On the Laplace Transform of the Aggregate Discounted Claims with Markovian Arrivals

Jiandong Ren*

Abstract

We present an explicit formula for the Laplace transform of the distribution of the aggregate discounted claims when interclaim times follow a Markovian arrival process. In addition, we derive explicit formulas for the first two moments and then show that the higher moments may be obtained by numerically solving a system of ordinary differential equations.

1. Introduction

This paper studies the distribution of the aggregate discounted claims in a fixed time period. We assume that claims occurring at time t are discounted by a factor $v(t)$ with $0 \leq v(t) \leq 1$. Then the aggregate discounted claims in time interval $(0, t]$ is given by

$$S(t) = \sum_{k=1}^{N(t)} X_k v(T_k), \qquad t \geq 0, \tag{1.1}$$

where $\{N(t), t \geq 0\}$ counts the number of claims in time interval $(0, t]$, $\{T_k, k \geq 1\}$ denote the time when the kth claim occurs, and $\{X_k, k \geq 1\}$ are claim size random variables.

The distribution of the aggregate discounted claims when $N(t)$ is a homogeneous Poisson process and claim sizes are identically and independently distributed has been studied extensively in the literature. See, for example, Taylor (1979), Waters (1989), Delbaen and Haezendonck (1987), Willmot (1989), and Jang (2004). For the related ruin problems, see Gerber (1971, a,b), Sundt and Teugels (1995), and Yang and Zhang (2001).

Allowing the claim rate and claim size distributions to change according to an independent continuous-time Markov chain, Kim and Kim (2007) studied the moments of the aggregate discounted claims in a Markovian random environment. When $N(t)$ is a renewal process, Léveillé and Garrido (2001a, b) presented recursive formulas for all the moments of the aggregate discounted claims. For an analysis of the tail probability of aggregate discounted claims from the point of view of the discounted random sums, see Goovaerts et al. (2005).

In this paper we assume that the claims occur according to a Markovian arrival process (see, e.g., Neuts 1979; Asmussen 2003) with representation $(\boldsymbol{\gamma}, \mathbf{D}_0, \mathbf{D}_1)$: that is, the claims occur according to a background Markov process $J(t)$ with $m < \infty$ states and initial distribution $\boldsymbol{\gamma}$. The intensity matrix of $J(t)$ is denoted by $\mathbf{D} = (d_{ij})_{i,j \in 1,2,\cdots,m}$. It is assumed to be irreducible and have limiting distribution $\boldsymbol{\pi}$. Further, $\mathbf{D}$ has the decomposition $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1$, where $\mathbf{D}_0 = (d_{0,ij})_{i,j \in 1,2,\cdots,m}$ gives the intensity of state changes without arrivals and $\mathbf{D}_1 = (d_{1,ij})_{i,j \in 1,2,\cdots,m}$ the intensity of state changes with arrivals. In addition, claims arriving when $J(t) = i$ are assumed to have probability distribution function P_i, density function p_i, kth moment $\mu_i^{(k)}$, and Laplace transform $\hat{p}_i(s) = \int_0^\infty e^{-sx} p_i(x)\, dx$.

The Markovian arrival process is very general (see, e.g., Asmussen 2003). On the one hand, it may represent a renewal process in which the interclaim times follow phase-type distributions, which is

* Jiandong Ren is an Assistant Professor in the Department of Statistical and Actuarial Sciences at the University of Western Ontario, London, Ontario N6A 5B7, Canada, jren@stats.uwo.ca.

dense in the set of distributions with nonnegative support. On the other hand, it allows for situations in which interclaim times and/or claim size random variables are dependent. For example, with $\mathbf{D}_0 = -\lambda$ and $\mathbf{D}_1 = \lambda$, it reduces to a Poisson process with rate λ; with $\mathbf{D}_0 = \mathbf{B}$ and $\mathbf{D}_1 = \mathbf{b}^\top \boldsymbol{\alpha}$, it reduces to a renewal process with the interclaim times following a phase-type distribution with representation $(\boldsymbol{\alpha}, \mathbf{B}, \mathbf{b}^\top)$; with $\mathbf{D}_0 = Q - diag(\lambda_i)$ and $\mathbf{D}_1 = diag(\lambda_i)$, where $diag(\lambda_i)$ denotes a diagonal matrix with λ_i on the diagonal, it reduces to a Markov modulated Poisson process with rate λ_i, $i \in \{1, 2, \ldots, m\}$ when an independent Markov process with state space $\{1, 2, \ldots, m\}$ and infinitesimal generator Q is in state i.

