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RECURSIVE CALCULATION OF THE DIVIDEND MOMENTS IN A MULTI-THRESHOLD RISK MODEL

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ABSTRACT

In this article, we consider the class of risk models with Markovian claim arrivals studied by Badescu et al. (2005) and Ramaswami (2006), among others. Under a multi-threshold dividend structure, we develop a recursive algorithm for the calculation of the moments of the discounted dividend payments before ruin. Capitalizing on the connection between an insurer's surplus process and its corresponding fluid flow process, our approach generalizes results obtained by Albrecher and Hartinger (2007) and Zhou (2006) in the framework of the classical compound Poisson risk model (with phase-type claim sizes). Contrary to the traditional analysis of the discounted dividend payments in risk theory, we develop a sample-path-analysis procedure that allows the determination of these moments with or without ruin occurrence (separately). Numerical examples are then considered to illustrate our main results and show the contribution of each component to the moments of the discounted dividend payments.

1. INTRODUCTION

In recent years surplus processes with different dividend strategies have been considered by many authors in ruin theory. For the so-called *single threshold dividend structure*, the analysis of the Gerber-Shiu discounted penalty function and/or the moments of the discounted dividend payments before ruin has been performed by Lin and Pavlova (2006) for the Cramer-Lundberg risk model, by Albrecher et al. (2005) and Li and Garrido (2004) for a subclass of Sparre Andersen risk models, and by Ahn et al. (2007) and Badescu et al. (2007a) for the more general class of risk models with Markovian claim arrivals. See references therein for additional papers on surplus processes with dividend payments.

In the context of a *multi-threshold dividend structure*, the class of surplus processes subject to a ruin theory analysis has been limited so far to the Cramer-Lundberg risk model. Lin and Sendova (2007), Albrecher and Hartinger (2007), and Zhou (2006) performed the analysis of some ruin-related quantities in the framework of the classical compound Poisson risk model. Using a similar approach, it is reasonable to believe that their results can be generalized to a subclass of Sparre Andersen risk models, although it is expected that the approach will likely result in a fairly cumbersome analysis of the ruin-related quantities of interest.

Recently, relying on a matrix analytic approach, Badescu et al. (2007b) have significantly enlarged the class of risk processes for which some ruin-related quantities can be analyzed in a multi-threshold dividend setting. Based on the connection between an insurer's surplus process and its corresponding fluid flow process, Badescu et al. (2007b) derived some recursive algorithms to compute various ruin-related quantities, such as the Laplace transform of the trivariate density of the time, surplus, and deficit of ruin, in a fairly general class of risk processes with Markovian arrivals. The use of well-known quantities developed in the fluid flow literature is the key to ensure that the ensuing analysis is kept

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at a manageable level. We recall that risk models with Markovian claim arrivals allow for a completely dependent structure between the claim sizes and the interclaim times (see, e.g., Badescu et al. (2005) and Ramaswami (2006)) and thus moves a long way from the stringent assumptions underlying the Cramer-Lundberg risk model and the Sparre Andersen risk model.

In this article, we consider the same class of Markovian risk models as in Badescu et al. (2007b) and derive a recursive algorithm to compute the higher-order moments of the discounted dividend payments. Traditionally these moments were computed without the distinction of whether ruin occurs or not. However, our approach allows us to calculate the moments of the discounted dividends with or without ruin occurrence (separately). The determination of the Laplace transform of an infinite buffer busy period, $\Psi(\delta)$ (see, e.g., Ahn and Ramaswami (2005) and Latouche and Ramaswami (1999) for its calculation) is central to our analysis and replaces the role played by the roots of the well-known generalized Lundberg equation in the traditional approach (see Badescu and Breuer (2007)). This article also constitutes an extension of Badescu et al. (2007b) in which the expected discounted dividend payments have been derived. As discussed next, the approach leading to the calculation of the higher-order moments significantly differs from the calculation of the first moment.

In Section 2 we introduce the class of surplus processes of interest and then define the equivalent class of fluid flow processes. A review of some important passage times in the fluid flow literature follows. The main results are derived in Section 3, where recursive algorithms to compute the moments of the discounted dividend payments before ruin are derived. The calculation of these moments is performed via the development of two recursive algorithms (with or without ruin occurrence). Numerical examples are then considered in Section 4 to show the contribution of each component to the moments of the discounted dividend payments.

2. PRELIMINARIES

2.1 Surplus Processes

We consider a surplus process with $n + 1$ layers of length b_i ($i = 1, \ldots, n + 1$) with $b_{n+1} = \infty$. Let $b_1 + \cdots + b_i$ be the ith barrier level ($i = 1, 2, \ldots, n + 1$). We refer to $[0, b_1)$ and $[b_1 + \cdots + b_{i-1}, b_1 + \cdots + b_i)$ ($i = 2, 3, \ldots, n + 1$) as the first and the ith surplus layer, respectively. We assume that the insurer pays dividends at rate d_i and collects net premiums at rate c_i (i.e., c_i = gross premium $- d_i$) whenever the surplus level resides in the ith surplus layer. Throughout this article, we refer to this insurance risk model as the *complete risk model.*

Here we propose a recursive algorithm to compute the moments of the discounted dividend payments. The starting point of the recursive procedure is the calculation of the moments of the discounted dividend payments in a threshold-free surplus process consisting of only the top (($n + 1$)–th) layer of the complete risk model. The recursion is constructed from top to bottom by adding the next lower layer at each iteration until the complete risk model is entirely recovered. For that purpose we introduce the family of surplus processes $\mathcal{R}_{\underline{c}_i,\underline{b}_i} = \{R_{\underline{c}_i,\underline{b}_i}(t),\ t \geq 0\}$ ($i = 1, \ldots, n$) where $\underline{c}_i = (c_i, c_{i+1}, \ldots, c_{n+1})$ and $\underline{b}_i = (b_i, b_{i+1}, \ldots, b_{n+1})$. The surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ is obtained by assuming that the ith barrier level in the complete risk model is the new origin (level 0) in the risk process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ (thus ignoring the structure below the ith barrier level in the complete risk model). The surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ is then defined as $R_{\underline{c}_i,\underline{b}_i}(0) = u$ and

$$dR_{\underline{c}_i,\underline{b}_i}(t) = \begin{cases} c_i dt - dS(t), & 0 \leq R_{\underline{c}_i,\underline{b}_i}(t) < b_i \\ c_{i+1} dt - dS(t), & b_i \leq R_{\underline{c}_i,\underline{b}_i}(t) < b_i + b_{i+1} \\ \vdots & \vdots \\ c_{n+1} dt - dS(t), & \sum_{k=i}^{n} b_k \leq R_{\underline{c}_i,\underline{b}_i}(t) < \infty \end{cases}, \tag{2.1}$$

for $i = 1, 2, \ldots, n + 1$, where u is the initial surplus level and $S(t) = \sum_{k=1}^{N(t)} X_k$ corresponds to the aggregate claim amount over the interval $(0, t]$. In this article we assume that the claim number process $\{N(t), t \geq 0\}$ follows a Markovian arrival process (MAP) with representation $MAP(\boldsymbol{\alpha}, D_0, D_1)$ (see Neuts (1989) for a detailed description). The distribution of the claim size r.v.'s $\{X_k\}_{k\geq 1}$ is such that a claim that occurs at the epoch of transition from state i to state j of the underlying continuous-time Markov chain of the MAP process is $PH(\alpha_{ij}, Q_{ij})$ distributed. Given a sample path of the continuous-time Markov chain, the sequence of claim sizes $\{X_k\}_{k\geq 1}$ are assumed to be mutually independent. We point out that the class of MAPs contains risk processes with dependent interarrival times and dependent claim sizes, and it provides a large class of processes with a complete dependence structure between the interarrival time and claim size r.v.'s (see Ramaswami (2006) and references therein).

