\mathbb{O}_{W_1} &= \{x \in \mathbb{O} : 0 \le a_W(x) \le 1, b_W(x) \le 1 + k\},\\ \mathbb{O}_{W_2} &= \{x \in \mathbb{O} : 0 \le a_W(x) \le 1, b_W(x) > 1 + k\},\\ \mathbb{O}_{W_3} &= \{x \in \mathbb{O} : a_W(x) > 1, b_W(x) \le 1 + k\},\\ \mathbb{O}_{W_4} &= \{x \in \mathbb{O} : a_W(x) > 1, b_W(x) > 1 + k\}.\end{aligned} \tag{5.2}$$

Next we give several lemmas.

Lemma 5.1

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = g_1(x), \tag{5.3}$$

where the function g_1 is given in (4.7), *then V solves HJB equation* (5.1) *on $\mathbb{O}_{V_1}$ and vice versa.*

Lemma 5.2

Suppose V is a twice continuously differentiable decreasing, convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = \tilde{g}_2(x), \tag{5.4}$$

where

$$\tilde{g}_2(x) = \frac{-\lambda^2/\sigma^2}{\sqrt{[\lambda - \mu - (\mu_1 + \mu_1 k - rk)x]^2 + (1+k)^2\lambda^2x^2\sigma_1^2/\sigma^2} + \mu - \lambda + (\mu_1 + \mu_1 k + rk)x}, \tag{5.5}$$

then V solves HJB equation (5.1) *on $\mathbb{O}_{V_2}$ and vice versa.*

Lemma 5.3

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = g_3(x), \tag{5.6}$$

where g_3 is given by (4.11), *then V solves HJB equation* (5.1) *on $\mathbb{O}_{V_3}$ and vice versa.*

Lemma 5.4

Suppose V is a twice continuously differentiable decreasing convex function on $\mathbb{O}$. If V solves

$$\frac{V''(x)}{V'(x)} = \tilde{g}_4(x), \tag{5.7}$$

where

$$\tilde{g}_4(x) = -\frac{2[\mu + (\mu_1 k - rk + \mu_1)x]}{\sigma^2 + (1+k)^2\sigma_1^2 x^2}, \tag{5.8}$$

then V solves HJB equation (5.1) *on* $\mathbb{O}_{V_4}$ *and vice versa.*

Lemma 5.5

Suppose V is a convex decreasing C^2 function that solves HJB equation (5.1) *on* $\mathbb{O}$ *with boundary conditions* $V(0) = 1$ *and* $V(x^*) = 0$; *then*

$$\begin{aligned}
\mathbb{O}_{V_1} &= [x_1, x^*) \cap [\tilde{x}_2, x^*) \cap \mathbb{O},\\
\mathbb{O}_{V_2} &= [\tilde{x}_2^-, \tilde{x}_2^+] \cap (0, \tilde{x}_2) \cap \mathbb{O},\\
\mathbb{O}_{V_3} &= [\tilde{x}_3, x_1) \cap \mathbb{O},\\
\mathbb{O}_{V_4} &= \{(0, \tilde{x}_2^-) \cup (\tilde{x}_2^+, x^*)\} \cap (0, \tilde{x}_3) \cap \mathbb{O},
\end{aligned} \tag{5.9}$$

where x_1, x^* *are given by* (4.14), (4.2) *respectively, and*

$$\begin{aligned}
\tilde{x}_2 &= \frac{2(\mu_1 - r)(\lambda - \mu)}{\mu_1^2 - r^2 + k(\mu_1 - r)^2 + (1+k)\lambda^2\sigma_1^2/\sigma^2},\\
\tilde{x}_2^{\pm} &= \frac{\mu_1 + \mu_1 k - rk \pm \sqrt{(\mu_1 + \mu_1 k - rk)^2 - (1+k)^2\lambda(\lambda - 2\mu)\sigma_1^2/\sigma^2}}{(1+k)^2\lambda\sigma_1^2/\sigma^2},\\
\tilde{x}_3 &= \frac{\sqrt{(1+k)^2\mu^2 + (\mu_1 - r)[(1+k)^2(\mu_1 - r) + 2(1+k)r]\sigma^2/\sigma_1^2} - (1+k)\mu}{(1+k)^2(\mu_1 - r) + 2(1+k)r}.
\end{aligned} \tag{5.10}$$

Noticing that inequalities $x_1 < x^*$, $\tilde{x}_2 < x^*$, $\tilde{x}_2 > 0$ always hold, and inequality $x_1 < \tilde{x}_2^+$ holds when $\tilde{x}_2^+$ exists, from Lemma 5.5 we have the following lemma.