With these assumptions, we next derive an explicit formula for the Laplace transform of the distribution of the aggregate discounted claims through a martingale argument. We then provide some formulas for the moments. These results extend those obtained by Léveillé and Garrido (2001a, b) for the discounted renewal process and Kim and Kim (2007) for the discounted risk process in a Markovian environment.

2. THE LAPLACE TRANSFORM OF THE AGGREGATE DISCOUNTED CLAIMS

Consider the Markov process $\{S(t), J(t), t\}$. Its generator acting on a function $f(S(t), J(t), t)$ belonging to its domain is given by

$$\begin{aligned}\mathscr{A}f(S(t), i, t) &= \frac{\partial f}{\partial t} + \int_0^\infty \left(\sum_{k=1}^n d_{1,ik}(f(S(t) + v(t)y, k, t) - f(S(t), i, t))\right) p_i(y)\, dy \\ &\quad + \sum_{k\neq i} d_{0,ik}(f(S(t), k, t) - f(S(t), i, t)) \\ &= \frac{\partial f}{\partial t} + \int_0^\infty \left(\sum_{k=1}^n d_{1,ik} f(S(t) + v(t)y, k, t)\right) p_i(y)\, dy \\ &\quad + \sum_{k=1}^n d_{0,ik} f(S(t), k, t), \qquad i = 1, \ldots, m. \end{aligned} \tag{2.1}$$

For $j = 1, 2, \ldots, m$, let $g_j(S(t), J(t), t) = e^{-\xi S(t)} I(J(t) = j)$, where ξ is a positive real number and $I(\cdot)$ is an indicator function. Then the infinitesimal generator acting on it is

$$\begin{aligned}\mathscr{A}g_j(S(t), i, t) &= \int_0^\infty d_{1,ij}(e^{-\xi(S(t)+v(t)y)}) p_i(y)\, dy + d_{0,ij} e^{-\xi S(t)}, \\ &= (d_{1,ij}\hat{p}_i(\xi v(t)) + d_{0,ij}) e^{-\xi S(t)}, \qquad i = 1, \ldots, m. \end{aligned} \tag{2.2}$$

By Dynkin's formula (see, e.g., Rolski et al. 1999, p. 442), the processes

$$M_j(t) = g_j(S(t), J(t), t) - g_j(S(0), J(0), 0) - \int_0^t \mathscr{A}g_j(S(s), J(s), s)\, ds, \qquad j = 1, 2, \ldots, m, \tag{2.3}$$

are martingales with zero means. In differential terms, we write

$$dg_j(S(t), J(t), t) = dM_j(t) + (\mathscr{A}g_j)(S(t), J(t), t)\, dt, \qquad j = 1, 2, \ldots, m, \tag{2.4}$$

with the initial conditions $g_j(S(0), J(0), 0) = I(J(0) = j)$.

Denote by $\mathbb{E}_i$ the conditional expectation given $\{J(0) = i\}$. For $i, j = 1, 2, \ldots, m$, let

$$\begin{aligned}L_{i,j}(\xi, t) &= \mathbb{E}_i[g_j(S(t), J(t), t)] \\ &= \mathbb{E}_i[e^{-\xi S(t)} I(J(t) = j)]. \end{aligned} \tag{2.5}$$

Then by conditioning on $J(0) = i$ and taking expectations on both sides of (2.4), we obtain

$$\begin{aligned}\frac{\partial L_{i,j}(\xi, t)}{\partial t} &= \mathbb{E}_i[(d_{1,J(t)j}\hat{p}_{J(t)}(\xi v(t)) + d_{0,J(t)j})e^{-\xi S(t)}] \\ &= \mathbb{E}_i\left[\sum_{k=1}^{m}(I(J(t)=k)(d_{1,kj}\hat{p}_k(\xi v(t)) + d_{0,kj})e^{-\xi S(t)})\right] \\ &= \sum_{k=1}^{m} L_{i,k}(\xi, t)(d_{1,kj}\hat{p}_k(\xi v(t)) + d_{0,kj}), \qquad i, j = 1, 2, \ldots, m, \end{aligned} \tag{2.6}$$

with initial conditions $L_{i,j}(\xi, 0) = 1$ if $i = j$ and $L_{i,j}(\xi, 0) = 0$ otherwise.