REMARK

In Definition (2.1) of the surplus process $R_{\underline{c}_i,\underline{b}_i}$, the aggregate claim amount process $\mathcal{S} = \{S(t), t \geq 0\}$ (as well as the r.v.'s used in the definition of $\mathcal{S}$) should carry an index i. For simplicity, this index will be absent from the definition of the aggregate claim amount process given that their processes are identically distributed.

Associated to the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i} = \{R_{\underline{c}_i,\underline{b}_i}(t), t \geq 0\}$ is the time to ruin $\sigma_{\underline{c}_i,\underline{b}_i}(u)$, defined as

$$\sigma_{\underline{c}_i,\underline{b}_i}(u) = \inf\{t \geq 0 : R_{\underline{c}_i,\underline{b}_i}(t) < 0 | R_{\underline{c}_i,\underline{b}_i}(0) = u\}$$

with $\sigma_{\underline{c}_i,\underline{b}_i}(u) = \infty$ if $R_{\underline{c}_i,\underline{b}_i}(t) \geq 0$, $\forall t \geq 0$ (ruin does not occur). In Badescu et al. (2007b), a general expression for the Laplace transform of the time to ruin $\sigma_{\underline{c}_i,\underline{b}_i}(u)$, as well as the Laplace transform (with respect to time) of the trivariate density of the time to ruin $\sigma_{\underline{c}_i,\underline{b}_i}(u)$, the surplus immediately prior to ruin $\mathcal{R}_{\underline{c}_i,\underline{b}_i}(\sigma_{\underline{c}_i,\underline{b}_i}(u)^-)$, and the deficit at ruin $|R_{\underline{c}_i,\underline{b}_i}(\sigma_{\underline{c}_i,\underline{b}_i}(u))|$, has been derived for the surplus process (2.1). Here we focus on the calculation of the discounted sum of dividend payments under (2.1). For that purpose we introduce next a family of fluid flow processes and emphasize their connection to the surplus processes defined in (2.1).

2.2 Fluid Flow Processes

Underlying the fluid flow processes, there is an irreducible continuous-time Markov chain (CTMC) $\mathcal{J} = \{J(t), t \geq 0\}$ that represents the environmental process governing the interclaim intervals and the claim sizes. The states of this process are referred to as *phases*. We assume that the CTMC $\mathcal{J}$ has a finite state space $S = S_1 \cup S_2$, where the set S_1 contains the phases for which the fluid flow increases (the interclaim intervals in the associated risk model) and the set S_2 contains the phases for which the fluid flow decreases (the claim sizes in the associated risk model). The infinitesimal generator associated with $\mathcal{J}$ is partitioned as

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{2.2}$$

where the matrices T_{ij} $(i, j = 1, 2)$ contain the transition rates from states in S_i to states in S_j (see, e.g., Badescu et al. (2005) for construction of the generator T). Leading to the analysis of the surplus processes (2.1) are their corresponding fluid flow processes

$$(\mathcal{F}_{\underline{c}_i,\underline{b}_i}, \mathcal{J}) = \{(F_{\underline{c}_i,\underline{b}_i}(t), J(t)), t \geq 0\}, \qquad i = 1, 2, \ldots, n + 1. \tag{2.3}$$

Because of the assumptions on the premium rates in (2.1), the fluid flow processes described in (2.3) have level-dependent fluid rates of increase/decrease. Mathematically the fluid flow processes (2.3) are defined as $F_{\underline{c}_i,\underline{b}_i}(0) = u$ and

$$dF_{\underline{c}_i,\underline{b}_i}(t) = (1_{\{J(t)\in S_1\}} - 1_{\{J(t)\in S_2\}}) \begin{cases} c_i dt, & 0 \le F_{\underline{c}_i,\underline{b}_i}(t) < b_i \\ c_{i+1} dt, & b_i \le F_{\underline{c}_i,\underline{b}_i}(t) < b_i + b_{i+1} \\ \vdots & \vdots \\ c_{n+1} dt, & \sum_{k=i}^{n} b_k \le F_{\underline{c}_i,\underline{b}_i}(t) < \infty \end{cases}, \tag{2.4}$$

for $i = 1, \ldots, n + 1$, where u is the initial fluid level. From (2.4) we observe that the fluid process $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ accumulates (depletes) at constant rate c_j whenever the fluid level resides in the interval $[\Sigma_{k=i}^{j-1} b_k, \Sigma_{k=i}^{j} b_k)$ $(i \le j \le n + 1)$, independently of the underlying phase in S_1 (S_2) of the CTMC $\mathscr{J}$. Due to their importance in the ensuing analysis, we introduce here some important first-passage times in the fluid flow process $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$:

- $\tau_{\underline{c}_i,\underline{b}_i}(x, y)$ = first passage time of $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ from level x to level y
- ${}^{b}\tau_{\underline{c}_i,\underline{b}_i}(x, y)$ = first passage time of $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ from level x to level y avoiding a visit to the levels in $[b, \infty)$ en route
- ${}_{a}\tau_{\underline{c}_i,\underline{b}_i}(x, y)$ = first passage time of $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ from level x to level y avoiding a visit to the levels in $[0, a]$ en route.

To analyze the fluid flow processes $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ defined in (2.4), we rely on quantities pertaining to the infinite buffer (threshold-free) fluid model

$$(\mathscr{F}_{c_j}, \mathscr{J}) = \{(F_{c_j}(t), J(t)), t \ge 0\}, \qquad j = 1, \ldots, n + 1, \tag{2.5}$$

with $F_{c_j}(0) = u$ and

$$dF_{c_j}(t) = \begin{cases} c_j dt, & F_{c_j}(t) \ge 0, J(t) \in S_1 \\ -c_j dt, & F_{c_j}(t) \ge 0, J(t) \in S_2 \end{cases}$$

(see Ramaswami (2006)). Closely related to the fluid flow model $(\mathscr{F}_{c_j}, \mathscr{J})$ is the reflected infinite buffer fluid flow process (e.g., see Ramaswami (2006)) operating at rate c_j, namely,

$$(\mathscr{F}^{r}_{c_j}, \mathscr{J}) = \{(F^{r}_{c_j}(t), J(t)), t \ge 0\}, \qquad j = 1, \ldots, n + 1, \tag{2.6}$$

with $F^{r}_{c_j}(0) = u$ and

$$dF^{r}_{c_j}(t) = \begin{cases} -c_j dt, & F^{r}_{c_j}(t) \ge 0, J(t) \in S_1 \\ c_j dt, & F^{r}_{c_j}(t) \ge 0, J(t) \in S_2 \end{cases},$$

in which the roles of the up and down environmental states are simply reversed. Note that quantities pertaining to the infinite buffer fluid flow processes (2.5) and (2.6) are extensively discussed in the fluid flow literature (see, e.g., Ramaswami (2006) and Badescu et al. (2007a)). For purposes of completeness, we list the relevant ones in Table 1.