Lemma 5.6

Suppose V is a convex decreasing C^2 function that solves HJB equation (5.1) *on* $\mathbb{O}$ *with boundary conditions* $V(0) = 1$ *and* $V(x^*) = 0$; *then the partition of* $\mathbb{O}$ *must have one of the following four forms:*

$$\begin{array}{lcccc}
 & \mathbb{O}_{V_1} & \mathbb{O}_{V_2} & \mathbb{O}_{V_3} & \mathbb{O}_{V_4}\\
Case1 & [\tilde{x}_2, x^*) & (0, \tilde{x}_2) & \emptyset & \emptyset\\
Case2 & [\tilde{x}_2, x^*) & [\tilde{x}_2^-, \tilde{x}_2) & \emptyset & (0, \tilde{x}_2^-)\\
Case3 & [x_1, x^*) & \emptyset & (0, x_1) & \emptyset\\
Case4 & [x_1, x^*) & \emptyset & [\tilde{x}_3, x_1) & (0, \tilde{x}_3)
\end{array}. \tag{5.11}$$

Accordingly, we have the following four theorems.

Theorem 5.1

If $\tilde{x}_2 \geq x_1$ *and* $\tilde{x}_2^- \leq 0$, *then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable* (*except at* x^*) *function given by*

$$V(x) = \begin{cases} 1 - e^{C_1}\int_0^x \exp\{\int_0^u \tilde{g}_2(v)\,dv\}\,du & 0 < x < \tilde{x}_2\\ e^{C_2}\int_x^{x^*} \exp\{\int_{\tilde{x}_2}^u g_1(v)\,dv\}\,du & \tilde{x}_2 \leq x < x^*\\ 0 & x \geq x^* \end{cases}, \tag{5.12}$$

where g_1 *and* $\tilde{g}_2$ *are given by* (4.7) *and* (5.5), *respectively,* $\tilde{x}_2^{\pm}$, *are given by* (5.10); x^* *and* x_1 *are given by* (4.2) *and* (4.14), *respectively; and constants* C_1, C_2 *are given by*

$$C_1 = -\ln\left[\int_0^{\tilde{x}_2} \exp\left\{\int_0^u \tilde{g}_2(v)\,dv\right\} du \exp\left\{\int_0^{\tilde{x}_2} \tilde{g}_2(v)\,dv\right\} \int_{\tilde{x}_2}^{x^*} \exp\left\{\int_{\tilde{x}_2}^u g_1(v)\,dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{\tilde{x}_2} \tilde{g}_2(v)\,dv + \int_{\tilde{x}_2}^{x^*} g_1(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} -\dfrac{\lambda}{\sigma^2 \tilde{g}_2(x)} & 0 < x < \tilde{x}_2 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & \tilde{x}_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \quad (5.13)$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + k & 0 < x < \tilde{x}_2 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & \tilde{x}_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \quad (5.14)$$

Theorem 5.2

If $\tilde{x}_2 \ge x_1$, and $\tilde{x}_2^- > 0$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u \tilde{g}_4(v)\,dv\}\,du & 0 < x < \tilde{x}_2^- \\ C_3 - e^{C_2} \int_{\tilde{x}_2^-}^x \exp\{\int_0^u \tilde{g}_2(v)\,dv\}\,du & \tilde{x}_2^- \le x < \tilde{x}_2 \\ e^{C_4} \int_x^{x^*} \exp\{\int_{\tilde{x}_2}^u g_1(v)\,dv\}\,du & \tilde{x}_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \quad (5.15)$$

where g_1, $\tilde{g}_2$, and $\tilde{g}_4$ are given by (4.7), (5.5), *and* (5.8), *respectively; $\tilde{x}_2^{\pm}$ and $\tilde{x}_2$ are given by* (5.10); *x^* and x_1 are given by* (4.2) *and* (4.14), *respectively; and constants C_1, C_2, C_3, and C_4 are given by*