Let $\mathbf{L}(\xi, t)$ denote the matrix with $L_{i,j}(\xi, t)$ being its ij-th element. Let $\Delta_{\hat{p}}(\xi, t)$ denote the diagonal matrix $diag(\hat{p}_i(\xi v(t)))$. Then (2.6) may be written as

$$\frac{\partial \mathbf{L}(\xi, t)}{\partial t} = \mathbf{L}(\xi, t)(\Delta_{\hat{p}}(\xi, t)\mathbf{D}_1 + \mathbf{D}_0) \tag{2.7}$$

with initial condition $\mathbf{L}(\xi, 0) = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix.

Equation (2.7) has the solution

$$\mathbf{L}(\xi, t) = \exp\left(\int_0^t \Delta_{\hat{p}}(\xi, s)ds\ \mathbf{D}_1 + \mathbf{D}_0 t\right). \tag{2.8}$$

With the distribution of $J(0)$ being $\boldsymbol{\gamma}$, this leads to the following result.

Theorem 2.1

Let $\mathbb{E}_\gamma$ denote the conditional expectation conditional on $J(0)$ distributed according to $\boldsymbol{\gamma}$. Let $l(\xi, t) = \mathbb{E}_\gamma[\exp(-\xi S(t))]$ denote the Laplace transform of the distribution of $S(t)$. Then it is given by

$$l(\xi, t) = \boldsymbol{\gamma} \exp\left[\int_0^t (\Delta_{\hat{p}}(\xi, s)\mathbf{D}_1 + \mathbf{D}_0)ds\right]\mathbf{e}^\top, \tag{2.9}$$

where $\mathbf{e}^\top$ is an m-dimensional column vector of ones. ■

Theorem 2.1 generalizes equation (13) in Taylor (1979) and equation (17) in Jang (2004).

One remark is due here. Recall that $N(t)$ counts the number of claims in $(0, t]$. Its probability-generating function is given by (see, e.g, Latouche and Ramaswami, 1999, p. 78)

$$\begin{aligned} P(z, N(t)) &= \sum_{k\geq 0} z^k \mathbb{P}_\gamma(N(t) = k) \\ &= \boldsymbol{\gamma} \exp[(\mathbf{D}_0 + z\mathbf{D}_1)t]\mathbf{e}^\top. \end{aligned} \tag{2.10}$$

It follows that the Laplace transform of the distribution of $S(t)$ can be rewritten as

$$l(\xi, t) = P\left(\frac{1}{t}\int_0^t \Delta_{\hat{p}}(\xi, s)\, ds, N(t)\right), \tag{2.11}$$

which is the probability-generating function of $N(t)$ evaluated at $1/t \int_0^t \Delta_{\hat{p}}(\xi, s)\, ds$.

Direct inversion formulas for the Laplace transform $l(\xi, t)$ exist for only a few cases. However, it may be numerically inverted to obtain the distribution of the aggregated discounted claims.

In the next section, we will focus on the moments of the aggregate discounted claims. We show that all the moments may be obtained by recursively solving a system of linear differential equations. In addition, we present explicit formulas for the first two moments of the aggregate discounted claims.

3. The Moments of the Aggregate Discounted Claims

The nth moment of $S(t)$ may be obtained by differentiating its Laplace transform $l(\xi, t)$ n times. However, direct differentiation of (2.9) with respect to ξ is difficult. Therefore, we evaluate the moments through the differential equations (2.7).

Let $\mathbf{L}_\xi^{(n)}(\xi, t)$ denote $\partial \mathbf{L}^n(\xi, t)/\partial \xi^n$, with the understanding that $\mathbf{L}_\xi^{(0)}(\xi, t) = \mathbf{L}(\xi, t)$. Let $\Delta_{\mu^{(k)}} = diag(\mu_i^{(k)})$. We have by differentiating both sides of (2.7) with respect to ξ successively and setting $\xi = 0$

$$\frac{d\mathbf{L}_\xi^{(1)}(0, t)}{dt} = \mathbf{L}_\xi^{(1)}(0, t)\mathbf{D} - v(t)\mathbf{L}(0, t)\Delta_{\mu^{(1)}}\mathbf{D}_1, \tag{3.1}$$

and for all $n \geq 2$

$$\frac{d\mathbf{L}_\xi^{(n)}(0, t)}{dt} = \mathbf{L}_\xi^{(n)}(0, t)\mathbf{D} + \sum_{k=0}^{n-1} \binom{n}{k} (-v(t))^{n-k}\mathbf{L}_\xi^{(k)}(0, t)\Delta_{\mu^{(n-k)}}\mathbf{D}_1. \tag{3.2}$$

The latter equation can easily be verified by induction.