Table 1

First-Passage Time Results in Fluid Flow Models

LST	First-Passage Time	Formula	Reference
$\hat{f}_{22,c}(x, 0, \delta)$	From (x, S_2) to $(0, S_2)$ in $\mathscr{F}_c$	$e^{H_c(\delta)x}$	Ahn and Ramaswami (2005), Th. 3
${}_0\hat{f}_{11,c}(0, x, \delta)$	From $(0, S_1)$ to (x, S_1) avoiding 0 en route in $\mathscr{F}_c$	$e^{H_c(\delta)x}[I + \Psi_c(\delta)U_c(\delta, x)]^{-1}$	Ramaswami (2006), Th. 1
${}^{x}\Psi_c(\delta) = {}^{x}\hat{f}_{12,c}(0, 0, \delta)$	From $(0, S_1)$ to $(0, S_2)$ avoiding x en route in $\mathscr{F}_c$	$\Psi_c(\delta) - {}_0\hat{f}_{11,c}(0, x, \delta)\Psi_c(\delta)e^{H_c(\delta)x}$	Ramaswami (2006), Th. 2
${}^{x}\hat{f}_{22,c}(x, 0, \delta)$	From (x, S_2) to $(0, S_2)$ avoiding x en route in $\mathscr{F}_c$	${}_0\hat{f}^{r}_{22,c}(0, x, \delta)$	Ramaswami (2006), Th. 3
${}^{x}\Psi^{r}_c(\delta) = {}_0\hat{f}_{22,c}(x, x, \delta)$	From $(0, S_2)$ to $(0, S_1)$ avoiding x en route in $\mathscr{F}^{r}_c$	$\Psi^{r}_c(\delta) - {}_0\hat{f}^{r}_{22,c}(0, x, \delta)\Psi^{r}_c(\delta)e^{H^{r}_c(\delta)x}$	Ramaswami (2006), Th. 4

2.3 Connection between Risk Processes and Fluid Flows

The connection between risk processes and fluid flows have been analyzed in numerous papers (see Asmussen (2000); Badescu et al. (2005); Ramaswami (2006), and references therein). For purposes of completeness, some important aspects of this connection are summarized below. To illustrate the connection, we consider here a fluid flow process whose rate of increase/decrease is c.

To see the risk model as a fluid flow process, the key idea is to pretend as though a claim of size x arrives continuously over an unseen time interval whose length is x/c at rate c per unit time. The paths of the risk process before ruin can be obtained from segments of the fluid process before the fluid level becomes empty, and this is achieved by replacing downward linear paths in the fluid flow model by downward jumps of appropriate sizes. Thus, the risk process is embedded into a fluid flow model by incising out times spent in the downward states of the fluid model. Given that the rate of increase and decrease of the fluid flow is the same, certain relationships between passage times in the two models can be established. For instance, the length of a busy period in a fluid flow starting at level 0 is twice the time to ruin in the insurance risk model with an initial surplus of 0. For additional relationships, see Section 3 in Ramaswami (2006). In our approach the analysis of the risk model described in (2.1) will be performed using quantities defined within each layer (given that the premium rate is constant within a layer) for which the well-known connections hold.

3. MOMENTS OF THE DISCOUNTED DIVIDEND PAYMENTS

Based on a sample path analysis of the corresponding fluid flow process, we propose a recursive algorithm to compute the moments of the discounted dividend payments before ruin in the fairly general class of surplus processes with Markovian claim arrivals. Let

$$D_{\underline{c}_i,\underline{b}_i}(u) = \sum_{j=i}^{n+1} d_j \int_0^{\sigma_{\underline{c}_i,\underline{b}_i}(u)} e^{-\delta t} 1_{\{\Sigma_{k=i}^{j-1} b_k \le R_{\underline{c}_i,\underline{b}_i}(t) < \Sigma_{k=i}^{j} b_k\}}\, dt$$

be the discounted sum of dividend payments before ruin for the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ at a force of interest $\delta > 0$.

REMARK

In what follows we assume that the force of interest δ is strictly positive. For $\delta = 0$ and under the assumption that ruin does not occur almost surely for the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ (common assumption in an insurance context), the total sum of dividend payments $D_{\underline{c}_i,\underline{b}_i}(u)$ will be infinite with a strictly positive probability. It is clear that the analysis of the moments of the total dividend payments in such a case is of no mathematical interest.

3.1 With Ruin

In this subsection we focus on the surplus paths for which ruin occurs and develop a recursive algorithm to compute the moments of the discounted dividend payments for this set of sample paths. The proposed algorithm is based on the recursive calculation of $W_{l,m,\underline{c}_i,\underline{b}_i}(u) (l, m \in \mathbb{N})$, an $|S_1| \times |S_2|$ matrix, whose components are given by

$$[W_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} = \mathrm{E}[e^{-l\delta\sigma_{\underline{c}_i,\underline{b}_i}(u)}(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_i,\underline{b}_i}(u)<\infty\}} 1_{\{J(\tau_{\underline{c}_i,\underline{b}_i}(u,0))=k\}} | J(0) = j]. \tag{3.1}$$

For simplicity, we adopt the following notation in this subsection, namely,

$$[W_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} = \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-l\delta\sigma_{\underline{c}_i,\underline{b}_i}(u)}(D_{\underline{c}_i,\underline{b}_i}(u))^m], \tag{3.2}$$

where the indices j and k represent the state of CTMC $\mathcal{J}$ at time 0 and time $\tau_{\underline{c}_i,\underline{b}_i}(u, 0)$, respectively, and the superscript *ruin* of E replaces the indicator $1_{\{\sigma_{\underline{c}_i,\underline{b}_i}(u)<\infty\}}$ in (3.1). Henceforth, we refer to (3.2) as the *generalized moments of the discounted dividend payments with ruin occurrence* given that the "ordinary" moments are obtained by letting $l = 0$. In Proposition 1 we first derive an expression for (3.2) in the threshold-free surplus process $\mathcal{R}_{\underline{c}_{n+1},\underline{b}_{n+1}}$ for which a dividend rate d_{n+1} is paid continuously from time 0 to the time to ruin $\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)$.