$$C_1 = -\ln\left[\int_0^{\tilde{x}_2^-} \exp\left\{\int_0^u \tilde{g}_4(v)\,dv\right\} du + \exp\left\{\int_0^{\tilde{x}_2^-} \tilde{g}_4(v)\,dv\right\} \int_{\tilde{x}_2^-}^{\tilde{x}_2} \exp\left\{\int_0^u \tilde{g}_2(v)\,dv\right\} du \right.$$
$$\left. + \exp\left\{\int_0^{\tilde{x}_2^-} \tilde{g}_4(v)\,dv + \int_{\tilde{x}_2^-}^{\tilde{x}_2} \tilde{g}_2(v)\,dv\right\} \int_{\tilde{x}_2}^{x^*} \exp\left\{\int_{\tilde{x}_2}^u g_1(v)\,dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{\tilde{x}_2^-} \tilde{g}_4(v)\,dv,$$

$$C_3 = 1 - e^{C_1} \int_0^{\tilde{x}_2^-} \exp\left\{\int_0^u \tilde{g}_4(v)\,dv\right\} du,$$

$$C_4 = C_1 + \int_0^{\tilde{x}_2^-} \tilde{g}_4(v)\,dv + \int_{\tilde{x}_2^-}^{\tilde{x}_2} \tilde{g}_2(v)\,dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < \tilde{x}_2^- \\ -\dfrac{\lambda}{\sigma^2 \tilde{g}_2(x)} & \tilde{x}_2^- \le x < \tilde{x}_2 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & \tilde{x}_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{5.16}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + k & 0 < x < \tilde{x}_2 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & \tilde{x}_2 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{5.17}$$

Theorem 5.3

If $x_1 > \tilde{x}_2$ and $\tilde{x}_3 \le 0$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u g_3(v)\, dv\}\, du & 0 < x < x_1 \\ e^{C_2} \int_x^{x^*} \exp\{\int_{x_1}^u g_1(v)\, dv\}\, du & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{5.18}$$

where g_1 and g_3 are given by (4.7) *and* (4.11), *respectively; $\tilde{x}_2$ and $\tilde{x}_3$ are given by* (5.10); *x^* and x_1 are given by* (4.2) *and* (4.14), *respectively; and constants C_1, C_2 are given by*

$$C_1 = -\ln \left[\int_0^{x_1} \exp\left\{ \int_0^u g_3(v)\, dv \right\} du \, \exp\left\{ \int_0^{x_1} g_3(v)\, dv\, dv \right\} \int_{x_1}^{x^*} \exp\left\{ \int_{x_1}^u g_1(v)\, dv \right\} du \right],$$

$$C_2 = C_1 + \int_0^{x_1} g_3(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{5.19}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} -\dfrac{\mu_1 - r}{x\sigma_1^2 g_3(x)} & 0 < x < x_1 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{5.20}$$

Theorem 5.4

If $x_1 > \tilde{x}_2$ and $\tilde{x}_3 > 0$, then the minimal ruin probability function V is a convex, decreasing, and twice continuously differentiable (except at x^*) function given by

$$V(x) = \begin{cases} 1 - e^{C_1} \int_0^x \exp\{\int_0^u \tilde{g}_4(v)\, dv\}\, du & 0 < x < \tilde{x}_3 \\ C_3 - e^{C_2} \int_{\tilde{x}_3}^x \exp\{\int_0^u g_3(v)\, dv\}\, du & \tilde{x}_3 \le x < x_1 \\ e^{C_4} \int_x^{x^*} \exp\{\int_{x_1}^u g_1(v)\, dv\}\, du & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{5.21}$$

where g_1, g_3, *and* $\tilde{g}_4$ *are given by* (4.7), (4.11), *and* (5.8), *respectively;* $\tilde{x}_2$ *and* $\tilde{x}_3$ *are given by* (5.10); x^* *and* x_1 *are given by* (4.2) *and* (4.14), *respectively; and constants* C_1, C_2, C_3, *and* C_4 *are given by*

$$C_1 = -\ln\left[\int_0^{\tilde{x}_3} \exp\left\{\int_0^u \tilde{g}_4(v)\, dv\right\} du + \exp\left\{\int_0^{\tilde{x}_3} \tilde{g}_4(v)\, dv\right\} \int_{\tilde{x}_3}^{x_1} \exp\left\{\int_0^u g_3(v)\, dv\right\} du \right.$$
$$\left. + \exp\left\{\int_0^{\tilde{x}_3} \tilde{g}_4(v)\, dv + \int_{\tilde{x}_3}^{x_1} g_3(v)\, dv\right\} \int_{x_1}^{x^*} \exp\left\{\int_{x_1}^u g_1(v)\, dv\right\} du\right],$$