For $n \geq 0$, let

$$\mathbf{m}_{\gamma,n}(t) = \boldsymbol{\gamma}\mathbf{L}_\xi^{(n)}(0, t) \tag{3.3}$$

be a vector with its ith element representing

$$(-1)^n\mathbb{E}_\gamma[S^n(t)I(J(t) = i)].$$

Then left-multiplying (3.1) and (3.2) by $\boldsymbol{\gamma}$ yields

$$\frac{d\mathbf{m}_{\gamma,1}(t)}{dt} = \mathbf{m}_{\gamma,1}(t)\mathbf{D} - v(t)\mathbf{m}_{\gamma,0}(t)\Delta_{\mu^{(1)}}\mathbf{D}_1 \tag{3.4}$$

and for all $n \geq 2$,

$$\frac{d\mathbf{m}_{\gamma,n}(t)}{dt} = \mathbf{m}_{\gamma,n}(t)\mathbf{D} + \sum_{k=0}^{n-1} \binom{n}{k} (-v(t))^{n-k}\mathbf{m}_{\gamma,k}(t)\Delta_{\mu^{(n-k)}}\mathbf{D}_1. \tag{3.5}$$

Thus, for $k \geq 1$, the vector $\mathbf{m}_{\gamma,k}(t)$ may be found by numerically solving the above system of differential equations, and then the moments of the aggregate discounted claims are given by

$$\begin{aligned} m_{\gamma,n}(t) &= \mathbb{E}_\gamma[S^n(t)] \\ &= (-1)^n\mathbf{m}_{\gamma,n}(t)\mathbf{e}^\top. \end{aligned} \tag{3.6}$$

In the following, we derive the explicit formulas for the first two moments. As will be seen, the results shed new lights on the nature of the aggregate discounted claim process.

3.1 The First Moment of the Aggregate Discounted Claims

First, from (2.8), we have

$$\mathbf{L}(0, t) = e^{\mathbf{D}t}. \tag{3.7}$$

Then, by left- and right-multiplying (3.1) $\boldsymbol{\gamma}$ and $\mathbf{e}^\top$, respectively, we obtain

$$\begin{aligned}\frac{dm_{\gamma,1}(t)}{dt} &= v(t)\boldsymbol{\gamma}\mathbf{L}(0, t)\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top \\ &= v(t)\boldsymbol{\gamma}e^{\mathbf{D}t}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top. \end{aligned} \tag{3.8}$$

Therefore,

$$m_{\gamma,1}(t) = \int_0^t v(s)\{\boldsymbol{\gamma}e^{\mathbf{D}s}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top\}\, ds, \tag{3.9}$$

which may be evaluated by, for example, considering Jordan canonical form of the matrix **D.**

Observe that the second factor in the integration in (3.9),

$$\boldsymbol{\gamma}e^{\mathbf{D}s}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top\, ds,$$

represents the expected amount of the claim occurring during a small time interval $(s, s + ds)$. Summing up the discounted individual *contributions* yields the expected aggregate discounted claims. Indeed, this observation alludes an illustrative proof of the formula for the first moment. A similar relationship for all the central moments of the aggregate discounted claims was pointed out in equation (14) of Taylor (1979) for the discounted compound Poisson processes. However, it is true only for the first moment in our case.

Formula (3.9) is particularly interesting when $\boldsymbol{\gamma} = \boldsymbol{\pi}$ so that $N(t)$ is stationary, in which case the first moment is simply given by

$$\begin{aligned} m_{\pi,1}(t) &= \mathbb{E}_{\pi}[S(t)] \\ &= \int_0^t v(s)\, ds(\boldsymbol{\pi}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top). \end{aligned} \tag{3.10}$$

Note that the last term of (3.10), $\boldsymbol{\pi}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top$, may be interpreted as the equilibrium claim intensity. Replacing it with the constant Poisson claim intensity yields formula (14) in Taylor (1979).

3.1.1 The First Moment of the Aggregate Discounted Claims with Constant Force of Interest

The most commonly used discounting function is the exponential discounting function $v(t) = e^{-\delta t}$, where δ is the constant force of interest. With this discounting function, equation (3.9) conveniently has the explicit form

$$\begin{aligned} m_{\gamma,1}(t) &= \int_0^t e^{-\delta s}\{\boldsymbol{\gamma}e^{\mathbf{D}s}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top\}\, ds, \\ &= \boldsymbol{\gamma}(e^{(\mathbf{D}-\delta\mathbf{I})t} - \mathbf{I})(\mathbf{D} - \delta\mathbf{I})^{-1}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top. \end{aligned} \tag{3.11}$$

Note that the matrix **D** is stochastic with zero being an eigenvalue, and all other eigenvalues have negative real parts. Therefore, matrix $\mathbf{D}-\delta\mathbf{I}$ is substochastic with all its eigenvalues having negative real parts, and so it is invertible.