Proposition 1

For the threshold-free surplus process $\mathcal{R}_{\underline{c}_{n+1},\underline{b}_{n+1}}$, a closed-form expression for the generalized moments of the discounted dividend payments with ruin occurrence is given by

$$W_{l,m,\underline{c}_{n+1},\underline{b}_{n+1}}(u) = \left(\frac{d_{n+1}}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \rho_{\delta(l+h),\underline{c}_{n+1},\underline{b}_{n+1}}(u), \tag{3.3}$$

for $l, m \in \mathbb{N}$, where

$$\rho_{\delta,\underline{c}_{n+1},\underline{b}_{n+1}}(u) = \Psi_{c_{n+1}}\left(\frac{\delta}{2}\right) e^{\delta u/2c_{n+1}} \hat{f}_{22,c_{n+1}}\left(u, 0, \frac{\delta}{2}\right). \tag{3.4}$$

PROOF

From the definition of $W_{l,m,\underline{c}_i,\underline{b}_i}(u)$ in (3.2), one deduces

$$\begin{aligned}
[W_{l,m,\underline{c}_{n+1},\underline{b}_{n+1}}(u)]_{jk} &= \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-l\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}(u)}}(d_{n+1}\bar{a}_{\overline{\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)}|})^m] \\
&= \left(\frac{d_{n+1}}{\delta}\right)^m \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-l\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}(u)}}(1 - e^{-\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}(u)}})^m],
\end{aligned}$$

where $\bar{a}_{\overline{t}|}$ holds for the present value of a continuous annuity paying at a rate of 1 from time 0 to time t. A binomial expansion of $(1 - e^{-\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}(u)}})^m$ yields

$$\begin{aligned}
[W_{l,m,\underline{c}_{n+1},\underline{b}_{n+1}}(u)]_{jk} &= \left(\frac{d_{n+1}}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \mathrm{E}_{jk}^{(\mathrm{ruin})} [e^{-(l+h)\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)}] \\
&= \left(\frac{d_{n+1}}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h [\rho_{\delta(l+h),\underline{c}_{n+1},\underline{b}_{n+1}}(u)]_{jk}.
\end{aligned}$$

Note that $\rho_{\delta,\underline{c}_{n+1},\underline{b}_{n+1}}(u)$ represents the Laplace transform of the time to ruin $\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)$ in the threshold-free risk process $\mathcal{R}_{\underline{c}_{n+1},\underline{b}_{n+1}}$:

$$\rho_{\delta,\underline{c}_{n+1},\underline{b}_{n+1}}(u) = \mathrm{E}[e^{-\delta\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)} 1_{\{\sigma_{\underline{c}_{n+1},\underline{b}_{n+1}}(u)<\infty\}}],$$

for which the general expression (3.4) is given in Ramaswami (2006). □

An equivalent representation for the generalized moment $W_{l,m,\underline{c}_{n+1},\underline{b}_{n+1}}(u)$ is easily found to be

$$W_{l,m,\underline{c}_{n+1},\underline{b}_{n+1}}(u) = \frac{(d_{n+1})^m}{\delta} (W_{l,m-1,\underline{c}_{n+1},\underline{b}_{n+1}}(u) - W_{l+1,m-1,\underline{c}_{n+1},\underline{b}_{n+1}}(u)), \tag{3.5}$$

using the decomposition $(\bar{a}_{\overline{t}|})^m = ((1 - e^{-\delta t})/\delta)(\bar{a}_{\overline{t}|})^{m-1}$ of the annuity factor. Despite the simplicity of (3.5), note that (3.3) is computationally more efficient to obtain the values of W because of its non-recursive structure in comparison to (3.5).

Let $\vec{1}_{|S_j|}$ be a column vector of 1 of size $|S_j|$ $(j = 1, 2)$ and $I_{|S_j|}$ be the identity matrix of size $|S_j|$ $(j = 1, 2)$.

Proposition 2

For the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$, the generalized moments of the discounted dividend payments with ruin occurrence satisfy the following:

(a) For $u \geq b_i$,

$$
\begin{aligned}
W_{l,m,\underline{c}_i,\underline{b}_i}(u) = & \sum_{\xi=0}^{m} \binom{m}{\xi} W_{l+\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) \\
& \left\{ \begin{array}{l} \displaystyle\sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \, {}^{b_i}\Psi^r_{c_i} \left(\frac{(\xi + l - x)\delta}{2}\right) W_{l,\xi-h,\underline{c}_i,\underline{b}i}(b_i) \\ \displaystyle + \left(\frac{d_i}{\delta}\right)^{\xi} \sum_{h=0}^{\xi} \binom{\xi}{h} (-1)^h \, e^{(l+h)\delta b_i/2c_i} \, {}^{b_i}\hat{f}_{22,c_i} \left(b_i, 0, \frac{(l+h)\delta}{2}\right) \end{array} \right\}
\end{aligned}
\tag{3.6}
$$

(b) For $u < b_i$,

$$
\begin{aligned}
W_{l,m,\underline{c}_i,\underline{b}_i(u)} = & \left(\frac{d_i}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \, e^{(l+h)\delta u/2c_i} \, {}^{b_i-u}\Psi_{c_i} \left(\frac{(l + h)\delta}{2}\right) {}^{b_i}\hat{f}_{22,c_i} \left(u, 0, \frac{(l + h)\delta}{2}\right) \\
& + \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d_i}{\delta}\right)^h \left(\begin{array}{l} \Sigma_{x=0}^{h} \binom{h}{x} (-1)^{h-x} e^{-(m-x+l)\delta/(b_i-u)2c_i} \\ \displaystyle \times \, {}_0\hat{f}_{11,c_i} \left(u, b_i, \frac{(m - x + l)\delta}{2}\right) W_{l,m-h,\underline{c}_i,\underline{b}_i}(b_i) \end{array} \right),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
& W_{l,m,\underline{c}_i,\underline{b}_i}(b_i) = \left(I_{|S_1|} - W_{l+m,0,\underline{c}_{i+1},\underline{b}_{i+1}}(0) \, {}^{b_i}\Psi^r_{c_i} \left(\frac{(m + l)\delta}{2}\right)\right)^{-1} \\
& \left\{ \begin{array}{l} \displaystyle \sum_{\xi=0}^{m} \binom{m}{\xi} \left(\frac{d_i}{\delta}\right)^{\xi} W_{l+\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(0) \sum_{h=0}^{\xi} \binom{\xi}{h} (-1)^h \, e^{(l+h)\delta b_i/2c_i} \, {}^{b_i}\hat{f}_{22,c_i} \left(b_i, 0, \frac{(l + h)\delta}{2}\right) \\ \displaystyle + \sum_{\substack{\xi=0 \\ (\xi,h)\neq(m,0)}}^{m} \sum_{h=0}^{\xi} \binom{m}{\xi} \binom{\xi}{h} W_{l+\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(0) \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \, {}^{b_i}\Psi^r_{c_i} \left(\frac{(\xi + l - x)\delta}{2}\right) W_{l,\xi-h,\underline{c}_i,\underline{b}_i}(b_i) \end{array} \right\},
\end{aligned}
\tag{3.8}
$$

for $l, m \in \mathbb{N}^+$.