$$C_2 = C_1 + \int_0^{\tilde{x}_3} \tilde{g}_4(v)\, dv,$$

$$C_3 = 1 - e^{C_1} \int_0^{x_3} \exp\left\{\int_0^u \tilde{g}_4(v)\, dv\right\} du,$$

$$C_4 = C_1 + \int_0^{\tilde{x}_3} \tilde{g}_4(v)\, dv + \int_{\tilde{x}_3}^{x_1} g_3(v)\, dv.$$

The optimal risk exposure feedback function is given by

$$a^*(x) = \begin{cases} 1 & 0 < x < x_1 \\ -\dfrac{\lambda}{\sigma^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}, \tag{5.22}$$

and the optimal investment feedback function is

$$b^*(x) = \begin{cases} 1 + k & 0 < x < \tilde{x}_3 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_3(x)} & \tilde{x}_3 \le x < x_1 \\ -\dfrac{\mu_1 - r}{x\sigma_1^2 g_1(x)} & x_1 \le x < x^* \\ 0 & x \ge x^* \end{cases}. \tag{5.23}$$

So far all the possible cases are considered, and the solution in each case is found. Under the constraint with limited proportion of borrowing, the monotonicity of the optimal feedback functions $a^*(x)$ and $b^*(x)$ in Theorems 5.1–5.4 also holds. In fact, we see that $\tilde{g}_2(x)$ decreases when $x < \hat{x}$, where $\hat{x} = 2(\lambda - \mu)(\mu_1 + \mu_1 k - rk)/((\mu_1 + \mu_1 k - rk)2 + (1 + k)^2\lambda^2\sigma_1^2/\sigma^2)$ and $\tilde{x}_2 < \hat{x}$. Further, it is easy to see that $g_1(x)$ and $g_3(x)$ are monotone on $\mathbb{O}$, and $a^*(x)$, $b^*(x)$ are continuous on $(0, \infty)$. Thus, the feedback control functions decrease.

6. ECONOMIC ANALYSIS AND CONCLUSION

In this paper we employ techniques of stochastic control to study the problem of how an insurance company should invest its surplus and buy proportional reinsurance to minimize the probability of ruin. Specifically, we study the model under three practical borrowing constraints. For each of the constraints, all the possible cases on exogenous parameters are considered, and closed-form solutions of the minimized ruin probability functions, as well as their associated optimal control policies, are provided. The borrowing constraints affect the optimal control policy only if the return rate in the risky asset μ_1 is higher than the risk-free rate(s) (α, β and r); otherwise, borrowing is not involved, and the constraints would have no effect.

Most of the results in this paper are intuitively appealing. For example, a more expensive reinsurance rate results in less reinsurance purchase. We also observe some interesting phenomena that are not financially obvious. For example, under constraint (B1) (Section 3) and when the surplus is near 0, the optimal strategy is heavily leveraged in the risky asset. More exactly, the insurance company tends to borrow a certain amount of money to invest in the risky asset together with all its surplus. This leveraging strategy at a low surplus level is also observed in Bayraktar and Young (2007) and Young (2004) when they study lifetime ruin probability minimization for an individual, and in Browne (1995, 1997) and Promislow and Young (2005) when they study ruin probability minimization for an entity without these borrowing constraints. However, leveraging at a low surplus level creates new risks, and it contradicts the idea of reducing risk in insurance. Hence, the other two borrowing constraints (B2) and (B3), which enforce lower levels of leveraging, are introduced. Another interesting observation is that when the surplus level increases, the insurance company tends to have a more conservative optimal control policy, that is, to invest more in risk-free assets and buy more reinsurance. It is well understood (see, e.g., Rolski et al. 1999, p. 15) that the proportional reinsurance is often favored at the start by smaller companies (i.e., companies with a low surplus level) to broaden their chances in underwriting policies. But the opposite is exhibited in our results.

We have provided a full consideration of the three practical borrowing constraints in this paper. To enhance the current work, one can study the borrowing constraints together with risky asset short-selling restrictions and generalize the model to the case with multiple risky assets.

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