Also note that equation (3.11) reduces to equation (9) of Kim and Kim (2007) for the discounted risk process in a Markovian environment.

Asymptotically, since

$$\lim_{t\to\infty} e^{(\mathbf{D}-\delta\mathbf{I})t} = \mathbf{0},$$

where **0** is a zero matrix, taking the limit as t goes to infinity in (3.11) yields

$$\lim_{t\to\infty} m_{\gamma,1}(t) = \boldsymbol{\gamma}(\delta\mathbf{I} - \mathbf{D})^{-1}\Delta_{\mu(1)}\mathbf{D}_1\mathbf{e}^\top. \tag{3.12}$$

In addition, it is obviously from (3.10) that in the stationary case

$$\lim_{t\to\infty} m_{\boldsymbol{\pi},1}(t) = \frac{1}{\delta}\,\boldsymbol{\pi}\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^{\top}. \tag{3.13}$$

3.1.2 Examples of the First Moment

This example shows that in some simple cases the first moment has more explicit expressions. Because analytical expressions for the first moment of the discounted risk process in a two-state Markovian environment are derived in Kim and Kim (2007), we present here only an example for the first moment of the discounted renewal process.

Assume that the claim size random variables are i.i.d. and have finite mean μ. Each of the interclaim time variables is the sum of two independent, exponentially distributed random variables with rates λ_1 and λ_2. Consequently,

$$\mathbf{D}_0 = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & -\lambda_2 \end{pmatrix}, \qquad \mathbf{D}_1 = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}, \qquad \text{and } \mathbf{D} = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

In the ordinary renewal case, $\boldsymbol{\gamma} = [1, 0]$. Plugging these values into equation (3.11) gives an explicit formula for the first moment of the aggregated discounted claims:

$$m_{\gamma,1}(t) = \lambda_1\lambda_2\mu\left[\frac{(\lambda_1+\lambda_2)(1-e^{-\delta t}) + \delta(e^{-(\delta+\lambda_1+\lambda_2)t} - e^{-\delta t})}{\delta(\lambda_1+\lambda_2)(\delta+\lambda_1+\lambda_2)}\right]. \tag{3.14}$$

As a check, setting $\lambda_1 = \lambda_2 = 100$, $\delta = 0.05$, and $\mu = 1$, we show some numerical values of the first moment of the aggregated discounted risk process at some values of t in Table 1. The results are identical to those in Table 2 of Léveillé and Garrido (2001b) for $n = 1$.

In the stationary case, since $\boldsymbol{\pi} = [\lambda_2/(\lambda_1+\lambda_2), \lambda_1/(\lambda_1+\lambda_2)]$, equation (3.10) gives

$$m_{\boldsymbol{\pi},1}(t) = \frac{1-e^{-\delta t}}{\delta}\,\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\,\mu. \tag{3.15}$$

With the same parameter values, we calculate the first moment for the stationary case in Table 1. Again, the numbers are identical to the results in Table 4 of Léveillé and Garrido (2001b) for $n = 1$.

3.2 The Second Moment of the Aggregate Discounted Claims

More calculations give the second moment of $S(t)$. Right-multiplying (3.1) by the matrix $e^{-\mathbf{D}t}$ and rearranging, we have

$$\frac{d}{dt}(\mathbf{L}_{\xi}^{(1)}(0, t)e^{-\mathbf{D}t}) = -v(t)\mathbf{L}(0, t)\Delta_{\mu^{(1)}}\mathbf{D}_1 e^{-\mathbf{D}t}. \tag{3.16}$$

Thus,

$$\mathbf{L}_{\xi}^{(1)}(0, t) = -\int_0^t v(s)e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1 e^{\mathbf{D}(t-s)}\,ds. \tag{3.17}$$

Table 1
The First Moment ($\lambda_1 = \lambda_2 = 100$)

	$t = 1$	$t = 10$	$t = \infty$
Ordinary	48.52	393.22	999.75
Stationary	48.77	393.47	1,000