PROOF

We first consider the case $u \geq b_i$. For ruin to occur, the corresponding fluid flow process $\mathcal{F}_{\underline{c}_i,\underline{b}_i}$ shall first make a transition from (u, S_1) to (b_i, S_2). Being at level b_i in S_2, ruin can occur with or without an eventual visit to level b_i in S_1. In the former, the corresponding fluid process reaches level b_i in S_1 before level 0 in S_2. Thus, it is immediate that the first passage time $\tau_{\underline{c}_i,\underline{b}_i}(u, 0)$ of $\mathcal{F}_{\underline{c}_i,\underline{b}_i}$ can be decomposed as

$$\tau_{\underline{c}_i,\underline{b}_i}(u, 0) = \tau_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i, 0) + {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i, b_i) + \tau^*_{\underline{c}_i,\underline{b}_i}(b_i, 0), \tag{3.9}$$

where $\tau^*_{\underline{c}_i,\underline{b}_i}(b_i, 0)$ is equal in distribution to $\tau_{\underline{c}_i,\underline{b}_i}(b_i, 0)$(i.e., $\tau^*_{\underline{c}_i,\underline{b}_i}(b_i, 0) \overset{d}{=} \tau_{\underline{c}_i,\underline{b}_i}(b_i, 0)$). Taking into consideration only the segments of time when the fluid flow increases and using Observation (c) in Ramaswami (2006), we derive the equivalent to (3.9) in the context of the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$, namely,

$$\sigma_{\underline{c}_i,\underline{b}_i}(u) = \sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + \frac{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i, b_i)}{2} + \sigma^*_{\underline{c}_i,\underline{b}_i}(b_i), \tag{3.10}$$

where $\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i) \overset{d}{=} \sigma_{\underline{c}_i,\underline{b}_i}(b_i)$. Based on the decomposition (3.10) of $\sigma_{\underline{c}_i,\underline{b}_i}(u)$, the discounted sum of dividend payments $D_{\underline{c}_i,\underline{b}_i}(u)$ can be subdivided as follows: (a) dividends from time 0 to $\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)$; (b) dividends from time $\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)$ to $\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + ({}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i, b_i)/2)$; (c) dividends after time $\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i, b_i)/2$. It is immediate that

$$D_{\underline{c}_i,\underline{b}_i}(u) = D_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + e^{-\delta\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)} \left(d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2}|} + e^{-\delta\, {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2}\, D^*_{\underline{c}_i,\underline{b}_i}(b_i) \right), \tag{3.11}$$

where $D^*_{\underline{c}_i,\underline{b}_i}(b_i) \overset{d}{=} D_{\underline{c}_i,\underline{b}_i}(b_i)$.

For the latter, the fluid process transits from level b_i in S_2 to level 0 in S_2 without revisiting level b_i en route. It follows that the first passage time $\tau_{\underline{c}_i,\underline{b}_i}(u, 0)$ of $\mathscr{F}_{\underline{c}_i,\underline{b}_i}$ can be decomposed as

$$\tau_{\underline{c}_i,\underline{b}_i}(u, 0) = \tau_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i, 0) + {}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i, 0). \tag{3.12}$$

Taking into consideration only the segments of time when the fluid flow increases, (3.12) can be converted in the context of the surplus process $\mathscr{R}_{\underline{c}_i,\underline{b}_i}$ to

$$\sigma_{\underline{c}_i,\underline{b}_i}(u) = \sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + \left(\frac{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i, 0)}{2} - \frac{b_i}{2c_i} \right), \tag{3.13}$$

which yields the following decomposition of $D_{\underline{c}_i,\underline{b}_i}(u)$,

$$D_{\underline{c}_i,\underline{b}_i}(u) = D_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) + e^{-\delta\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)} d_i\bar{a}_{\overline{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i}|}. \tag{3.14}$$

By combining the former and the latter with their respective definitions of $\sigma_{\underline{c}_i,\underline{b}_i}(u)$ and $D_{\underline{c}_i,\underline{b}_i}(u)$, (3.1) becomes

$$\begin{aligned}
[W_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} = {} & \sum_{\xi=0}^{m} \binom{m}{\xi} \mathrm{E}^{(\text{ruin})}_{jk} \left[\begin{matrix} e^{-(l+\xi)\delta\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)} (D_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i))^{m-\xi} \\ \times\, e^{-l\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2+\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i))} (d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2}|} + e^{-\delta{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2} D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{\xi} \end{matrix} \right] \\
& + \sum_{\xi=0}^{m} \binom{m}{\xi} \mathrm{E}^{(\text{ruin})}_{jk} \left[\begin{matrix} e^{-(l+\xi)\delta\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)} (D_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i))^{m-\xi} \\ \times\, e^{-l\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i)} (d_i\bar{a}_{\overline{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i}|})^{\xi} \end{matrix} \right],
\end{aligned} \tag{3.15}$$

using a binomial expansion on the $D_{\underline{c}_i,\underline{b}_i}(u)$ terms. We recall that, for a surplus process with Markovian claim arrivals and phase-type distributed claim sizes, quantities defined on disjoint intervals are mutually independent given the sample path of the underlying continuous-time Markov chain of the MAP process. Thus, using a matrix representation, (3.15) can be rewritten as

$$W_{l,m,\underline{c}_i,\underline{b}_i}(u) = \sum_{\xi=0}^{m} \binom{m}{\xi} W_{l+\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)\bar{W}_{l,\xi,c_i}(b_i), \tag{3.16}$$

where $\bar{W}_{l,\xi,c_i}(b_i)$ is an $|S_2| \times |S_2|$ matrix with

$$\begin{aligned}
[\bar{W}_{l,\xi,c_i}(b_i)]_{jk} = {} & \mathrm{E}^{(\text{ruin})}_{jk}[e^{-l\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2+\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i))} (d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2}|} + e^{-\delta{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2} D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{\xi}] \\
& + \mathrm{E}^{(\text{ruin})}_{jk}[e^{-l\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i)} (d_i\bar{a}_{\overline{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i}|})^{\xi}].
\end{aligned} \tag{3.17}$$

Note that the argument j in (3.17) represents the state (in S_2) of the underlying CTMC $\mathscr{J}$ at the passage time $\tau_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)$. Simple modifications of (3.17) yield

$$\begin{aligned}
[\bar{W}_{l,\xi,c_i}(b_i)]_{jk} &= \sum_{h=0}^{\xi} \binom{\xi}{h} \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-(\xi-h+l)\delta\ {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2} (d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2|}})^h e^{-l\delta\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)} (D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{\xi-h}] \\
&\quad + \left(\frac{d_i}{\delta}\right)^{\xi} \sum_{h=0}^{\xi} \binom{\xi}{h} (-1)^h\ \mathrm{E}_{jk}[e^{-(l+h)\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2-b_i/2c_i)}] \\
&= \sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x}\ \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-(\xi+l-x)\delta\ {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2} e^{-l\delta\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)} (D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{\xi-h}] \\
&\quad + \left(\frac{d_i}{\delta}\right)^{\xi} \sum_{h=0}^{\xi} \binom{\xi}{h} (-1)^h e^{(l+h)\delta b_i/2c_i}\ \mathrm{E}_{jk}[e^{-(l+h)\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i,0)/2)}],
\end{aligned} \tag{3.18}$$

where the superscript (*ruin*) is dropped from the second term in (3.18) given that ruin occurs almost surely with the passage time ${}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(b_i, 0)$. Using a matrix representation, it follows

$$\begin{aligned}
\bar{W}_{l,\xi,c_i}(b_i) &= \sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x}\ {}^{b_i}\Psi^r_{c_i} \left(\frac{(\xi + l - x)\delta}{2}\right) W_{l,\xi-h,\underline{c}_i,\underline{b}_i}(b_i) \\
&\quad + \left(\frac{d_i}{\delta}\right)^{\xi} \sum_{h=0}^{\xi} \binom{\xi}{h} (-1)^h e^{(l+h)\delta b_i/2c_i}\ {}^{b_i}\hat{f}_{22,c_i}\left(b_i, 0, \frac{(l+h)\delta}{2}\right).
\end{aligned} \tag{3.19}$$

Combining (3.16) and (3.19) yields (3.6).

For $u < b_i$, the corresponding fluid process $\mathcal{F}_{\underline{c}_i,\underline{b}_i}$ either reaches level b_i (in S_1) or level 0 (in S_2) first. For the former, the first passage of $\mathcal{F}_{\underline{c}_i,\underline{b}_i}$ to level 0 in S_2 can be decomposed as

$$\tau_{\underline{c}_i,\underline{b}_i}(u, 0) = {}_0\tau_{\underline{c}_i,\underline{b}_i}(u, b_i) + \tau^*_{\underline{c}_i,\underline{b}_i}(b_i, 0). \tag{3.20}$$

By eliminating the segments of time when the fluid flow decreases in (3.20), the time to ruin $\sigma_{\underline{c}_i,\underline{b}_i}(u, 0)$ for the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}(t)$ can be expressed as

$$\sigma_{\underline{c}_i,\underline{b}_i}(u, 0) = \left(\frac{{}_0\tau_{\underline{c}_i,\underline{b}_i}(u, b_i)}{2} + \frac{b_i - u}{2c_i}\right) + \sigma^*_{\underline{c}_i,\underline{b}_i}(b_i, 0), \tag{3.21}$$

which yields the following decomposition of the future dividend payments:

$$D_{\underline{c}_i,\underline{b}_i}(u) = d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i|}} + e^{-\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)} D^*_{\underline{c}_i,\underline{b}_i}(b_i). \tag{3.22}$$

For the latter, the fluid process $\mathcal{F}_{\underline{c}_i,\underline{b}_i}$ transits to level 0 before visiting level b_i, which implies $\tau_{\underline{c}_i,\underline{b}_i}(u, 0) = {}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u, 0)$. It follows that the quantities pertaining to the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}(t)$ can be expressed as

$$\sigma_{\underline{c}_i,\underline{b}_i}(u) = \frac{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u, 0)}{2} - \frac{u}{2c_i},$$

and

$$D_{\underline{c}_i,\underline{b}_i}(u) = d_i\bar{a}_{\overline{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u,0)/2-u/2c_i|}}.$$

By combining the former and the latter with their respective definitions of $\sigma_{\underline{c}_i,\underline{b}_i}(u)$ and $D_{\underline{c}_i,\underline{b}_i}(u)$, we find

$$\begin{aligned}
[W_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} &= \mathrm{E}_{jk}^{(\mathrm{ruin})}[e^{-l\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u,0)/2-u/2c_i)} (d_i\bar{a}_{\overline{{}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u,0)/2-u/2c_i|}})^m] \\
&\quad + \mathrm{E}_{jk}^{(\mathrm{ruin})}\left[\begin{array}{c} e^{-l\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i+\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i,0))} \\ \times\ (d_i\bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i|}} + e^{-\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)} D^*_{\underline{c}_i,\underline{b}_i}(b_i))^m \end{array}\right].
\end{aligned} \tag{3.23}$$

Simple modifications of (3.23) yield

$$\begin{aligned}
[W_{l,m,\underline{c}_i,\underline{b}_i}(u)]_{jk} &= \left(\frac{d_i}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \ \mathrm{E}_{jk}[e^{-(l+h)\delta({}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u,0)/2-u/2c_i)}] \\
&\quad + \sum_{h=0}^{m} \binom{m}{h} \mathrm{E}_{jk}^{(\mathrm{ruin})} \begin{bmatrix} e^{-(m-h+l)\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)}(d_i\bar{a}_{\overline{({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)|}})^h \\ \times\, e^{-l\delta\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)}(D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{m-h} \end{bmatrix} \\
&= \left(\frac{d_i}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \ e^{(l+h)\delta u/2c_i} \ \mathrm{E}_{jk}[e^{-(l+h)\delta/2\ {}^{b_i}\tau_{\underline{c}_i,\underline{b}_i}(u,0)}] \\
&\quad + \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \ \mathrm{E}_{jk}^{(\mathrm{ruin})} \begin{bmatrix} e^{-(m-x+l)\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)} \\ \times\, e^{-l\delta\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)}(D^*_{\underline{c}_i,\underline{b}_i}(b_i))^{m-h} \end{bmatrix}. \qquad (3.24)
\end{aligned}$$

Under matrix notation, (3.24) becomes

$$\begin{aligned}
W_{l,m,\underline{c}_i,\underline{b}_i}(u) &= \left(\frac{d_i}{\delta}\right)^m \sum_{h=0}^{m} \binom{m}{h} (-1)^h \ e^{(l+h)\delta u/2c_i} \ {}^{b_i-u}\Psi_{c_i}\left(\frac{(l+h)\delta}{2}\right)^{b_i} \hat{f}_{22,c_i}\left(u,\, 0,\, \frac{(l+h)\delta}{2}\right) \\
&\quad + \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d_i}{\delta}\right)^h \begin{pmatrix} \displaystyle\sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \ e^{-(m-x+l)\delta(b_i-u)/2c_i} \\ \times\ {}_0\hat{f}_{11,c_i}\left(u,\, b_i,\, \dfrac{(m-x+l)\delta}{2}\right) W_{l,m-h,\underline{c}_i,\underline{b}_i}(b_i) \end{pmatrix}, \qquad (3.25)
\end{aligned}$$

which completes the proof of (3.7).

Finally, (3.8) is obtained from (3.6) at $u = b_i$. □

3.2 Without Ruin

In this section we develop a recursive algorithm to compute the higher moments of the discounted dividend payments without ruin occurrence. In nature, the analysis follows along the same lines than for the moments of the discounted dividends with ruin occurrence. The obvious difference is that there is no state of the CTMC $\mathcal{J}$ at the time to ruin (does not exist). Thus, the proposed recursive algorithm is based on the recursive calculation of $\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u)(m \in \mathbb{N}^+)$, a $|S_1|$ column vector for which the jth component is defined as

$$[\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u)]_j = \mathrm{E}_j[(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_i,\underline{b}_i}(u)=\infty\}}]. \qquad (3.26)$$

The starting point of the recursive procedure is the well-known expression of (3.26) in the threshold-free risk model $\mathcal{R}_{\underline{c}_{n+1},\underline{b}_{n+1}}$,

$$\vec{\chi}_{m,\underline{c}_{n+1},\underline{b}_{n+1}}(u) = \left(\frac{d_{n+1}}{\delta}\right)^m (\vec{1}_{|S_1|} - \rho_{0,\underline{c}_{n+1},\underline{b}_{n+1}}(u)\vec{1}_{|S_2|}), \qquad (3.27)$$

for $u \geq 0$.