With $n = 2$, left- and right-multiplying (3.2) by γ and $\mathbf{e}^\top$, respectively, we have

$$\frac{dm_{\gamma,2}(t)}{dt} = -2v(t)\gamma\mathbf{L}_\xi^{(1)}(0, t)\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^\top + (v(t))^2\gamma\mathbf{L}_\xi^{(0)}(0, t)\Delta_{\mu^{(2)}}\mathbf{D}_1\mathbf{e}^\top. \tag{3.18}$$

Applying (3.7) and (3.17) to this and integrating yields

$$m_{\gamma,2}(t) = 2\boldsymbol{\gamma}\int_0^t\int_0^x v(s)v(x)e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1e^{\mathbf{D}(x-s)}\,ds\,dx\,\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^\top + \boldsymbol{\gamma}\int_0^t (v(s))^2e^{\mathbf{D}s}\,ds\,\Delta_{\mu^{(2)}}\mathbf{D}_1\mathbf{e}^\top. \tag{3.19}$$

Equation (3.19) is in general difficult to evaluate analytically. However, as will be shown in the following subsection, with $v(t) = e^{-\delta t}$, an explicit formula can be found.

3.2.1 The Second Moment of the Discounted Aggregate Claims with Constant Force of Interest

With exponential discounting function, the double integral in (3.19) can be evaluated further by changing the order of integration as follows:

$$\begin{aligned}
&\int_0^t\int_0^x v(s)v(x)e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1e^{\mathbf{D}(x-s)}\,ds\,dx\\
&= \int_0^t v(s)e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1e^{\mathbf{D}(-s)}\int_s^t v(x)e^{\mathbf{D}x}\,dx\,ds\\
&= \int_0^t e^{-\delta s}e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1e^{\mathbf{D}(-s)}[e^{(\mathbf{D}-\delta\mathbf{I})t} - e^{(\mathbf{D}-\delta\mathbf{I})s}](\mathbf{D} - \delta\mathbf{I})^{-1}\,ds\\
&= e^{-\delta t}\int_0^t e^{-\delta s}e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1e^{\mathbf{D}(t-s)}\,ds\,(\mathbf{D} - \delta\mathbf{I})^{-1}\\
&\quad - \int_0^t e^{-2\delta s}e^{\mathbf{D}s}\Delta_{\mu^{(1)}}\mathbf{D}_1\,ds\,(\mathbf{D} - \delta\mathbf{I})^{-1}\\
&= -e^{-\delta t}\mathbf{L}_\xi^{(1)}(0, t)\,(\mathbf{D} - \delta\mathbf{I})^{-1}\\
&\quad - [e^{(\mathbf{D}-2\delta\mathbf{I})t} - \mathbf{I}]\,(\mathbf{D} - 2\delta\mathbf{I})^{-1}\,\Delta_{\mu^{(1)}}\mathbf{D}_1\,(\mathbf{D} - \delta\mathbf{I})^{-1}.
\end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) yields

$$\begin{aligned}
m_{\gamma,2}(t) &= (-2e^{-\delta t}\boldsymbol{\gamma}\mathbf{L}_\xi^{(1)}(0, t) - 2\boldsymbol{\gamma}[e^{(\mathbf{D}-2\delta\mathbf{I})t} - \mathbf{I}](\mathbf{D} - 2\delta\mathbf{I})^{-1}\Delta_{\mu^{(1)}}\mathbf{D}_1)(\mathbf{D} - \delta\mathbf{I})^{-1}\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^\top\\
&\quad + \boldsymbol{\gamma}[e^{(\mathbf{D}-2\delta\mathbf{I})t} - \mathbf{I}](\mathbf{D} - 2\delta\mathbf{I})^{-1}\Delta_{\mu^{(2)}}\mathbf{D}_1\mathbf{e}^\top.
\end{aligned} \tag{3.21}$$

The expression is simplified in the stationary case with $\boldsymbol{\gamma} = \boldsymbol{\pi}$. Some algebra applied to (3.21) leads to

$$\begin{aligned}
m_{\pi,2}(t) &= 2e^{-\delta t}\boldsymbol{\pi}\Delta_{\mu^{(1)}}\mathbf{D}_1(\mathbf{D} + \delta\mathbf{I})^{-1}\,[e^{\mathbf{D}t} - e^{-\delta t}\mathbf{I}](\mathbf{D} - \delta\mathbf{I})^{-1}\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^\top\\
&\quad - 2\,\frac{1 - e^{-2\delta t}}{2\delta}\,\boldsymbol{\pi}\Delta_{\mu^{(1)}}\mathbf{D}_1(\mathbf{D} - \delta\mathbf{I})^{-1}\,\Delta_{\mu^{(1)}}\mathbf{D}_1\mathbf{e}^\top\\
&\quad + \frac{1 - e^{-2\delta t}}{2\delta}\,\boldsymbol{\pi}\Delta_{\mu^{(2)}}\mathbf{D}_1\mathbf{e}^\top.
\end{aligned} \tag{3.22}$$