Proposition 3

For the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$, the moments of the discounted dividend payments without ruin occurrence satisfy the following:

(a) For $u \geq b_i$,

$$\begin{aligned}\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u) &= \vec{\chi}_{m,\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)\\ &\quad + \sum_{\xi=0}^{m} \binom{m}{\xi} W_{\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i) \left(\begin{array}{l} \displaystyle\sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \\ \times\ {}^{b_i}\Psi^r_{c_i}\left(\dfrac{(\xi - x)\delta}{2}\right) \vec{\chi}_{\xi-h,\underline{c}_i,\underline{b}_i}(b_i) \end{array} \right),\end{aligned}$$

(b) For $u < b_i$,

$$\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u) = \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x}\, e^{-(m-x)\delta(b_i-u)/2c_i}\ {}_0\hat{f}_{11,c_i}\left(u,\, b_i,\, \frac{m - x}{2}\,\delta\right) \vec{\chi}_{m-h,\underline{c}_i,\underline{b}_i}(b_i), \tag{3.28}$$

where

$$\begin{aligned}\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(b_i) &= \left(I_{|S_1|} - W_{m,0,\underline{c}_{i+1},\underline{b}_{i+1}}(0)\ {}^{b_i}\Psi^r_{c_i}\left(\frac{m\delta}{2}\right)\right)^{-1}\\ &\left(\begin{array}{l} \vec{\chi}_{m,\underline{c}_{i+1},\underline{b}_{i+1}}(0) \\ \displaystyle + \sum_{\substack{\xi=0 \\ (\xi,h)\neq(m,0)}}^{m} \sum_{h=0}^{\xi} \binom{m}{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} W_{\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(0)\ {}^{b_i}\Psi^r_{c_i}\left(\frac{\xi-x}{2}\,\delta\right) \vec{\chi}_{\xi-h,\underline{c}_i,\underline{b}_i}(b_i) \end{array} \right),\end{aligned} \tag{3.29}$$

for $m \in \mathbb{N}^+$.

PROOF

For an initial capital $u \geq b_i$, if ruin does not occur, the surplus process can either visit or not visit level b_i in a decreasing phase (in S_2). For the latter, the surplus process will never visit levels in the surplus layer $(0, b_i]$. For the former, the surplus process eventually will visit level b_i in an increasing phase allowing for the decomposition (3.10) and (3.11) of $\sigma_{\underline{c}_i,\underline{b}_i}(u)$ and $D_{\underline{c}_i,\underline{b}_i}(u)$, respectively.

Combining these two cases, (3.26) can be written as

$$\begin{aligned}[\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u)]_j &= \mathrm{E}_j[(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)=\infty\}}]\\ &\quad + \mathrm{E}_j[(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)<\infty\}} 1_{\{\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)=\infty\}}],\end{aligned} \tag{3.30}$$

where, by definition,

$$\mathrm{E}_j[(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)=\infty\}}] = [\vec{\chi}_{m,\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i)]_j, \tag{3.31}$$

and, using (3.11) followed by a binomial expansion,

$$\begin{aligned}&\mathrm{E}_j[(D_{\underline{c}_i,\underline{b}_i}(u))^m 1_{\{\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)<\infty\}} 1_{\{\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)=\infty\}}]\\ &\quad = \sum_{k=0}^{m} \binom{m}{k} \mathrm{E}_j \left[\begin{array}{l} e^{-k\delta\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)} (D_{\underline{c}_{i+1},\underline{b}_{i+1}}(u - b_i))^{m-k} 1_{\{\sigma_{\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i)<\infty\}} \\ \times\ (d_i \bar{a}_{\overline{{}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2|}} + e^{-\delta\ {}_0\tau_{\underline{c}_i,\underline{b}_i}(b_i,b_i)/2} (D^*_{\underline{c}_i,\underline{b}_i}(b_i)))^k 1_{\{\sigma^*_{\underline{c}_{i+1},\underline{b}_{i+1}}(b_i)=\infty\}} \end{array} \right].\end{aligned} \tag{3.32}$$

Note that (3.32) is of the same form than the first term on the right-hand side of (3.15) (with $l = 0$). Following a similar line of logic, one obtains the matrix representation

$$\sum_{\xi=0}^{m} \binom{m}{\xi} W_{\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i) \sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x}\, {}^{b_i}\Psi^r_{c_i}\left(\frac{(\xi-x)\delta}{2}\right) \vec{\chi}_{\xi-h,\underline{c}_i,\underline{b}_i}(b_i), \tag{3.33}$$

for (3.32). Combining (3.31) and (3.33) with (3.30), one concludes

$$\begin{aligned} \vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u) &= \vec{\chi}_{m,\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i) \\ &\quad + \sum_{\xi=0}^{m} \binom{m}{\xi} W_{\xi,m-\xi,\underline{c}_{i+1},\underline{b}_{i+1}}(u-b_i) \left(\begin{array}{l} \displaystyle\sum_{h=0}^{\xi} \binom{\xi}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} \\ \displaystyle\times\, {}^{b_i}\Psi^r_{c_i}\left(\frac{(\xi-x)\delta}{2}\right) \vec{\chi}_{\xi-h,\underline{c}_i,\underline{b}_i}(b_i) \end{array} \right). \end{aligned} \tag{3.34}$$

For $u < b_i$, the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}$ must reach level b_i before level 0 to avoid ruin. Using the decomposition (3.21) and (3.22) for the time to ruin and the discounted dividend payments, respectively, one finds

$$[\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u)]_j = \mathrm{E}_j[(d_i \bar{a}_{\overline{({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)|}} + e^{-\delta({}_0\tau_{\underline{c}_i,\underline{b}_i}(u,b_i)/2+(b_i-u)/2c_i)} D^*_{\underline{c}_i,\underline{b}_i}(b_i))^m 1_{\{\sigma^*_{\underline{c}_i,\underline{b}_i}(b_i)=\infty\}}]. \tag{3.35}$$

Note that (3.35) is of a similar form as the second term on the right-hand side of (3.23) at $l = 0$. Using a similar line of logic, one concludes

$$\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u) = \sum_{h=0}^{m} \binom{m}{h} \left(\frac{d_i}{\delta}\right)^h \sum_{x=0}^{h} \binom{h}{x} (-1)^{h-x} e^{-(m-x)\delta(b_i-u)/2c_i}\, {}_0\hat{f}_{11,c_i}\left(u, b_i, \frac{(m-x)\delta}{2}\right) \vec{\chi}_{m-h,\underline{c}_i,\underline{b}_i}(b_i), \tag{3.36}$$

for $u < b_i$ which proves (3.28).

Finally, (3.29) is obtained from (3.34) at $u = b_i$. ☐

We next provide additional details regarding the two recursive formulas derived in Propositions 2 and 3.

3.3 Computational Issues

In Badescu et al. (2007b) it is shown that the calculation of the expected discounted dividend payments implies a one-level recursion in terms of layers. However, from Propositions 2 and 3 we observe that the higher-order moments are calculated using a two-level recursion, one in terms of the layers and the other in terms of the order of the moments. As described in the Preliminaries section, under both scenarios (with and without ruin), the starting point of the recursive algorithms is the expression of the moments in the threshold-free risk model given by (3.3) and (3.27). We then proceed by adding up an extra layer at each iteration and calculating quantities of interest (i.e., the W's and $\vec{\chi}$'s) in the interim via Propositions 2 and 3.

Note that the execution time of these recursive formulas grows exponentially with the number of layers and the order of the desired moment. To reduce the computational time, we suggest the creation of a multidimensional array in which the values of the recursive function will be stored at each iteration to avoid redundancy. In our case the multidimensional matrix is of dimension 4 (3) in the with (without) ruin case, containing all the relevant values of the parameters l, i, m, u (i, m, u).

Finally, one can obtain the higher-order moments of the discounted dividend payments in the surplus process $\mathcal{R}_{\underline{c}_i,\underline{b}_i}(t)$ by multiplying $W_{0,m,\underline{c}_i,\underline{b}_i}(u)$ by $\vec{1}_{|S_2|}$, and then summing it up to $\vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u)$:

$$\mathrm{E}[(D_{\underline{c}_i,\underline{b}_i}(u))^m] = W_{0,m,\underline{c}_i,\underline{b}_i}(u)\, \vec{1}_{|S_2|} + \vec{\chi}_{m,\underline{c}_i,\underline{b}_i}(u).$$

4. NUMERICAL ILLUSTRATION

To illustrate the applicability of our results, we consider an example where the claim arrival process follows a Markovian arrival process. We assume a background Poisson process of claims that occur at a rate $\lambda = 1$ and exponentially distributed claim sizes with mean $1/\mu = 0.01$. At each instant when the process is in the background mode, there is a probability of $\varepsilon/(\varepsilon + \lambda) = 0.1$ that an infectious environment will appear. While in this environment, the insurer will encounter some trains of big claims that may be caused by some external factors. It is assumed that each train of claims will have a geometrically distributed number of claims, their amounts following a mixture of five exponentials with the following parameters: the transition rates $\mu_1 = 3.675472$, $\mu_2 = 0.7116063$, $\mu_3 = 0.09447445$, $\mu_4 = 0.009322986$, $\mu_5 = 0.0004965620$, and the respective mixing probabilities $p_1 = 0.6635948$, $p_2 = 0.3114878$, $p_3 = 0.02405664$, $p_4 = 0.0008425574$, $p_5 = 0.00001820254$. (Note that this claim size distribution is taken from Thorin and Wikstad (1973) and is constructed in such a way that it approximates a log-normal distribution.) We further assume that the interclaim arrivals within the same claim train are i.i.d. random variables that occur at a rate $\theta = 100$ ($\theta \gg \lambda$). Under this assumption the MAP parameters can be written as

$$D_0 = \begin{bmatrix} -(\varepsilon + \lambda) & \varepsilon \\ 0 & -(\lambda + \theta) \end{bmatrix}, \qquad D_1 = \begin{bmatrix} \lambda & 0 \\ (1 - p)\theta & \lambda + p\theta \end{bmatrix}.$$

A closer look at our MAP representation reveals that transitions from the infectious environment to the background environment will happen at the end of each train of infectious claims at a rate $\theta(1 - p)$, $0 < p < 1$, where p is the probability that a new infectious claim will appear. We further assume that the initial probability distribution for the MAP is given by $\alpha = (1, 0)$, so the process will start in the background environment. The infinitesimal generator of the associated fluid queue can be partitioned as

$$T = \begin{bmatrix} -(\varepsilon + \lambda) & \varepsilon & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(\lambda + \theta) & (1 - p)\,\theta & \lambda & p_1\theta p & p_2\theta p & p_3\theta p & p_4\theta p & p_5\theta p \\ \mu & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & -\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 & -\mu_1 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & -\mu_2 & 0 & 0 & 0 \\ 0 & \mu_3 & 0 & 0 & 0 & 0 & -\mu_3 & 0 & 0 \\ 0 & \mu_4 & 0 & 0 & 0 & 0 & 0 & -\mu_4 & 0 \\ 0 & \mu_5 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_5 \end{bmatrix}.$$

To illustrate our results, we consider four different dividend strategies summarized in Table 2. As is observed, the first strategy assumes no barrier, the dividends being paid at any point in time at rate 0.25. Moving further to the other strategies, the barriers are added one by one, and the dividend rate is increased in each layer. Table 3 presents the first moment and the standard deviation of the discounted dividend payments ($\delta = 4\%$), under all the possible scenarios for different levels of the initial surplus, assuming that the probability of an infectious claim within each train of claims is $p = 0.5$.

As expected, within the same strategy an increase in the initial surplus leads to an increase of the expected amount of dividend payments. However, the standard deviation exhibits the opposite behavior, this being explained by the fact that extremes (ruin vs. non-ruin) are more likely for small values of the initial capital.

Table 2

Four Dividend Strategies under Consideration

Rates	Strategy 1	Strategy 2	Strategy 3	Strategy 4
$\underline{c}_1$	(1, 1, 1, 1)	(1, 0.75, 0.75, 0.75)	(1, 0.75, 0.5, 0.5)	(1, 0.75, 0.5, 0.25)
$\underline{d}_1$	(0.25, 0.25, 0.25, 0.25)	(0.25, 0.5, 0.5, 0.5)	(0.25, 0.5, 0.75, 0.75)	(0.25, 0.5, 0.75, 1)

Table 3

Expected (Standard Deviation) Discounted Dividends in the Multi-threshold Risk Model ($\delta = 0.04$, $p = 0.5$)

u	Strategy 1	Strategy 2	Strategy 3	Strategy 4
3	6.03 (1.06)	11.25 (2.07)	15.28 (2.97)	18.46 (3.81)
8	6.15 (0.68)	12.24 (1.47)	17.52 (2.27)	21.69 (3.15)
11	6.17 (0.63)	12.31 (1.26)	18.33 (2.03)	23.52 (2.98)
15	6.19 (0.55)	12.36 (1.11)	18.47 (1.75)	24.41 (2.52)

Comparing the four strategies with the discount rate being fixed at 4%, we observe that Strategy 4 will maximize the expected discounted dividend payments for a given initial surplus. However, a closer look at the standard deviation reveals the highest value among all strategies. This suggests that an insurer in the process of choosing the optimal strategy (among all four) will have to consider more than the first moment.

In Table 4 we assume that the infectious claims are more probable within the same train of claims ($p = 0.7$). For a given initial surplus, the ruin probabilities increase with p within the same strategy. Given the structure of the four strategies (lower dividend rates for small initial surplus levels) together with the previous comment, we anticipate the expected discounted dividend payments to decrease as p increases, which is in accordance with Table 4.

Table 4

Expected (Standard Deviation) Discounted Dividends in the Multi-threshold Risk Model ($\delta = 0.04$, $p = 0.7$)

u	Strategy 1	Strategy 2	Strategy 3	Strategy 4
3	5.61 (1.66)	10.22 (3.36)	13.53 (4.74)	15.81 (5.85)
8	5.93 (1.17)	11.65 (2.52)	16.27 (3.93)	19.47 (5.15)
11	6.01 (1.02)	11.88 (2.17)	17.36 (3.48)	21.51 (4.90)
15	6.08 (0.84)	12.06 (1.83)	17.81 (2.95)	22.94 (4.29)

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