Table 2
The Second Moment ($\lambda_1 = \lambda_2 = 100$)

	$t = 1$	$t = 10$	$t = \infty$
Ordinary	2,425.43	155,095.36	1,000,249.94
Stationary	2,450.06	155,292.28	1,000,750.063

Asymptotically, letting $t \to \infty$ in (3.21) yields

$$\lim_{t\to\infty} m_{\gamma,2}(t) = 2\boldsymbol{\gamma}(\mathbf{D} - 2\delta\mathbf{I})^{-1}\,\Delta_{\mu(1)}\mathbf{D}_1\,(\mathbf{D} - \delta\mathbf{I})^{-1}\,\Delta_{\mu(1)}\mathbf{D}_1\,\mathbf{e}^{\top} - \boldsymbol{\gamma}(\mathbf{D} - 2\delta\mathbf{I})^{-1}\,\Delta_{\mu(2)}\mathbf{D}_1\mathbf{e}^{\top}. \quad (3.23)$$

If $\boldsymbol{\gamma} = \boldsymbol{\pi}$ so that $N(t)$ is stationary, (3.13) simplifies to

$$\lim_{t\to\infty} m_{\pi,2}(t) = \frac{\boldsymbol{\pi}\Delta_{\mu(2)} - 2\boldsymbol{\pi}\Delta_{\mu(1)}\mathbf{D}_1\,(\mathbf{D} - \delta\mathbf{I})^{-1}\,\Delta_{\mu(1)}}{2\delta}\,\mathbf{D}_1\mathbf{e}^{\top}. \quad (3.24)$$

3.2.2 Examples of the Second Moment

Again, for some simple cases as in the example in Section 3.1.2, it is possible to write (3.21) and (3.22) more explicitly. In particular, for this case we pursued an explicit expression of the second moment with the MATLAB symbolic computation toolbox. However, the resulting expressions are quite long, and we will not present them here. Instead, we apply them only to calculate the second moment and show numerical numbers in Table 2. The results are exactly the same as the values in Tables 2 and 4 of Léveillé and Garrido (2001b) for $n = 2$.

4. CONCLUSIONS

We have derived an explicit formula for the Laplace transform of the distribution of the aggregate discounted claims when claims follow a Markovian arrival process. Some explicit formulas for the first two moments are also provided. These results extend those obtained by Léveillé and Garrido (2001a, b) for the discounted renewal process and by Kim and Kim (2007) for the discounted risk process in a Markovian environment.

In addition to calculating the first two moments, one may numerically invert the Laplace transform to obtain the distribution of the aggregate discounted claims or numerically solve the system of differential equations (3.6) to obtain the higher moments. Future research may investigate the ruin probabilities under the framework of this model.

5. ACKNOWLEDGMENTS

This research was funded by a grant from the Natural Sciences and Engineering Council of Canada. The author is very grateful to the anonymous referees for their constructive comments. Thanks also go to Professor José Garrido for his helpful comments on the paper when an earlier version was presented in the 11th IME Congress in Piraeus, Greece, in 2007.

REFERENCES

ASMUSSEN, SØREN. 2003. *Applied Probability and Queues*. New York: Springer.

DELBAEN, F., AND J. HAEZENDONCK. 1987. Classical Risk Theory in an Economic Environment. *Insurance: Mathematics and Economics* 6: 85–116.

GERBER, HANS U. 1971a. The Discounted Central Limit Theorem and Its Berry-Esseen Analogue. *Annals of Mathematical Statistics* 40(1): 389–92.

———. 1971b. Der Einfluss von Zins auf die Ruinwahrscheinlichkeit. *Bulletin of the Swiss Association of Actuaries* 71(1): 63–70.

GOOVAERTS, MARC, ROB KAAS, ROGER LAEVEN, QIHE TANG, AND RALUCA VERNIC. 2005. The Tail Probability of Discounted Sums of Pareto-like Losses in Insurance. *Scandinavian Actuarial Journal* 446–61.

JANG, JI-WOOK. 2004. Martingale Approach for Moments of Discounted Aggregate Claims. *Journal of Risk and Insurance* 71(2): 201–11.

KIM, BARA, AND HWA-SUNG KIM. 2007. Moments of Claims in a Markovian Environment. *Insurance: Mathematics and Economics* 40(3): 485–97.

LATOUCHE, G., AND V. RAMASWAMI. 1999. *Introduction to Matrix Analytic Methods in Stochastic Modeling.* Philadelphia: SIAM.

LÉVEILLÉ, GHISLAIN, AND JOSÉ GARRIDO. 2001a. Moments of Compound Renewal Sums with Discounted Claims. *Insurance: Mathematics and Economics* 28(2): 217–31.

———. 2001b. Recursive Moments of Compound Renewal Sums with Discounted Claims. *Scandinavian Actuarial Journal* 98–110.

NEUTS, MARCEL F. 1979. A Versatile Markovian Point Process. *Journal of Applied Probability* 16(4): 764–79.

ROLSKI, TOMASZ, HANSPETER SCHMIDLI, VOLKER SCHMIDT, AND JOZEF TEUGELS. 1999. *Stochastic Processes for Insurance and Finance.* Chichester: Wiley.

SUNDT, BJØRN, AND JOZEF L. TEUGELS. 1995. Ruin Estimates under Interest Force. *Insurance: Mathematics and Economics* 16: 7–22.

TAYLOR, GREGORY C. 1979. Probability of Ruin under Inflationary Conditions or under Experience Rating. *ASTIN Bulletin* 10: 149–62.

WATERS, H. 1989. Probability of Ruin for a Risk Process with Claims Cost Inflation. *Scandinavian Actuarial Journal* 148–64.

WILLMOT, GORDON E. 1989. The Total Claims Distribution under Inflationary Conditions. *Scandinavian Actuarial Journal* 1–12.

YANG, HAILIANG, AND LIHONG ZHANG. 2001. On the Distribution of Surplus Immediately after Ruin under Interest Force. *Insurance: Mathematics and Economics* 29: 247–55.

DISCUSSION

ELIAS S. W. SHIU*

Dr. Ren is to be congratulated for this elegant paper. Many actuarial students would have seen a special case of (1.1),

$$S(t) = \sum_{k=1}^{N(t)} X_k e^{-\delta T_k}, \tag{D.1}$$

in Example 5.19 of Ross (2003, p. 311), where $\{N(t)\}$ is a Poisson process and $\{X_k\}$ are i.i.d. random variables independent of the Poisson process. Until 2007 various editions of Ross's book had been on the Society of Actuaries' examination syllabus.

Ross (1970) is a more advanced book. Exercise 9 at the end of its Chapter 2, "The Poisson Process," asks for the *characteristic function* of

$$S(t) = \sum_{k=1}^{N(t)} g(X_k, T_k), \tag{D.2}$$

which generalizes (D.1). To solve this problem, we can use the following result for Poisson processes. Given that $N(t) = n$, the n arrival times $T_1, T_2, \ldots, T_n$ have the same distribution as the *order statistics* corresponding to n independent random variables uniformly distributed on the interval $(0, t)$. In other words, under the condition that n claims have occurred in $(0, t)$, the times $T_1, T_2, \ldots, T_n$ at which claims occur, considered as unordered random variables, are distributed independently and uniformly in $(0, t)$. It follows that, conditional on $N(t) = n$, the random variable $S(t)$ is the sum of n independent random variables, each having the same distribution as

$$g(X, tU),$$

where U is a random variable uniformly distributed on the interval (0, 1) and independent of X. Thus, $S(t)$ has a compound Poisson distribution. With the definition

* Elias S. W. Shiu, ASA, PhD, is Principal Financial Group Foundation Professor of Actuarial Science in the Department of Statistics and Actuarial Science, University of Iowa, Iowa City, IA 52242-1409, shiu@stat.uiowa.edu.

$$\phi_Y(r) = \mathrm{E}[e^{irY}] \tag{D.3}$$

for random variable Y, the characteristic function of $S(t)$ is

$$\phi_{S(t)}(r) = \exp\{\lambda t[\phi_{g(X,tU)}(r) - 1]\}. \tag{D.4}$$

REFERENCES

ROSS, SHELDON M. 1970. *Applied Probability Models with Optimization Applications*. San Francisco: Holden-Day. Reprint, Mineola, NY: Dover, 1992.

———. 2003. *Introduction to Probability Models*. 8th edition. San Diego: Academic Press.

Discussions on this paper can be submitted until October 1, 2008. The author reserves the right to reply to any discussion. Please see the Submission Guidelines for Authors on the inside back cover for instructions on the submission of discussions.